CONTINUITY AND DIFFERENTIABILITY

3.1 BASIC CONCEPTS AND IMPORTANT RESULTS

(a) Continuity of a real function at a point

A function f is said to be left continuous or continuous from the left at x = c iff

(i) f(c) exists (ii) $\operatorname{Lt}_{x \to c^-} f(x)$ exists and (iii) $\operatorname{Lt}_{x \to c^-} f(x) = f(c)$.

A function f is said to be right continuous or continuous from the right at x = c iff

(i) f(c) exists (ii) $\operatorname{Lt}_{x \to c^+} f(x)$ exists and (iii) $\operatorname{Lt}_{x \to c^+} f(x) = f(c)$.

A function f is said to be continuous at x = c iff

(i) f(c) exists (ii) $\operatorname{Lt}_{x \to c} f(x)$ exists and (iii) $\operatorname{Lt}_{x \to c} f(x) = f(c)$.

Hence, a function is continuous at x = c iff it is both left as well as right continuous at x = c.

When $\operatorname{Lt}_{x \to c} f(x)$ exists but either f(c) does not exist or $\operatorname{Lt}_{x \to c} f(x) \neq f(c)$, we say that f has a removable discontinuity; otherwise, we say that f has non-removable discontinuity.

(b) Continuity of a function in an interval

A function f is said to be continuous in an open interval (a, b) iff f is continuous at every point of the interval (a, b) ; and f is said to be continuous in the closed interval [a, b] iff f is continuous in the open interval (a, b) and it is continuous at a from the right and at b from the left.

Continuous function. A function is said to be a continuous function iff it is continuous at every point of its domain. In particular, if the domain is a closed interval, say [a, b], then f must be continuous in (a, b) and right continuous at a and left continuous at b.

The set of all point where the function is continuous is called its domain of continuity. The domain of continuity of a function may be a proper subset of the domain of the function.

3.2 PROPERTIES OF CONTINUOUS FUNCTIONS

Property 1. Let f, g be two functions continuous at x = c, then

(i) αf is continuous at x = c, $\forall\ \alpha \in R$ (ii) f + g is continuous at x = c

(iii) f – g is continuous at x = c (iv) fg is continuous at x = c

(v) $\frac{f}{g}$ is continuous at x = c, provided $g(c) \neq 0$.

Property 2. Let D_1 and D_2 be the domains of continuity of the functions f and g respectively then

(i) αf is continuous on D_1 for all $\alpha \in R$ (ii) f + g is continuous on $D_1 \cap D_2$

(iii) f – g is continuous on $D_1 \cap D_2$ (iv) fg is continuous on $D_1 \cap D_2$

(v) $\frac{f}{g}$ is continuous on $D_1 \cap D_2$ except those points where g(x) = 0.

Property 3. A polynomial function is continuous everywhere.
In particular, every constant function and every identity function is continuous.

Property 4. A rational function is continuous at every point of its domain.

Property 5. If f is continuous at c, then | f | is also continuous at x = c.
In particular, the function | x | is continuous for every $x \in R$.

Property 6. Let f be a continuous one-one function defined on [a, b] with range [c, d], then the inverse function f^{-1} : [c, d] $\to$ [a, b] is continuous on [c, d]

Property 7. If f is continuous at c and g is continuous at f(c), then gof is continuous at c.

Property 8. All the basic trigonometric functions i.e. sin x, cos x, tan x, cot x, sec x and cosec x are continuous.

Property 9. All basic inverse trigonometric functions i.e. $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $\text{cosec}^{-1} x$ are continuous (in their respective domains).

Property 10. Theorem. If a function is differentiable at any point, it is necessarily continuous at that point.

The converse of the above theorem may not be true i.e. a function may be continuous at a point but may not be derivable at that point.

3.3 DERIVATIVE OF VARIOUS FUNCTIONS

(a) Derivative of composite functions

Theorem. If u = g(x) is differentiable at x and y = f(u) is differentiable at u, then y is differentiable at x and $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

If g is differentiable at x and f is differentiable at g(x), then the composite function h(x) = f(g(x)) is differentiable at x and h′(x) = f′(g(x))· g′(x).

Chain Rule. The above rule is called the chain rule of differentiation, since determining the derivative of y = f(g(x)) at x involves the following chain of steps :

(i) First, find the derivative of the outer function f at g(x).

(ii) Second, find the derivative of the inner function g at x.

(iii) The product of these two derivatives gives the required derivative of the composite function fog at x .

(i) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, provided $\frac{dx}{dt} \neq 0$.

(ii) $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, provided $\frac{dx}{dy} \neq 0$.

(iii) $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$

(iv) $\frac{d}{dx}(|x|) = \frac{x}{|x|}$, $x \neq 0$.

