APPLICATIONS OF DERIVATIVES

4.1 Derivative as a rate measure

If y = f(x) is a function of x, then $\frac{dy}{dx}$ or f '(x) represents the rate-measure of y with respect to x.

(a) $\left(\frac{dy}{dx}\right)_{x=x_0}$ or $f'(x_0)$ represents the rate of change of y w.r.t. x at $x = x_0$.

(b) If y increases as x increases, then $\frac{dy}{dx}$ is positive and if y decreases as x increases, then $\frac{dy}{dx}$ is negative.

(c) Marginal Cost (MC) is the instantaneous rate of change of total cost with respect to the number of items produced at an instant.

(d) Marginal Revenue (MR) is the instantaneous rate of change of total revenue with respect to the number of items sold at an instant.

4.2 Tangents and normal

(a) To find the equation of the tangent to the curve y = f(x) at the given point $P(x_1, y_1)$, proceed as under :

(i) Find $\frac{dy}{dx}$ from the given equation y = f(x).

(ii) Find the value of $\frac{dy}{dx}$ at the given point P (x_1, y_1), let $m = \left(\frac{dy}{dx}\right)_{\substack{x=x_1\\y=y_1}}$.

(iii) The equation of the required tangent is $y - y_1 = m(x - x_1)$.

In particular, if $= \left(\frac{dy}{dx}\right)_{\substack{x=x_1\\y=y_1}}$ does not exist, then the equation of the tangent is $x = x_1$.

(b) To find the equation of the normal to the curve y = f(x) at the given point $P(x_1, y_1)$, proceed as under :

(i) Find $\frac{dy}{dx}$ from the given equation y = f(x).

(ii) Find the value of $\frac{dy}{dx}$ at the given point $P(x_1, y_1)$.

(iii) If m is the slope of the normal to the given curve at P, then $m = -\frac{1}{\left(\frac{dy}{dx}\right)_{\substack{x=x_1\\y=y_1}}}$.

(iv) The equation of the required normal is $y - y_1 = m(x - x_1)$.

In particular if, $\left(\frac{dy}{dx}\right)_{\substack{x=x_1\\y=y_1}} = 0$, then the equation of the normal at P is $x = x_1$; and if $= \left(\frac{dy}{dx}\right)_{\substack{x=x_1\\y=y_1}}$ does not exist, then the equation of the normal at P is $y = y_1$.

Angle of intersection of two curves

The angle of intersection of two curves is the angle between the tangents to the two curves at their point of intersection.

If m_1 and m_2 are the slopes of the tangents to the given curves at their point of intersection $P(x_1, y_1)$, then the (acute) angle θ between the curves is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

In particular :

(i) if $m_1 m_2 = -1$, then curves are said to cut orthogonally.

(ii) if $m_1 = m_2$, the curves touch each other.

4.3 Increasing and decreasing functions

If f is a real valued function defined in an interval D (a subset of R), then f is called an increasing function in an interval D_1 (a subset of D) iff for all $x_1, x_2 \in D_1$, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ and f is called a strictly increasing function in D_1 iff for all $x_1, x_2 \in D_1$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

Similarly, f is called a decreasing function in an interval D_2 (a subset of D) iff for all $x_1, x_2 \in D_2$, $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ and f is called a strictly decreasing function in D_2 iff for all $x_1, x_2 \in D_2$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

In particular, if $D_1 = D$ then f is called an increasing function iff for all $x_1, x_2 \in D$, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$; and f is called strictly increasing function iff for all $x_1, x_2 \in D$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. Analogously, if $D_2 = D$ then f is called a decreasing function iff for all $x_1, x_2 \in D$, $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$; and f is called strictly decreasing function iff for all $x_1, x_2 \in D$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

A function which is either (strictly) increasing or (strictly) decreasing is called a (strictly) monotonic function.

Conditions for an increasing or a decreasing function

Theorem 1. If a function f is continuous in [a, b], and derivable in (a, b) and

(i) $f'(x) \ge 0$ for all $x \in (a, b)$, then f is increasing in [a, b]

(ii) $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing in [a, b].

Theorem 2. If a function f is continuous in [a, b], and derivable in (a, b) and

(i) $f'(x) \le 0$ for all $x \in (a, b)$, then f is decreasing in [a, b]

(ii) $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing in [a, b].

In fact, if a function f(x) is continuous in [a, b], derivable in (a, b) and

(i) $f'(x) > 0$ for all $x \in (a, b)$ except for a finite number of points where $f'(x) = 0$, then f(x) is strictly increasing in [a, b].

