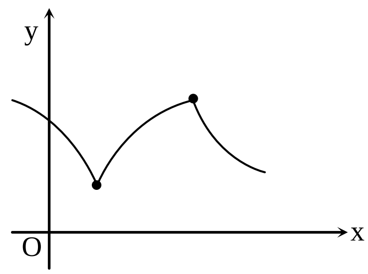
Application of Derivative (Maxima & Minima)

1. Introduction

In this chapter we shall study those points of the domain of a function where its graph changes its direction from upwards to downwards or from downwards to upwards. At such points the derivative of the function, if it exists, is necessarily zero.

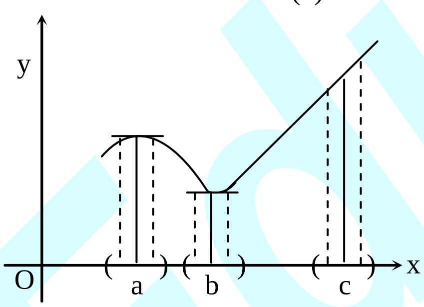


2. Maximum & Minimum Points

The value of a function f (x) is said to be maximum at x = a, if there exists a very small positive number h, such that

$f(x) < f(a)\ \forall\ x \in (a-h, a+h)\ ,\ x \neq a$

In this case the point x = a is called a point of maxima for the function f(x).



Similarly, the value of f(x) is said to the minimum at x = b, If there exists a very small positive number, h, such that

$f(x) > f(b), \forall\ x \in (b-h, b+h), x \neq b$

In this case x = b is called the point of minima for the function f(x).

Here we find that,

(i) x = a is a maximum point of f(x)

$$\begin{cases} f(a) - f(a+h) > 0 \\ f(a) - f(a-h) > 0 \end{cases}$$

(ii) x = b is a minimum point of f(x)

$$\begin{cases} f(b) - f(b+h) < 0 \\ f(b) - f(b-h) < 0 \end{cases}$$

(iii) x = c is neither a maximum point nor a minimum point

$$\left.\begin{array}{l} f(c) - f(c+h) \\ \text{and} \\ f(c) - f(c-h) > 0 \end{array}\right\} \text{have opposite signs.}$$

Note :

(i) The maximum and minimum points are also known as extreme points.

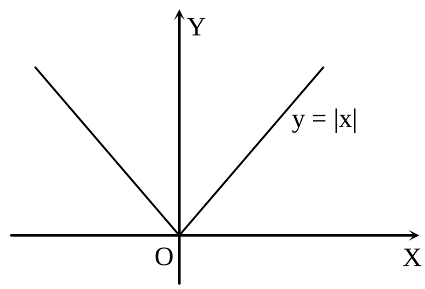
(ii) A function may have more than one maximum and minimum points.

(iii) A maximum value of a function f(x) in an interval [a,b] is not necessarily its greatest value in that interval. Similarly, a minimum value may not be the least value of the function. A minimum value may be greater than some maximum value for a function.

(iv) If a continuous function has only one maximum (minimum) point, then at this point function has its greatest (least) value.

(v) Monotonic functions do not have extreme points.

Ex. f(x) = |x| has a minimum point at x = 0. It can be easily observed from its graph.



3. Conditions For Maxima & Minima of a Function

A. Necessary Condition : A point x = a is an extreme point of a function f(x) if $f'(a) = 0$, provided $f'(a)$ exists. Thus if $f'(a)$ exists, then

x = a is an extreme point $\Rightarrow f'(a) = 0$ or $f'(a) \neq 0 \Rightarrow$ x = a is not an extreme point.

But its converse is not true i.e.

$f'(a) = 0$ x = a is an extreme point.

For example if $f(x) = x^3$, then $f'(0) = 0$ but $x = 0$ is not an extreme point.

B. Sufficient Condition :

(i) The value of the function f(x) at x = a is maximum, if $f'(a) = 0$ and $f''(a) < 0$.

(ii) The value of the function f(x) at x = a in minimum if $f'(a) = 0$ and $f''(a) > 0$.

Note:

(i) If $f'(a) = 0$, $f''(a) = 0$, $f'''(a)$ 0 then x = a is not an extreme point for the function f(x).