(b) Derivatives of inverse trigonometric functions

(i) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$ i.e. $|x| < 1$

(ii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, $x \in (-1, 1)$ i.e. $|x| < 1$

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, for all $x \in R$

(iv) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, for all $x \in R$

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$, $x > 1$

(vi) $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$, $x > 1$

(c) Derivatives of algebraic and trigonometric functions

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$

(ii) $\frac{d}{dx}(x^x) = x^x \log ex$

(iii) $\frac{d}{dx}(\sin x) = \cos x$

(iv) $\frac{d}{dx}(\cos x) = -\sin x$

(v) $\frac{d}{dx}(\tan x) = \sec^2 x$

(vi) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(vii) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$.

(d) Derivatives of exponential and logarithmic functions

(i) $\frac{d}{dx}(e^x) = e^x$, for all $x \in R$

(vi) $\frac{d}{dx}(\log_a |x|) = \frac{1}{x\log a}$, $x \neq 0$, $a > 0$, $a \neq 1$.

(ii) $\frac{d}{dx}(a^x) = a^x \log a$, $a > 0$, $a \neq 1$, $x \in R$

(v) $\frac{d}{dx}(\log |x|) = \frac{1}{x}$, $x \neq 0$

(iii) $\frac{d}{dx}(\log x) = \frac{1}{x}$, $x > 0$

(iv) $\frac{d}{dx}(\log_a x) = \frac{1}{x\log a}$, $x > 0$, $a > 0$, $a \neq 1$

(e) Logarithmic differentiation

If u, v are differentiable functions of x, then $\frac{d}{dx}(u^v) = u^v \frac{d}{dx}(v \log u)$.

(f) Derivatives of functions in parametric form

If x and y are two variables such that both are explicitly expressed in terms of a third variable, say t, i.e. if $x = f(t)$ and $y = g(t)$, then such functions are called parametric functions and

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ provided } \frac{dx}{dt} \neq 0.$$

(g) Derivative of second order

If a function f is differentiable at a point x, then its derivative f′ is called the first derivative or derivative of first order of the function f. If f′ is further differentiable at the same point x, then its derivative is called the second derivative or derivative of the second order of f at that point and is denoted by f″.

If the function f is defined by $y = f(x)$, then its first and second derivatives are denoted by $f'(x)$ and $f''(x)$ or by $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ or by y_1 and y_2 or by y' and y'' respectively.

3.4 ROLLE'S THEOREM AND LAGRANGE'S MEAN VALUE THEOREM

(i) Rolle's theorem

If a function f is

(i) continuous in the closed interval [a, b]

(ii) derivable in the open interval (a, b) and

(iii) $f(a) = f(b)$,

then there exists atleast one real number c in (a, b) such that $f'(c) = 0$.

Thus converse of Rolle's theorem may not be true.

(ii) Lagrange's mean value theorem

If a function f is

(i) continuous in the closed interval [a, b] and

(ii) derivable in the open interval (a, b),

then there exists atleast one real number c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

The converse of Lagrange's mean value theorem may not be true.

SOLVED PROBLEMS

Ex.1 ***Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?***

Sol. Here, $f(\pi) = (\pi)^2 - \sin \pi - 5 = \pi^2 - 5$

$$\lim_{x\to\pi^+} f(x) = \lim_{h\to 0} f(\pi+h) = \lim_{h\to 0} [(\pi+h)^2 - \sin(\pi+h) - 5]$$

$$= \lim_{h\to 0} [(\pi + h)^2 + \sin h - 5] = \pi^2 - 5$$

and $\lim_{x\to\pi^-} f(x) = \lim_{h\to 0} f(\pi-h) = \lim_{h\to 0} [(\pi-h)^2 - \sin(\pi-h) - 5]$

$$= \lim_{h\to 0} [(\pi - h)^2 - \sin h - 5] = \pi^2 - 5$$

Since, $\lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^-} f(x) = f(\pi)$,

the function f is continuous at $x = \pi$.

Ex.2 ***Discuss the continuity of the cosine, cosecant, secant and cotangent functions.***

Sol. Continuity of $f(x) = \cos x$

Let a be an arbitrary point of the domain of the function $f(x) = \cos x$.

Then, $f(a) = \cos a$

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a + h) = \lim_{h\to 0} \cos(a + h)$$

$$= \lim_{h\to 0} [\cos a \cos h - \sin a \sin h]$$

$$= \cos a \times 1 - \sin a \times 0 = \cos a$$

and $\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a - h) = \lim_{h\to 0} \cos(a - h)$

$$= \lim_{h\to 0} [\cos a \cos h + \sin a \sin h]$$

$$= \cos a \times 1 + \sin a \times 0 = \cos a$$

Since, $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$, the function is continuous at $x = a$.

As a is an arbitrary point of the domain, the function is continuous on the domain of the functions,

Proceed as above and prove yourself the continuity of other trigonometric

Ex.3 ***Find all points of discontinuity of f, where***

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \ge 0 \end{cases}$$

Sol. The point of discontinuity of f can at most be $x = 0$.