(ii) $f'(x) < 0$ for all $x \in (a, b)$ except for a finite number of points where $f'(x) = 0$, then f(x) is strictly decreasing in [a, b].

4.4 Maxima and Minima

If f is a real valued function defined on D (subset of R) then

(i) f is said to have a maximum value in D, if there exists a point x = c in D such that $f(c) \ge f(x)$ for all $x \in D$. The number f(c) is called the (absolute) maximum value of f in D and the point c is called point of maxima of f in D.

(ii) f is said to have a minimum value in D, if there exists a point x = d in D such that $f(d) \le f(x)$ for all $x \in D$. The number f(d) is called the (absolute) minimum value of f in D and the point d is called point of minima of f in D.

(a) Local maxima and local minima

If f is a real valued function defined on D (subset of R), then

(i) f is said to have a local (or relative) maxima at x = c (in D) iff there exists a positive real number δ such that $f(c) \geq f(x)$ for all x in $(c - \delta, c + \delta)$ i.e. $f(c) \geq f(x)$ for all x in the immediate neighbourhood of c, and c is called point of local maxima and f(c) is called local maximum value.

(ii) f is said to have local (or relative) minima at x = d(in D) iff there exists some positive real number δ such that $f(d) \leq f(x)$ for all $x \in (d - \delta, d + \delta)$ i.e. $f(d) \leq f(x)$ for all x in the immediate neighbourhood of d, and d is called point of local minima and f(d) is called local minimum value.

Geometrically, a point c in the domain of the given function f is a point of local maxima or local minima according as the graph of f has a peak or trough (cavity) at c.

(iii) a point (in D) which is either a point of local maxima or a point of local minima is called an extreme point, and the value of the function at this point is called an extreme value.

(b) Critical (or turning) point

If f is a real valued function defined on D (subset of R), then a point c (in D) is called a critical (or turning or stationary) point of f iff f is differentiable at x = c and $f'(c) = 0$.

(c) Point of inflection

A point P(c, f(c)) on the curve y = f(x) is called a point of inflection iff on one side of P the curve lies below the tangent at P and on the other side it lies above the tangent at P. Thus, a point where the curve crosses the tangent is called a point of inflection.

Theorem. A function f (or the curve y = f(x)) has a point of inflection at x = c iff $f'(c) = 0$, $f''(c) = 0$ and $f'''(c) \neq 0$.

(d) Working rules for finding (absolute) maximum and minimum

If a function f is differentiable in [a, b] except (possibly) at finitely many points, then to find (absolute) maximum and minimum values adopt the following procedure :

(i) Evaluate f(x) at the points where $f'(x) = 0$

(ii) Evaluate f(x) at the points where derivative fails to exist.

(iii) Find f(a) and f(b).

Then the maximum of these values is the absolute maximum and minimum of these values is the absolute minimum of the given function f.

(e) Working rules to find points of local maxima and minima

1. Locate the points where the given function 'f' is likely to have extreme values :
   (i) The points where the derivative fails to exist.
   (ii) The turning (critical) points i.e. the points where the derivative is zero.
   (iii) End points of the domain if f is defined in a closed interval [a, b].
   These are the only points where f may have extreme values, let c be any one such point.
2. If $f'(c)$ does not exist but f' exists in neighbourhood of c, then the following table describes the behaviour of the function f at c :

x	slightly < c	slightly > c	Nature of point
f '(x)	+ ve	– ve	Maxima
f '(x)	– ve	+ ve	Minima

3. If c is a turning point i.e. $f'(c)$ exists and $f'(c) = 0$. Let $n \geq 2$ be the smallest positive integer such that $f^{(n)}(c) \neq 0$, then the following table describes the behaviour of the function f at c :

n	sign of $f^{(n)}(c)$	Nature of the turning point c
odd	+ ve or – ve	Neither maxima nor minima
even	+ ve	Minima
even	– ve	Maxima

Alternatively, if it is difficult to find the derivatives of higher order then the following table describes the behaviour of the function f at c :

x	slightly < c	slightly > c	Nature of the turning point
f '(x)	+ ve	– ve	Maxima
f '(x)	– ve	+ ve	Minima
f '(x)	+ ve	+ ve	Neither maxima nor minima
f '(x)	– ve	– ve	Neither maxima nor minima

4. The following tables describe the behaviour at the end points ;
Let end point a.

x	slightly > a	Nature of point
f '(x)	+ ve	Minima
f '(x)	– ve	Maxima

Right end point b.

x	slightly < b	Nature of point
f '(x)	+ ve	Maxima
f '(x)	– ve	Minima

5. Finally, to find the absolute maximum or absolute minimum of a function f in closed interval [a, b] ; find the values of f at the points of all the three categories listed above. The maximum of these values is the absolute maximum and the minimum of these values is the absolute minimum of the function f in [a, b].