(ii) If $f'(a) = 0$, $f''(a) = 0$, $f'''(a) = 0$ then the sign of $f^{(iv)}(a)$ will determine the maximum and minimum value of function i.e. f(x) is maximum, if $f^{(iv)}(a) < 0$ and minimum if $f^{(iv)}(a) > 0$.

4. Working Rule For Finding Maxima & Minima

(I) Find the differential coefficient of f(x) with respect to x, i.e. $f'(x)$ and equate it to zero.

(ii) Find different real values of x by solving the equation $f'(x) = 0$. Let its roots be a, b, c,......

(iii) Find the value of $f''(x)$ and substitute the value of a, b, c ....in it and get the sign of $f'(x)$ for each value of x.

(iv) If $f'(a) < 0$ then the value of f (x) is maximum at x = a and if $f'(a) > 0$ then value of f(x) will be minimum at x = a. Similarly by getting the signs of $f''(x)$ at other points b, c.... we can find the points of maxima and minima.

5. Greatest & Least Values of a Function in a Given Functional

If a function f(x) is defined in an interval [a, b], then greatest or least values of this function occurs either at x = a or x = b or at those values of x where $f'(x) = 0$.

Remember that a maximum value of the function f(x) in any interval [a, b] is not necessarily its greatest value in that interval. Thus greatest value of f(x) in interval [a, b]

$$= \max. [f(a), f(b), f(c)]$$

Least value of f(x) in interval [a, b]

$$= \text{Min}. [f(a), f(b), f(c)]$$

Where x = c is a point such that $f'(c) = 0$

6. Properties of Maxima & Minima

If f (x) is a continuous function and the graph of this function is drawn, then-

(i) Between two equal values of f(x), there lie at least one maxima or minima.

(ii) Maxima and minima occur alternately. For example if x = – 1,0,2,3 are extreme points of a continuous function and if x = 0 is a maximum point then x = – 1,2 will be minimum points.

(iii) When x passes a maximum point, the sign of $f'(x)$ changes from + ve to – ve, whereas x passes through a minimum point, the sign of $f'(x)$ changes from – ve to + ve.

(iv) If there is no change in the sign of dy/dx on two sides of a point, then such a point is not an extreme point.

(v) If f(x) is a maximum (minimum) at a point x = a, then 1/f (x), $[f(x) \neq 0]$ will be minimum (maximum) at that point.

(vi) If f(x) is maximum (minimum) at a point x = a, then for any $\lambda \in R$, $\lambda + f(x)$, log f(x) and for any k > 0, k f (x), $[f(x)]^k$ are also maximum (minimum) at that point.

7. Maxima & Minima of Functions of Two Variables

If a function is defined in terms of two variables and if these variables are associated with a given relation then by eliminating one variable, we convert function in terms of one variable and then find the maxima and minima by known methods.

8. Some Standard Geometrical Results Related to Maxima & Minima

The following results can easily be established.

(i) The area of rectangle with given perimeter is greatest when it is a square.

(ii) The perimeter of a rectangle with given area is least when it is a square.

(iii) The greatest rectangle inscribed in a given circle is a square.

(iv) The greatest triangle inscribed in a given circle is equilateral.

(v) The semi vertical angle of a cone with given slant height and maximum volume is $\tan^{-1}\sqrt{2}$.

(vi) The height of a cylinder of maximum volume inscribed in a sphere of radius a is a $2a/\sqrt{3}$.

9. Some Important Results

(i) Equilateral triangle :

Area = $(\sqrt{3}/4)\,x^2$, where x is its side.

(ii) Square :

Area = a^2 , perimeter = 4a,

where a is its side.

(iii) Rectangle:

Area = a b, perimeter = 2 (a + b)

where a, b are its sides

(iv) Trapezium :

Area = 1/2 (a+ b) h

Where a, b are lengths of parallel sides and h be the distance between them.

(v) Circle :

Area = πa^2 , perimeter = $2\pi a$,

where a is its radius.

(vi) Sphere :

Volume = $4/3\ \pi a^3$, surface $4\pi a^2$

where a is its radius

(vii) Right Circular cone :

Volume = $1/3\ \pi r^2 h$, curved surface = $\pi r \lambda$

Where r is the radius of its base, h be its height and λ be its slant heights

(viii) Cylinder :

Volume = $\pi r^2 h$

whole surface = $2\pi r(r + h)$

where r is the radius of the base and h be its height.