Let us examine the continuity of f at $x = 0$.

Here, $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} [(0+h)+1] = 1$

and $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0 - h) = \lim_{h\to 0} \dfrac{\sin(0-h)}{0-h}$

$$= \lim_{h\to 0} \frac{-\sin h}{-h} = 1$$

Also, $f(0) = 0 + 1 = 1$

Since, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$, f is continuous at $x = 0$.

Hence, there is no point of discontinuity of f.

Ex.4 ***Determine if f defined by***

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function.

Sol. It is sufficient to examine the continuity of the function f at $x = 0$.

Here $f(0) = 0$

Also, $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0 + h)$

$$= \lim_{h\to 0} \left[(0+h)^2 \sin\frac{1}{0+h}\right] = \lim_{h\to 0} \left[h^2 \sin\frac{1}{h}\right] = 0$$

and $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0 - h)$

$$= \lim_{h\to 0} \left[(0-h)^2 \sin\frac{1}{0-h}\right] = \lim_{h\to 0} \left[h^2 \sin\left(\frac{1}{-h}\right)\right] = 0$$

$$\left[\because \left|\sin\frac{1}{h}\right| \le 1\right]$$

Hence, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$

So, f is continuous at $x = 0$

This implies that f is a continuous function at all $x \in R$.

Ex.5 ***Examine the continuity of f, where f is defined by***

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$$

Sol. Here, f(0) = –1

Also, $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} [\sin(0+h) - \cos(0+h)]$

$= \lim_{h\to 0} [\sin h - \cos h] = -1$

and $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} [\sin(0-h) - \cos(0-h)]$

$= \lim_{h\to 0} [-\sin h - \cos h]$

[$\because$ sin (–h) = –sin h]

$= -0 - 1 = -1$

and cos (–h) = cos h]

Hence, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$

So, f is continuous at x = 0; and hence continuous at all $x \in R$.

Ex.6 ***Find the value of k so that the following function f is continuous at the indicated point :***

(i) $f(x) = \begin{cases} kx+1, & \text{if } x \le 5 \\ 3x-5, & \text{if } x > 5 \end{cases}$ **at x = 5**

(ii) $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$ **at x = 2**

Sol. (i) Since f is given to be continuous at x = 5, we have

$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^-} f(x) = f(5)$

$\Rightarrow \lim_{h\to 0} f(5+h) = \lim_{h\to 0} f(5-h) = f(5)$

$\Rightarrow \lim_{h\to 0} [3(5+h)-5] = \lim_{h\to 0} [k(5-h)+1] = 5k+1$

$\Rightarrow 10 = 5k + 1 \quad \Rightarrow k = \frac{9}{5}$

(ii) Since f is given to be continuous at x = 2, we have

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$

$\Rightarrow \lim_{h\to 0} f(2+h) = \lim_{h\to 0} f(2-h) = f(2)$

$\Rightarrow \lim_{h\to 0} (3) = \lim_{h\to 0} [k(2-h)^2] = 4k$

$\Rightarrow 3 = 4k \quad \Rightarrow k = \frac{3}{4}$

Ex.7 ***Find the values of a and b such that the function defined by***

$$f(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax+b, & \text{if } 2 < x \le 10 \\ 21, & \text{if } x \ge 10 \end{cases}$$

is a continuous function.

Sol. Since the function f is continuous, it is continuous at x = 2 as well as at x = 10.

So, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$

i.e., $\lim_{h\to 0} f(2+h) = \lim_{h\to 0} f(2-h) = f(2)$

i.e., $2a + b = 5$ (......1)

and $\lim_{x\to 10^+} f(x) = \lim_{x\to 10^-} f(x) = f(10)$

i.e., $\lim_{h\to 0} f(10+h) = \lim_{h\to 0} f(10-h) = f(10)$

i.e., $21 = 10a + b$ (......2)

From (1) and (2), we find that

a = 2 and b = 1

Ex.8 ***Show that the function defined by g(x) = x – [x] is discontinuous at all integral points. Here, [x] denotes the greatest integer less than or equal to x.***

Sol. The function f(x) = x – [x] can be written as

$f(x) = \begin{cases} x-(k-1), & \text{if } k-1 < x < k \\ x-k, & \text{if } k < x < k+1 \end{cases}$, where k is an arbitrary integer.

Now, $\lim_{x\to k^+} f(x) = \lim_{h\to 0} f(k+h) = \lim_{h\to 0} [(k+h) - k] = 0$

and $\lim_{x\to k^-} f(x) = \lim_{h\to 0} f(k-h) = \lim_{h\to 0} [(k-h) - (k-1)] = 1$

Since, $\lim_{x\to k^+} f(x) \neq \lim_{x\to k^-} f(x)$, the function f is not continuous at x = k.