(f) For practical problems on maxima and minima

If a function is continuous in an open interval (a, b) and it has only one extreme point in (a, b), then it is a point of absolute maxima or a point of absolute minima according as it is a point of local maxima or a point of local minima.

4.5 Approximations

Let y = f(x) be a function of x and δx be a small change in the value of x and δy be the corresponding change in the value of y, then $\delta x = \frac{dy}{dx}\,\delta x$.

* **Some important terms**

Absolute error. The increment δx in x is called the absolute error in x.

SOLVED PROBLEMS

Ex.1 ***The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm ?***

Sol. Let r cm be the radius of an air bubble and V cm^3 be its volume. Then,

$$V = \frac{4}{3}\pi r^3 \qquad \text{..........(i)}$$

It is given that $\frac{dr}{dt} = \frac{1}{2}$ cm/s

Now, from (i), we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi r^2 \times \frac{1}{2} = 2\pi r^2$$

When r = 1 cm,

$$\frac{dV}{dt} = 2\pi\,(1)^2 = 2\pi \text{ cm}^3/\text{s}$$

Thus, the rate of change of volumes of the air bubble is 2π cm^3/s.

Ex.2 ***A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.***

Sol. Let r cm denote the radius of the spherical balloon and V be its volume. Then,

$$V = \frac{4}{3}\pi r^3 \qquad \text{........(i)}$$

It is given that $\frac{dV}{dt} = 900$ cm^3/s

From (i), we have

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{900}{4\pi r^2} = \frac{225}{\pi r^2}$$

When r = 15 cm,

$$\frac{dr}{dt} = \frac{225}{\pi(15)^2} = \frac{1}{\pi} = \frac{7}{22} \text{ cm/s}$$

Thus, the rate at which the radius of the balloon increases is $\frac{7}{22}$ cm/s, when r = 15 cm.

Ex.3 ***Find the least value of a such that the function f given by f(x) = x^2 + ax + 1 is strictly increasing in (1, 2).***

Sol. Here, $f'(x) = 2x + a$

Now, $1 < x < 2 \Rightarrow 2 < 2x < 4$

$\Rightarrow 2 + a < 2x + a < 4 + a$ ....(1)

Now, f(x) is an increasing function, so $f'(x) > 0$ for $x \in (1, 2)$

$\Rightarrow 2x + a > 0$ for $x \in (1, 2)$

$\Rightarrow 0 < 2x + a$ for $x \in (1, 2)$

$\Rightarrow 0 \le 2 + a$ [using (1)]

$\Rightarrow a \ge -2$

So, the least value of a is –2.

Ex.4 ***Prove that the function f given by f(x) = $x^2 - x + 1$ is neither strictly increasing nor strictly decreasing in (–1, 1).***

Sol. We have, $f(x) = x^2 - x + 1$

$\Rightarrow f'(x) = 2x - 1$

The critical value of f(x) is $\frac{1}{2}$

Case 1 When $-1 < x < \frac{1}{2}$

In this case, $-1<x<\frac{1}{2} \Rightarrow -3<2x-1<0$

$\Rightarrow f'(x) < 0$

Thus, f(x) is decreasing for $-1 < x < \frac{1}{2}$

Case 2 When $\frac{1}{2} < x < 1$

In this case, $\frac{1}{2} < x < 1 \Rightarrow 1 < 2x < 2$

$\Rightarrow 0 < 2x - 1 < 1$

Hence, $f'(x) > 0$

So, f(x) is increasing for $\frac{1}{2} < x < 1$

So, in the interval (–1, 1), f(x) is neither increasing nor decreasing.

Ex.5 ***Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.***

Sol. Let the point of tangency be (x_1, y_1).

The, slope of the tangent at (x_1, y_1) is given by $\left.\frac{dy}{dx}\right]_{x=x_1}$, which is $\left[3x^2\right]_{x=x_1}$

i.e., $3x_1^2$.