Since k is an arbitrary integer, we can easily conclude that the function is discontinuous at all integral points.

Ex.9 ***Verify LMV Theorem for the function***

$$f(x) = \begin{cases} x^3+2, & \text{when } x \le 1 \\ 3x, & \text{when } x > 1 \end{cases} \text{ on } [-1, 2].$$

Sol. Both $x^3 + 2$ and $3x$ are polynomial functions. So, f (x) is continuous and differentiable everywhere except at x = 1.

Here, $\lim_{x\to 1^+} f(x) = 3.1 = 3$

$\lim_{x\to 1^-} f(x) = 1^3 + 2 = 3$

As

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$, f(x) is continuous at x = 1.

Obviously, then f(x) is continuous on [–1, 2]. Again to test the differentiability of f(x) at x = 1, we have

$$L f'(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{(x^3+2)-(1^3+2)}{x-1}$$

$$= \lim_{x\to 1^-} \frac{x^3-1}{x-1} = \lim_{x\to 1^-} (x^2 + x + 1) = 3$$

$$R f'(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{3x-3.1}{x-1}$$

$$= \lim_{x\to 1^+} (3) = 3$$

As L f '(1) = R f '(1), the function f (x) is differentiable at x = 1. Hence, f is differentiable in (–1, 2).

Thus, both the conditions required for the applicability of the LMV Theorem are satisfied and hence, there exists at least one $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2)-f(-1)}{2-(-1)} \Rightarrow f'(c) = \frac{6-1}{3} = \frac{5}{3}$$

Now, in x > 1, f '(x) = 3. So, f '(c) cannot be $\frac{5}{3}$ in this interval.

In $x \le 1$. f '(x) = $3x^2$

$\Rightarrow$ f '(c) = $3c^2$

Obviously, $3c^2 = \frac{5}{3}$ gives $c^2 = \frac{5}{9}$ or $c = \pm\frac{\sqrt{5}}{3}$

Both $\frac{\sqrt{5}}{3}$ and $-\frac{\sqrt{5}}{3}$ lie in (–1, 2). Thus, LMV is verified for f(x) and [–1, 2].

Ex.10 Verify Rolle's theorem for the function f (x) = x (x – 3)² in the closed interval 0 ≤ x ≤ 3.

Sol. (i) Here, $f(x) = x(x-3)^2$

$= x(x^2 - 6x + 9)$

$= x^3 - 6x^2 + 9x$

Since f(x) is a polynomial function of x, it is continuous in [0, 3]

(ii) $f'(x) = 3x^2 - 12x + 9$

exists uniquely in the open interval (0, 3)

(iii) $f(0) = (0)^3 - 6(0)^2 + 9(0)$

$= 0 - 0 + 0 = 0$

$f(3) = (3)^3 - 6(3)^2 + 9(3)$

$= 27 - 54 + 27 = 0$

$f(0) = f(3)$

Thus, all the three conditions are satisfied, Hence, Rolle's Theorem is applicable.

Let us now solve f '(c) = 0

i.e. $3c^2 - 12c + 9 = 0$

$3(c^2 - 4c + 3) = 0$

$(c - 3)(c - 1) = 0$

c = 3, 1

SInce, c = 1 $\in$ (0, 3), the Rolle's Theorem is verified for the function.

$f(x) = x(x-3)^2$ in the closed interval [0, 3].

Ex.11 Verify Rolle's Theorem for the function f(x) = (x – a)ᵐ (x – b)ⁿ in [a, b] ; m, n being positive integers.

Sol. Here, f(x) is a polynomial function of degree (m + n). So, it is a continuous function in [a, b].

$f'(x) = (x-a)^{m-1}(x-b)^{n-1}[m(x-b) + n(x-a)]$ exists uniquely in (a, b). So, it is derivable m (a, b).

Further, f(a) = 0 and f(b) = 0. So, f(a) = f(b)

Thus, all the three conditions of Rolle's Theorem are satisfied. Hence, Rolle's Theorem is applicable.

Let us now solve f '(c) = 0

$\Rightarrow (c-a)^{m-1}(c-b)^{n-1}[m(c-b) + n(c-a)] = 0$

$\Rightarrow$ c = a or c = b or $c = \frac{mb+na}{m+n}$

Since $c = \frac{mb+na}{m+n} \in (a, b)$, the Rolle's Theorem is verified.

Ex.12 ***Show that the function*** $f(x) = \begin{cases} 1+x, & \text{if } x \le 2 \\ 5-x, & \text{if } x > 2 \end{cases}$ ***is continuous at x = 2, but not differentiable at x = 2.***

Sol. At x = 2,

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} [5-(2+h)] = 3$$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} [1+(2-h)] = 3$$

Also, $f(2) = 1 + 2 = 3$

Since, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$, f(x) is continuous at x = 2.