Since, the slope is given to be equal to y_1,we have

$$3x_1^2 = y_1 \qquad ......(1)$$

As (x_1, y_1) lies on the curve $y = x^3$, we have

$$y_1 = x_1^3 \qquad ....(2)$$

From (1) and (2), we have

$$3x_1^2 = x_1^3$$

$\Rightarrow$ $x_1 = 0, 3$ and so $y_1 = 0^3 = 0$ or $y_1 = 3^3 = 27$

Thus, the required points are (0, 0) and (3, 27).

Ex.6 ***Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.***

Sol. Slope of the tangent to the curve $ay^2 = x^3$ at the point (am^2, am^3) is given by

$$\left.\frac{dy}{dx}\right]_{(am^2, am^3)}$$

Given $\qquad ay^2 = x^3$

We have, $\quad 2ay\frac{dy}{dx} = 3x^2$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

$$\left.\frac{dy}{dx}\right]_{(am^2, am^3)} = \frac{3}{2a}\frac{a^2m^4}{am^3} = \frac{3m}{2}$$

$\therefore$ Slope of the normal to the curve at (am^2, am^3) is $-\frac{2}{3m}$.

Hence, the equation of the normal at the point (am^2, am^3) to the curve $ay^2 = x^3$ is

$$y - am^3 = -\frac{2}{3m}(x - am^2)$$

i.e., $\qquad 3my - 3am^4 = -2x + 2am^2$

i.e., $\qquad 2x + 3my - am^2 (2 + 3m^2) = 0$

Ex.7 ***A circular metal plate expands under heating so that its radius increases by 2%. Find the approximate increase in the area of the plate if the radius of the plate before heating is 10 cm.***

Sol. Let r cm be the radius of the circular plate and A be its area. Then, $A = \pi r^2$

$$\Rightarrow \qquad \frac{dA}{dr} = 2\pi r$$

Given $\qquad r = 10$ cm (before heating)

Let Δr be the increase in the radius of the circular plate (after heating) and ΔA be the corresponding increase in area.

Then, $\qquad \Delta r = 2\%$ of 10 cm

$$= \frac{2}{100} \times 10 \text{ cm} = 0.2 \text{ cm}$$

For, $r = 10$ cm, $\Delta r = 0.2$ cm

$$\Delta A = \frac{dA}{dr}.\Delta r$$

$= 2\pi r \,.\, (0.2)$ sq cm

$= 2\pi\,(10)\,.\,(0.2)$ sq cm

$= 4\pi$ sq cm

Ex.8 ***Find the approximate change in the total surface area of a right circular cone, when the radius r remains constant while the altitude 'h' changes by α.***

Sol. Let s denote the total surface area of the right circular cone. Then,

$$s = \pi r(r^2 + h^2)^{\frac{1}{2}} + \pi r^2$$

Since r remains constant, s can be regarded as a function of h only.

We have $\Delta s = \frac{ds}{dh}\,.\,\Delta h$

$$\Rightarrow \Delta s = \left(\frac{\pi rh}{\sqrt{r^2 + h^2}} + 0\right).\alpha \qquad (\because \Delta h = \alpha)$$

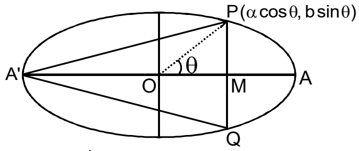
$$= \frac{\pi rh\alpha}{\sqrt{r^2 + h^2}}$$

Ex.9 ***Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.***

Sol. Let the given ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Let P (a cos θ, b sin θ) be a point on the ellipse Then,

PQ=2PM=2b sinθ
A′ M = A′ O + OM
= a + a cos θ



Now, Area (A) of A′PQ $= \frac{1}{2}(a + a\cos\theta)(2b\sin\theta)$

$= ab\sin\theta(1 + \cos\theta)$

We shall now find for what value of θ A is maximum.