Next, $Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$

$$= \lim_{h\to 0} = \frac{(1+2-h)-(1+2)}{-h} = 1$$

$$Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} = \frac{5-(2+h)-(1+2)}{h} = -1$$

Since, $Lf'(2) \neq Rf'(2)$, the function f is not differentiable at x = 2.

Ex.13 ***Show that the function*** $f(x) = \begin{cases} 1-x, & \text{if } x < 1 \\ x^2-1, & \text{if } x \ge 1 \end{cases}$ ***is continuous at x = 1, but not differentiable thereat.***

Sol. The function is continuous at x = 1, because $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$ as shown below :

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} [(1+h)^2 - 1]$$

$$= \lim_{h\to 0} (h^2 + 2h) = 0 ;$$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} [1-(1-h)]$$

$$= \lim_{h\to 0} (h) = 0$$

and $f(1) = (1)^2 - 1 = 1 - 1 = 0$

Further, $Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

$$= \lim_{h\to 0} \frac{[(1+h)^2-1]-[0]}{h} = 2$$

$$Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{h} = \lim_{h\to 0} \frac{[(1-h)^2-1]-[0]}{h}$$

$$= \lim_{h\to 0} \left(\frac{h}{-h}\right) = \lim_{h\to 0} (-1) = -1$$

Since, $Rf'(1) \neq Lf'(1)$, the function is not differentiable at x = 1.

Ex.14 ***Show that the function f defined as***

$$f(x) = \begin{cases} 3x-2, & \text{if } 0 < x \le 1 \\ 2x^2-x, & \text{if } 1 < x \le 1 \\ 5x-4, & \text{if } x > 2 \end{cases}$$

is continuous at x = 2, but not differentiable thereat.

Sol. At x=2, $\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} [5(2+h)-4] = 6$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} [2(2-h)^2 - (2-h)]$$

$$= \lim_{h\to 0} [2(4-4h+h^2) - (2-h)]$$

$$= \lim_{h\to 0} [6 - 7h + 2h^2] = 6$$

and $f(2) = 2(2)^2 - 2 = 8 - 2 = 6$

Since $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$, the function f is continuous at x = 2.

Next, $Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h}$

$$= \lim_{h\to 0} \frac{2(2-h)^2-(2-h)-[5(2)-4]}{-h}$$

$$= \frac{6-7h+2h^2-6}{-h} = 7$$

$$Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{[5(2+h)-4]-[5(2)-4]}{h}$$

$$= \lim_{h\to 0} \frac{[5(2+h)-4]-[5(2)-4]}{h}$$

$$= \lim_{h\to 0} \left[\frac{6+5h-6}{h}\right] = 5$$

Since, $Lf'(2) = Rf'(2)$, the function f is not differentiable at x = 2.

EXERCISE – I UNSOLVED PROBLEMS

Q.1 Show that the function $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$ is discontinuous at x = 0.

Q.2 Show that the function $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$ is continuous at x = 0

Q.3 Is the following function continuous at the origin ?

$f(x) = \dfrac{\cos ax - \cos bx}{x^2}$, when $x \neq 0$

$f(0) = \dfrac{b^2 - a^2}{2}$, when x = 0

Q.4 If the function defined by $f(x) = \begin{cases} \dfrac{e^x - 1}{\log(1+2x)}, & \text{when } x \neq 0 \\ k, & \text{when } x = 0 \end{cases}$ is continuous at x = 0, find the value of k.

Q.5 if $f(x) = \begin{cases} \dfrac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1}-1}, & \text{when } x \neq 0 \\ k, & \text{when } x = 0 \end{cases}$ is continuous at x = 0, find k.

Q.6 Determine the value of k so that the function $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{when } x < 0 \\ k, & \text{when } x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{when } x > 0 \end{cases}$ is continuous at x = 0.

Q.7 Discuss the continuity of the function f(x) at x = 0 if $f(x) = \begin{cases} \dfrac{x-|x|}{2}, & \text{when } x \neq 0 \\ 2, & \text{when } x = 0 \end{cases}$

Q.8 Let $f(x) = \begin{cases} 2, & \text{if } x \leq 3 \\ ax+b, & \text{if } 3 < x < 5 \\ a, & \text{if } x \geq 5 \end{cases}$ find the value of a and b, so that f(x) is continuous.

Q.9 Find the derivative of $f(x) = \begin{cases} \dfrac{x-1}{2x^2-7x+5}, & x \neq 1 \\ \dfrac{-1}{3}, & x = 1 \end{cases}$ at x = 1

Q.10 Find the value of 'a' and 'b', so that the function $f(x) = \begin{cases} x^2+3x+a, & \text{if } x \leq 1 \\ bx+2, & \text{if } x > 1 \end{cases}$ is differentiable at each $x \in R$.

Q.11 Differentiate the following w.r.t x :

(i) $\log\sqrt{\dfrac{1-\cos x}{1+\cos x}}$ (ii) $\log_a\left(x+\sqrt{x^2+a^2}\right)$ (iii) log (sec x + tan x)

(iv) $\log\left(\sin\dfrac{x}{2}+\cos\dfrac{x}{2}\right)$ (v) $\log\left(\dfrac{1+x\sin x}{1-x\sin x}\right)$ (vi) $\log\left(\dfrac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}\right)$

Q.12 (i) If $y = \sqrt{\frac{1-x}{1+x}}$, prove that $(1 - x^2)\frac{dy}{dx} + y = 0$

(ii) If $y = \left(\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}\right)$, prove that $\frac{dy}{dx} = \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}$

(iii) If $y = \frac{\cos x + \sin x}{\cos x - \sin x}$, show that $\frac{dy}{dx} = \sec^2\left(x + \frac{\pi}{4}\right)$

(iv) If $y = \sqrt{\frac{\sec x + \tan x}{\sec x - \tan x}}$, show that $\frac{dy}{dx} = \sec x\,(\sec x + \tan x)$

Q.13 (i) If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, show that $2x\frac{dy}{dx} + y = 2\sqrt{x}$ (ii) If $y = x \sin y$, prove that $x\frac{dy}{dx} = \frac{y}{(1 - x\cos y)}$

(iii) If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, prove that $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$, $x \neq y$

Q.14 If $y = \sqrt{x+\sqrt{x+\sqrt{x+\ldots\ldots}}}$, prove that $\frac{dy}{dx} = \frac{1}{(2y-1)}$

Q.15 Given that $\cos\frac{x}{2} \cdot \cos\frac{x}{4} \cdot \cos\frac{x}{8} \ldots\ldots = \frac{\sin x}{x}$, prove that $\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{4} + \ldots\ldots = \text{cosec}^2 x - \frac{1}{x^2}$

Q.16 If $x = \tan^{-1}\frac{2t}{1-t^2}$ and $y = \sin^{-1}\frac{2t}{1+t^2}$, show that $\frac{dy}{dx} = 1$.

Q.17 Differentiate : (i) $\sin^{-1}\left\{x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2}\right\}$ (ii) $\tan^{-1}\left(\frac{2x}{1+15x^2}\right)$

(iii) $\tan^{-1}\left(\frac{1}{x^2+x+1}\right) + \tan^{-1}\left(\frac{1}{x^2+3x+3}\right) + \tan^{-1}\left(\frac{1}{x^2+5x+7}\right) + \ldots\ldots\ldots$ to n terms.

Q.18 Differentiate : (i) $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) + \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right)$ (ii) $\cos^{-1}\left(\frac{3\cos x - 4\sin x}{5}\right)$

Q.19 Differentiate : $\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) + \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$.

Q.20 Differentiate : (i) $\sin^{-1}\left(\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right)$ (ii) $\cos^{-1}\left(\frac{x-x^{-1}}{x+x^{-1}}\right)$

Q.21 Discuss the continuity of the function $f(x) = \begin{cases} 2x-1, & \text{if } x < 0 \\ 2x+1, & \text{if } x \geq 0 \end{cases}$

Q.22 If a function f(x) is defined as $f(x) = \begin{cases} \frac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$ show that f is everywhere continuous except at x = 4.

Q.23 Discuss the continuity of the function $f(x) = |x| + |x - 1|$ in the interval $[-1, 2]$

Q.24 Show that $f(x) = |x|$ is not differential at $x = 1$.

Q.25 Let $f(x) = \begin{cases} 2+x, & \text{if } x \geq 0 \\ 2-x, & \text{if } x < 0 \end{cases}$, show that f(x) is not derivable at x = 0.

Q.26 Show that the function $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is differential at x = 0 and f '(0) = 0.

EXERCISE – II BOARD PROBLEMS

Q.1 If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Q.2 If $x^p y^q = (x + y)^{p+q}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

Q.3 Find $\frac{d^2y}{dx^2}$ when $y = \log\left(\frac{x^2}{e^x}\right)$.

Q.4 If $y = ae^{2x} + be^{-x}$, prove that $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$.

Q.5 If y = A cos nx + B sin nx show that $\frac{d^2y}{dx^2} + n^2y = 0$.

Q.6 Discuss the continuity of the function f(x) at x = 0 if $f(x) = \begin{cases} 2x-1, & x<0 \\ 2x+1, & x\ge 0 \end{cases}$

Q.7 Show that the function f(x) = 2x – | x | is continuous at x = 0.

Q.8 If the function $f(x) = \begin{cases} 3ax+b, & x>1 \\ 11, & x=1 \\ 5ax-2b, & x<1 \end{cases}$ is continuous at x = 1, find the values of a and b.

Q.9 If $y = \sqrt{\frac{1-\sin 2x}{1+\sin 2x}}$, prove that $\frac{dy}{dx} + \sec^2\left(\frac{\pi}{4}-x\right) = 0$.