$\frac{dA}{d\theta} = ab\cos\theta(1 + \cos\theta) - ab\sin^2\theta = ab\cos\theta + ab(\cos^2\theta - \sin^2\theta)$

$= ab[\cos\theta + \cos 2\theta]$

$= ab\left[2\cos\frac{3\theta}{2}\cos\frac{\theta}{2}\right]$

Equating $\frac{dA}{d\theta}$ to 0, we have

$\cos\frac{3\theta}{2}\cos\frac{\theta}{2} = 0$

$\Rightarrow \cos\frac{3\theta}{2} = 0$ or $\cos\frac{\theta}{2} = 0$

$\Rightarrow \frac{3\theta}{2} = \frac{\pi}{2}$ or $\frac{\theta}{2} = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{3}$ or $\theta = \pi$

But $0 < \theta < \frac{\pi}{2}$ $\therefore \theta = \frac{\pi}{3}$

Now, $\frac{d^2A}{d\theta^2} = ab[-\sin\theta - 2\sin 2\theta]$

$\left.\frac{d^2A}{d\theta^2}\right]_{\theta=\frac{\pi}{3}} = ab\left[-\frac{\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2}\right] < 0$

$\Rightarrow$ A is maximum when $\theta = \frac{\pi}{3}$

Maximum value of the area of the isosceles triangle

$= ab\sin\left(\frac{\pi}{3}\right)\left[1 + \cos\frac{\pi}{3}\right] = ab\left(\frac{\sqrt{3}}{2}\right)\left(1 + \frac{1}{2}\right)$

$= \frac{3\sqrt{3}}{4}$ ab sq units.

Ex.10 Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi-vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2\alpha$

Sol. Let ABC be the cone whose semi-vertical angle is α.

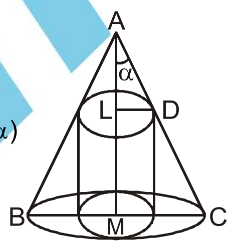
AM = h

A cylinder is inscribed in it whose radius LD of the base is r and height LM = h′.

From the figure, h′ = AM – AL

= h – γ cot α ...(1)

Volume (V) of the cylinder is given by



$V = \pi r^2 h'$

$= \pi r^2(h - \gamma\cot\alpha)$ ...(2)

$\therefore \frac{dV}{dr} = 2\pi r(h - \gamma\cot\alpha) + \pi r^2(-\cot\alpha)$

$= 2\pi\gamma h - 3\pi r^2\cot\alpha$

$= \pi r(2h - 3\gamma\cot\alpha)$

For max. or min. value of V, we must have $\frac{dV}{dr} = 0$

$r = \frac{2}{3}h\tan\alpha$

Now,

$\frac{d^2V}{dr^2} = \pi(2h - 3\gamma\cot\alpha) + \pi r(-3\cot\alpha)$

$= 2\pi h - 6\pi r\cot\alpha$

$\left.\frac{d^2V}{dr^2}\right]_{r=\frac{2}{3}h\tan\alpha} = \pi(2h - 4h) < 0$

Thus, V is maximum, when $r = \frac{2}{3}h\tan\alpha$

From (1), we have

$h' = h - \frac{2}{3}h\tan\alpha\cot\alpha = h - \frac{2}{3}h = \frac{1}{3}h$

Greatest volume of the cylinder = $[V]_{r=\frac{2}{3}h\tan\alpha}$

$= \pi\left(\frac{2}{3}h\tan\alpha\right)^2\left(h - \frac{2}{3}h\tan\alpha\cot\alpha\right)$

$= \frac{4}{2\pi}\pi h^3\tan^2\alpha$

EXERCISE – I UNSOLVED PROBLEMS

Q.1 The volume of a cube is increasing at a constant rate. Prove that the increase in the surface area varies inversely as the length of the edge of the cube.

Q.2 A man 2m high walks at a uniform speed of 6m/min away from a lamp-post, 5m high. Find the rate at which the length of his shadow increases.

Q.3 Two men A and B start at the same time from the junction of two roads, one on each road with uniform sped v. Show that the rate at which AB, the distance between them increases is equal to $\sqrt{3}$ v ; if the roads meet at an angle of 120º.

Q.4 Find the intervals on which function is (i) Increasing (ii) Decreasing $f(x) = x^3 - 6x^2 + 9x + 15$

Q.5 Find the intervals on which function is (i) Increasing (ii) Decreasing, If $f(x) = \log(1 + x) - \frac{x}{(1+x)}$

Q.6 Separate the interval $\left[0, \frac{\pi}{2}\right]$ into subinterval in which $f(x) = (\sin^4 x + \cos^4 x)$ is (i) Increasing (ii) Decreasing.

Q.7 Prove that the function f given by If $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in (–1, 1)

Q.8 Prove that the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$ at the point (a, b), whatever be the value of n.

Q.9 Find the equations of the tangent and the normal at the point 't' on the curve $x = a \sin^3 t$, $y = b \cos^3 t$.