Q.10 If $y = \log\left(\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)$, show that $\frac{dy}{dx} - \sec x = 0$.

Q.11 Verify Lagrange's mean value theorem for the following functions in the given intervals.

Also find 'c' of this theorem : (i) $f(x)=x^2+x-1$ in [0, 4] (ii) $f(x)=\sqrt{x^2-4}$ on [2, 4]

Q.12 If $y = e^x(\sin x + \cos x)$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

Q.13 Differentiate the following w.r.t. x

(i) $\log\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$ (ii) $\log\left(x + \sqrt{1+x^2}\right)$

Q.14 If x = a(t + sin t), y = a(1 – cos t), find $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$.

Q.15 Differentiate the following functions w.r.t x :

(i) $\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$. (ii) $\cot^{-1}\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)$ (iii) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

(iv) $\sin^{-1}\left(\frac{5x+12\sqrt{1-x^2}}{13}\right)$ (v) $\tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ (vi) $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

Q.16 Prove that $\frac{d}{dx}\left(\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{2}\right) = \sqrt{a^2-x^2}$.

Q.17 If $y = (\sin x)^x + (\cos x)^{\tan x}$, find $\frac{dy}{dx}$

Q.18 Find $\frac{dy}{dx}$, when $x = a\,\frac{1-t^2}{1+t^2}$, $y = \frac{2bt}{1+t^2}$

Q.19 Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ w.r.t. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Q.20 If $f(x) = \left(\frac{3+x}{1+x}\right)^{2+3x}$, find $f'(0)$.

Q.21 Find $\frac{dy}{dx}$ if : $x = a\left(\frac{1+t^2}{1-t^2}\right)$, $y = \frac{2t}{1-t^2}$

Q.22 If $y = x\log\left(\frac{x}{a+bx}\right)$, prove that $\frac{d^2y}{dx^2} = \frac{1}{x}\left(\frac{a}{a+bx}\right)^2$.

Q.23 If the function f defined by $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2} & , x<0 \\ a & , x=0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & , x>0 \end{cases}$ is continuous at x = 0, find the value of a.

Q.24 If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, then show that $2x\frac{dy}{dx} + y = 2\sqrt{x}$.

Q.25 Differentiate w.r.t. x : $\tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

Q.26 If $y\sqrt{x^2+1} = \log(\sqrt{x^2+1} - x)$, prove that $(x^2 + 1)\frac{dy}{dx} + xy + 1 = 0$.

Q.27 If $x = a\sin 2t\,(1 + \cos 2t)$ and $y = b\cos 2t\,(1 - \cos 2t)$, show that $\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{b}{a}$.

Q.28 If $y = \operatorname{cosec} x + \cot x$, show that $\sin x \cdot \frac{d^2y}{dx^2} = y^2$.

Q.29 Verify LMV ; find 'c' $f(x) = x^2 + 2x + 3$ in $[4, 6]$

Q.30 If $f(x) = \begin{cases} \frac{x^2-25}{x-5} & , x\neq 5 \\ k & , x=5 \end{cases}$ is continuous at x = 5, find the value of k.

Q.31 If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$.

Q.32 If y = A e^{mx} + B e^{nx}, prove that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$.

Q.33 If y = sin (log x), prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

Q.34 For what value of k is the following function continuous at x = 2 ?

$$f(x) = \begin{cases} 2x+1, & x<2 \\ k, & x=2 \\ 3x-1, & x>2 \end{cases}$$

Q.35 Discuss the continuity of the following function at x = 0 :

$$f(x) = \begin{cases} \dfrac{x^4+2x^3+x^2}{\tan^{-1}x}, & x \neq 0 \\ 0, & x=0 \end{cases}.$$

Q.36 Let $f(x) = \begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x < \dfrac{\pi}{2} \\ a, & \text{if } x = \dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2}, & \text{if } x > \dfrac{\pi}{2} \end{cases}$. If f(x) be a continuous function at $x = \frac{\pi}{2}$, find a and b.

Q.37 If f(x), defined by the following, is continuous at x = 0, find the values of a, b and c.

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x+\sin x}{x}, & \text{if } x<0 \\ c, & \text{if } x=0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}, & \text{if } x>0 \end{cases}$$

Q.38 If $y = (x + \sqrt{x^2+a^2})^n$, prove that $\frac{dy}{dx} = \frac{ny}{\sqrt{x^2+a^2}}$.

Q.39 If $x\sqrt{1+y}+y\sqrt{1+x}=0$, find $\frac{dy}{dx}$.

Q.40 If $y = \sqrt{x^2+1} - \log\left(\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}\right)$, find $\frac{dy}{dx}$.

Q.41 If $x = a\left(\cos\theta+\log\tan\frac{\theta}{2}\right)$ and y = a sin θ, find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$.

Q.42 If $y = (\log(x+\sqrt{x^2+1}))^2$, show that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2 = 0$.

Q.43 If sin y = x sin (a + y), prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

Q.44 If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$.

Q.45 If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, show that $(1 - x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$

Q.46 Differentiate the following function w.r.t. x : $x^{\sin x} + (\sin x)^{\cos x}$

Q.47 $x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right) + (\sin x)^{\cos x}(\cos x \cot x - \sin x \log\sin x)$

Q.48 Find $\frac{dy}{dx}$ if $(x^2 + y^2)^2 = xy$.

Q.49 If $y = 3\cos(\log x) + 4\sin(\log x)$, then show that $x^2 . \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Q.50 If $y = \cos^{-1}\left(\frac{3x + 4\sqrt{1-x^2}}{5}\right)$, find $\frac{dy}{dx}$.

Q.51 If $y = \operatorname{cosec}^{-1} x$, $x > 1$, then show that $x(x^2 - 1)\frac{d^2y}{dx^2} + (2x^2 - 1)\frac{dy}{dx} = 0$

Q.52 If $x^y = e^{x-y}$, show that $\frac{dy}{dx} = \frac{\log x}{\{\log(xe)\}^2}$.

Q.53 If $x = \tan\left(\frac{1}{a}\log y\right)$, show that $(1 + x^2)\frac{d^2y}{dx^2} + (2x - a)\frac{dy}{dx} = 0$

Q.54 If $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$.

Q.55 Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2} - 1}{x}\right]$ with respect to x.

Q.56 If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, $0 < t < \frac{\pi}{2}$, find $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$.

Q.57 If $y^x = e^{y-x}$, prove that $\frac{dy}{dx} = \frac{(1+\log)^2}{\log y}$.

Q.58 Differentiate the following with respect to x :

$$\sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right)$$

Q.59 Find the value of k, for which

$$f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \frac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$$

is continuous at x = 0.

OR

If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{6}$.

Answers

EXERCISE – 1 (UNSOLVED PROBLEMS)

3. yes **4.** 1/2 **5.** –4 **6.** 8 **7.** discontinuous **8.** 7/5 and –17/2

9. –2/9 **10.** 3 and 5 **11.** **(i)** cosec x **(ii)** $\frac{1}{\log a\sqrt{x^2+a^2}}$ **(iii)** sec x **(iv)** $\frac{1}{2}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$

(v) $\frac{2(x\cos x+\sin x)}{1-x^2\sin^2 x}$ **(vi)** $\frac{2}{\sqrt{x^2+a^2}}$ **17. (i)** $\frac{1}{\sqrt{1-x^2}}+\frac{1}{2\sqrt{x-x^2}}$ **(ii)** $\frac{5}{1+25x^2}-\frac{3}{1+9x^2}$ **(iii)** $\frac{1}{1+(x+x)^2}-\frac{1}{1+x^2}$

18. (i) $\frac{3}{2\sqrt{1-x^2}}$ **(ii)** 1 **19.** $\frac{-1}{2(1+x^2)}$ **20. (i)** $\frac{-1}{2\sqrt{1-x^2}}$ **(ii)** $\frac{-2}{1+x^2}$

EXERCISE – 2 (BOARD PROBLEMS)

3. $\frac{-2}{x^2}$ **6.** Discontinuous **8.** 3,2 **11. (i)** 2 **(ii)** $\sqrt{6}$ **13. (i)** cosec x **(ii)** $\frac{1}{\sqrt{1+x^2}}$

14. 1 **15. (i)** $\frac{1}{2}$ **(ii)** $\frac{1}{2}$ **(iii)** $\frac{1}{2(1+x^2)}$ **(iv)** $\frac{1}{\sqrt{1-x^2}}$ **(v)** $\frac{x}{\sqrt{1-x^4}}$ **(vi)** $\frac{-1}{2\sqrt{1-x^2}}$ **18.** $\frac{-b(1-t^2)}{2at}$

19. 1 **21.** $\frac{1+t^2}{2at}$ **23.** 8 **25.** –1/2 **29.** 5 **30.** 10 **32.** 5/2 **34.** 5

35. continuous **36.** $\frac{1}{2}$, 4 **37.** $-\frac{3}{2}$, any real number, $\frac{1}{2}$ **39.** $\frac{-1}{(1+x)^2}$ **40.** $\frac{\sqrt{x^2+1}}{x}$ **41.** 1

44. $\frac{\log\sin y+y\tan x}{\log\cos x-\cot y}$ **48.** $\frac{y-4x^3-4xy^2}{4x^2y+4y^3-x}$ **50.** $\frac{-1}{\sqrt{1-x^2}}$ **55.** $\frac{1}{2(1+x^2)}$ **56.** at cos t, at sin t and $\frac{\sec^3 t}{at}$