Q.10 For the curve $y = 4x^3 - 2x^5$, find all points at which the tangent passes through the origin.

Q.11 Prove that the sum of the intercepts on the coordinate axes of any tangent to $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is constant.

Q.12 Using differentials, find the approximate value of $(127)^{1/3}$

Q.13 If is a triangle ABC, the side c and the angle C remain constant, while the remaining elements are changed slightly, differentials show that $\frac{da}{\cos A} + \frac{db}{\cos B} = 0$

Q.14 The combined resistance R of two resistors R_1 and R_2, where $R_1, R_2 > 0$ is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If $R_1 + R_2 = C$ (Constant) show that the maximum resistance R is obtained by choosing $R_1 = R_2$

Q.15 Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi vertical angle α is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

Q.16 Show that a cone of greatest volume which can be inscribed in a given sphere is such that three times it altitude is twice the diameter of the sphere. Find the volume of the largest cone inscribed in sphere of radius R.

Q.17 A square tank of capacity 250 cubic meters has to be dug out. The cost of the land is Rs. 50 per sq. m. The cost of digging increases with the depth and for the whole tank cost is Rs $400 \times h^2$, where h metres is the depth of the tank. What should be the dimensions of the tank so that the cost be minimum ?

Q.18 A jet of an enemy is flying along the curve $y = x^2 + 2$. A soldier is placed at the point (3, 2). What is the nearest distance between the soldier and the jet ?

Q.19 A manufacture can sell x items at a price of Rs (250 – x) each. The cost of producing x items is Rs $(2x^2 - 50x + 12)$. Determine the number of items to be sold so that he can make maximum.

Q.20 The cost of fuel for running a bus is proportional to the square of the speed generated in km/hr. It costs Rs 48 per hour when the bus is moving with a speed of 20 km/hr. What is the most economical speed if the fixed charges are Rs 108 for one hour, over and above the running charges ?

EXERCISE – II BOARD PROBLEMS

Q.1 Find the intervals in which the function $f(x) = x^3 - 6x^2 + 9x + 15$ is increasing or decreasing.

Q.2 Find the largest possible area of right angled triangle whose hypotenuse is 5 cm long.

Q.3 Using differentials, find the approximate value of $\sqrt{0.37}$.

Q.4 Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x + 30$ is
(i) increasing (ii) decreasing.

Q.5 Using differentials, find approximate value of $\sqrt{0.26}$.

Q.6 Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

Q.7 A window is in the form of rectangle above which there is semi-circle. If perimeter of window is p cm, show that window will allow the maximum possible light only when the radius of semi-circle is $\frac{p}{\pi+4}$ cm.

Q.8 If $y = x^4 + 10$ and x changes from 2 to 1.99. Find the approximate change in y.

Q.9 Show that rectangle of maximum area that can be inscribed in a circle of radius r is a square of side $\sqrt{2}r$.

Q.10 A balloon which always remains spherical is being inflated by pumping in gas at the rate of 900 cm^3/sec. Find the rate at which the radius of balloon is increasing when radius of balloon is 15 cm.

Q.11 A window is in the form of a rectangle surmounted by a semi-circle. If the perimeter of the window is 100 m, find the dimensions of the window so that maximum light enters through the window.

Q.12 A particle moves along the curve $y = \frac{2}{3}x^3 + 1$. FInd the points on the curve at which y-coordinate is changing twice as fast as x-coordinate.

Q.13 For the curve $y = 4x^3 - 2x^5$, find all points at which tangent passes through origin.

Q.14 A square piece of tin of side 18 cm is to be made into a box without a top by cutting a square piece from each corner and folding up the flaps. What should be the side square to be cut off so that volume of the box be maximum ? Also find the maximum volume of the box.

Q.15 Find the intervals on which function $f(x) = \frac{x}{1+x^2}$ is (i) Increasing (ii) Decreasing.

Q.16 Find the intervals in which the function $f(x) = x^3 - 6x^2 + 9x + 15$ is
(i) increasing (ii) decreasing.

Q.17 Find equations of tangent and normal to the curve $x = 1 - \cos\theta$, $y = \theta - \sin\theta$ at $\theta = \frac{\pi}{4}$.

Q.18 Find the intervals in which the function $f(x) = \frac{4x^2+1}{x}$, $x \neq 0$ is
(i) increasing (ii) decreasing.

Q.19 Find the equation of tangent and normal to the curve $y = x^2 + 4x + 1$ at the point whose x-coordinate is 3.

Q.20 The volume of spherical balloon is increasing at the rate of 25 cm^3/sec. Find the rate of change of its surface area at the instant when its radius is 5 cm.

Q.21 Find the intervals in which function $f(x) = 2x^3 - 6x^2 - 48x + 17$ is

(i) increasing (ii) decreasing.

Q.22 Using differentials, find the approximate value of $\sqrt{0.037}$, correct upto three decimal places.

Q.23 The surface area of a spherical bubble is increasing at the rate of 2 cm^2/s. Find the rate at which the volume of the bubble is increasing at the instant its radius is 6 cm.

Q.24 A wire of length 36 cm is cut into two pieces one of the pieces is turned in the form of a square and the other into a circle. What should be the lengths of each piece so that the sum of the areas of two be minimum ?

Q.25 Prove that the line $\frac{x}{a} + \frac{y}{b} = 1$ is a tangent to the curve $y = be^{-x/a}$, at the point where the curve cuts y-axis.

Q.26 An open box with a square base is to be made out of a given iron sheet of area 27 sq. m. Show that the maximum volume of the box is 13.5 cu. m.

Q.27 Find the intervals in which the function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

(i) strictly increasing (ii) strictly decreasing.

Q.28 Find the equations of the tangent and normal to the curve $16x^2 + 9y^2 = 144$ at (x_1, y_1) where $x_1 = 2$ and $y_1 > 0$. Also find the points of intersection where both tangent and normal cut the x-axis.

Q.29 Find a point on the parabola $f(x) = (x - 3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

Q.30 Find the equation of the line through the point (3, 4) which cuts from the first quadrant a triangle of minimum area.

Q.31 Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ is 500π cm^3.

Q.32 Given the sum of the perimeters of a square and a circle prove that the sum of their area is least when the side of the square is equal to the diameter of the circle.

Q.33 Find the equation of the tangent to the curve $x = \theta + \sin\theta$, $y = 1 + \cos\theta$ at $\theta = \frac{\pi}{4}$.

Q.34 Prove that the tangents to the curve $y = x^2 - 5x + 6$ at the point P(2, 0) and Q(3, 0) are at the right angles to each other.

Q.35 A point source of light along a straight road is at a height of 'a' metre. A boy 'b' metre in height is walking along the road. How fast is his shadow increasing if he is walking away from the light at the rate of c metre per minute ?

Q.36 The volume of a cube is increasing at the rate of 7 cubic centimeter per second. How fast the surface area of the cube increasing when the length of an edge is 12 centimeter ?

Q.37 Find the equations of the tangent and the normal to the curve $y = x^3$ at the point P(1, 1)

Q.38 Prove that the curve $x = y^2$ and $xy = k$ cut each other at right angles if $8k^2 = 1$.

Q.39 Show that the height of cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$.

Q.40 Prove that volume of largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of sphere.

Q.41 A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area of the square and the circle is minimum.

Q.42 Find the point on the curve $x^2 = 4y$ which is nearest to the point $(-1, 2)$

Q.43 Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

Q.44 Show that the rectangle of maximum area that can be inscribed in a circle is a square.

Q.45 Show that the height of the cylinder of maximum volume that can be inscribed in a cone of height h is $\frac{1}{3}h$.

Q.46 The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When x = 8 cm and y = 6 cm, find the rate of change of (a) the perimeter, (b) the area of the rectangle.

Q.47 Find the intervals in which the function f given by $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$, is strictly increasing or strictly decreasing.

Q.48 If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

Q.49 A manufacturer can sell x items at a price of Rs $\left(5-\frac{x}{100}\right)$ each. The cost price of x items is Rs $\left(\frac{x}{5}+100\right)$. Find the number of items he should sell to earn maximum profit.

Q.50 Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Q.51 Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}$, $x \neq 0$ is

(i) increasing (ii) decreasing.

Q.52 Find the volume o the largest cylinder that can be inscribed in a sphere of radius r.

Q.53 A tank with rectangular base and rectangular sides, open at the top is to be constrcuted so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs 70 per sq. metre for the base and Rs 45 per sq. metre for sides, what is the cost of least expensive tank ?

Q.54 Show that the volume of the greatest cylinder that can be inscribed in a cone of height 'h' and semi-vertical angle 'α' is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

Q.55 Find the intervals in which the following function is
(a) strictly increasing, (b) strictly decreasing.

Q.56 Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function in $\left[0, \frac{\pi}{2}\right]$.

Q.57 If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area.

Q.58 Show that the right-circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

Q.59 A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window.

Q.60 A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?

Q.61 Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of the at of the cone.

Q.62 An open box with a square base is to be made out of a given quantity of carboard of area c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

Q.63 Find the area of the greatest rectange that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

OR

Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which pass through the point $\left(\frac{4}{3}, 0\right)$.

Answers

EXERCISE – 1 (UNSOLVED PROBLEMS)

2. 4m/min **4.** In. $(-\infty, 1) \cup (3, \infty)$, Dec. $(1, 3)$ **5.** In. $(0, \infty)$, Dec. $(-1, 0)$

6. In. $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, Dec. $\left(0, \frac{\pi}{4}\right)$ **9.** tangent $\frac{x}{a\sin t} + \frac{y}{b\cos t} = 1$, normal ax sint – by cost = $a^2 \sin^4 t - b^2 \cos^4 t$

10. (0, 0) ; (1, 2) and (–1, –2) **12.** 5.026 **16.** $\frac{32\pi R^3}{81}$ **17.** side = 10 m depth = 2.5 m

18. (1, 3) **19.** 50 items **20.** speed = 30 km/hr.

EXERCISE – 2 (BOARD PROBLEMS)

1. Increasing in $(-\infty, -1) \cup (3, \infty)$ and decreasing in $(1, 3)$ **2.** $\frac{25}{4}$ sq. units **3.** 0.683

5. 0.51 **8.** – 0.32 **10.** $\frac{7}{22}$ cm/sec **11.** $\frac{200}{\pi+4}, \frac{100}{\pi+4}$ **12.** $\left(1, \frac{5}{3}\right), \left(-1, \frac{1}{3}\right)$

13. (0,0), (1,2) and (–1,–2) **14.** 3 cm, 432 cm^3 **15.** Increasing in $(-1, 1)$ and decreasing in $(-\infty, -1) \cup (1, \infty)$

16. Increasing in $(-\infty, -1) \cup (3, \infty)$ and decreasing in $(1, 3)$

17. $y - \frac{\pi}{4} + \frac{1}{\sqrt{2}} = \sqrt{2} - 1\left(x - 1 + \frac{1}{\sqrt{2}}\right)$, $y - \frac{\pi}{4} + \frac{1}{\sqrt{2}} = \frac{-1}{\sqrt{2}-1}\left(x - 1 + \frac{1}{\sqrt{2}}\right)$

18. Increasing in $\left(-\infty, \frac{-1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$ and decreasing in $\left(\frac{-1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$

19. $10x - y - 8 = 0$, $x + 10y - 223 = 0$ **20.** 10 cm^2/s

21. Increasing in $(-\infty, -2) \cup (4, \infty)$ and decreasing in $(-2, 4)$ **22.** 0.1924 **23.** 6 cm^3/s

24. $\frac{144\sqrt{3}}{9+4\sqrt{3}}$ cm, $\frac{324}{9+4\sqrt{3}}$ cm **27.** Increasing in $(-\infty, -2) \cup (3, \infty)$ and decreasing in $(-2, 3)$

28. $y - \frac{4}{3}\sqrt{5} = -\frac{8}{3\sqrt{5}}(x - 2)$ **29.** $\left(\frac{7}{2}, \frac{1}{4}\right)$ **30.** $4zx + 3y - 24 = 0$

33. $x + (\sqrt{2} + 1)y - \frac{3+2\sqrt{2}}{\sqrt{2}} - \frac{\pi}{4} - \frac{1}{\sqrt{2}} = 0$ **35.** $\frac{bc}{a-b}$ m/min. **36.** $\frac{7}{3}$ cm^2/s

37. $3x - y - 2 = 0$, $x + 3y - 4 = 0$ **41.** $C = \frac{28\pi}{\pi+4}$, $S = \frac{112}{\pi+4}$ **42.** (–2, 1) **46. (a)** –2 cm/min **(b)** 2 cm^2/min

47. In. $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$, De. $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ **49.** 240 **50.** $48x - 24y = 23$ **51.** In. $(-\infty, -1) \cup (1, \infty)$, De. $(-1, 1)$

52. $\frac{4\pi R^3}{3\sqrt{3}}$ **53.** Rs 1000 **57.** 2.16π **59.** $x = \frac{4(6+\sqrt{3})}{11}$, $y = \frac{30-6\sqrt{3}}{11}$ **60.** $\frac{8}{3}$