Matrices

1. Definition

A rectangular arrangement of numbers in rows and columns, is called a Matrix. This arrangement is enclosed by small ( ) or big [ ] brackets. A matrix is represented by capital letters A, B, C etc. and its element are by small letters a, b, c, x, y etc.

2. Order of a Matrix

A matrix which has m rows and n columns is called a matrix of order m × n.

A matrix A of order m × n is usually written in the following manner-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & ...a_{1j} & ...a_{1n} \\ a_{21} & a_{23} & a_{23} & ...a_{2j} & ...a_{2n} \\ ..... & ..... & ..... & ..... & ..... \\ a_{i1} & a_{i2} & a_{i3} & ...a_{ij} & ...a_{in} \\ ..... & ..... & ..... & ..... & ..... \\ a_{m1} & a_{m2} & a_{m3} & ...a_{mj} & ...a_{mn} \end{bmatrix} \text{ or}$$

$$A = [a_{ij}]_{m \times n} \text{ where } \begin{matrix} i = 1, & 2,......m \\ i = 1, & 2,......n \end{matrix}$$

Here a_{ij} denotes the element of i^{th} row and j^{th} column.

3. Types of Matrices

3.1 Row matrix :

If in a Matrix, there is only one row, then it is called a Row Matrix.

Thus $A = [a_{ij}]_{m \times n}$ is a row matrix if m = 1.

3.2 Column Matrix :

If in a Matrix, there is only one column, then it is called a Column Matrix.

Thus $A = [a_{ij}]_{m \times n}$ is a Column Matrix if n = 1.

3.3 Square Matrix :

If number of rows and number of column in a Matrix are equal, then it is called a Square Matrix.

Thus $A = [a_{ij}]_{m \times n}$ is a Square Matrix if m = n

Note :

(a) If m ≠ n then Matrix is called a Rectangular Matrix.

(b) The elements of a Square Matrix A for which i = j i.e. $a_{11}, a_{22}, a_{33}, ... a_{nn}$ are called diagonal elements and the line joining these elements is called the principal diagonal or of leading diagonal of Matrix A.

(c) **Trance of a Matrix :** The sum of diagonal elements of a square matrix . A is called the trance of Matrix A which is denoted by tr A.

$$\text{tr } A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ...a_{nn}$$

3.4 Singleton Matrix :

If in a Matrix there is only one element then it is called Singleton Matrix. Thus

$A = [a_{ij}]_{m \times n}$ is a Singleton Matrix if m = n = 1.

3.5 Null or Zero Matrix :

If in a Matrix all the elements are zero then it is called a zero Matrix and it is generally denoted by O.

Thus $A = [a_{ij}]_{m \times n}$ is a zero matrix if $a_{ij} = 0$ for all i and j.

3.6 Diagonal Matrix :

If all elements except the principal diagonal in a **Square Matrix** are zero, it is called a Diagonal Matrix. Thus a Square Matrix

$A = [a_{ij}]$ is a Diagonal Matrix if $a_{ij} = 0$, when i ≠ j

Note :

(a) No element of Principal Diagonal in diagonal Matrix is zero.

(b) Number of zero in a diagonal matrix is given by $n^2 - n$ where n is a order of the Matrix.

3.7 Scalar Matrix :

If all the elements of the diagonal of a **diagonal matrix** are equal , it is called a scalar matrix. Thus a Square Matrix $A = [a_{ij}]$ is a Scalar Matrix is

$$a_{ij} = \begin{cases} 0 & i \neq j \\ k & i = j \end{cases}$$ where k is a constant.

3.8 Unit Matrix :

If all elements of principal diagonal in a **Diagonal Matrix** are 1, then it is called Unit Matrix. A unit Matrix of order n is denoted by I_n.

Thus a square Matrix

A = $[a_{ij}]$ is a unit Matrix if

$$a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Note :

Every unit Matrix is a Scalar Matrix.

3.9 Triangular Matrix :

A Square Matrix $[a_{ij}]$ is said to be triangular matrix if each element above or below the principal diagonal is zero it is of two types-

(a) Upper Triangular Matrix : A Square Matrix $[a_{ij}]$ is called the upper triangular Matrix, if $a_{ij} = 0$ when $i > j$.

(b) Lower Triangular Matrix : A Square Matrix $[a_{ij}]$ is called the lower Triangular Matrix, if

$a_{ij} = 0$ when $i < j$

Note :

Minimum number of zero in a triangular matrix is given by $\frac{n(n-1)}{2}$ where n is order of Matrix.

3.10 Equal Matrix :

Two Matrix A and B are said to be equal Matrix if they are of same order and their corresponding elements are equal.

3.11 Singular Matrix :

Matrix A is said to be singular matrix if its determinant |A| = 0, otherwise non- singular matrix i.e.

If det $|A| = 0 \Rightarrow$ Singular

and det $|A| \neq 0 \Rightarrow$ non-singular

4. Addition and Subtraction of Matrices

If A $[a_{ij}]_{m \times n}$ and $[b_{ij}]_{m \times n}$ are two matrices of the same order then their sum A + B is a matrix whose each element is the sum of corresponding element.

i.e. $A + B = [a_{ij} + b_{ij}]_{m \times n}$

Similarly their subtraction A – B is defined as

$$A - B = [a_{ij} - b_{ij}]_{m \times n}$$

Note :

Matrix addition and subtraction can be possible only when Matrices are of same order.

4.1 Properties of Matrices addition :

If A, B and C are Matrices of same order, then-

(i) A + B = B + A (Commutative Law)

(ii) (A+ B) + C = A + (B + C) (Associative Law)

(iii) A + O = O + A = A, where O is zero matrix which is additive identity of the matrix.

(iv) A + (– A) = 0 = (–A) + A where (–A) is obtained by changing the sign of every element of A which is additive inverse of the Matrix

(v) $\left.\begin{array}{l} A + B = A + C \\ B + A = C + A \end{array}\right\} \Rightarrow B = C$ (Cancellation Law)

(vi) tr (A ± B) = tr (A) ± tr (B)

5. Scalar Multiplication of Matrices

Let A = $[a_{ij}]_{m \times n}$ be a matrix and k be a number then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it is denoted by

kA thus if A = $[a_{ij}]_{m \times n}$ then

$$kA = Ak = [ka_{ij}]_{m \times n}$$

5.1 Properties of Scalar Multiplication :

If A, B are Matrices of the same order and λ, μ are any two scalars then -

(i) $\lambda(A + B) = \lambda A + \lambda B$

(ii) $(\lambda + \mu) A = \lambda A + \mu A$

(iii) $\lambda(\mu A) = (\lambda\mu) A = \mu(\lambda A)$

(iv) $(-\lambda A) = -(\lambda A) = \lambda(-A)$

(v) tr $(kA) = k$ tr (A)

6. Multiplication of Matrices

If A and B be any two matrices, then their product AB will be defined only when number of column in A is equal to the number of rows in B. If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ then their product $AB = C = [c_{ij}]$, will be matrix of order $m \times p$, where

$$(AB)_{ij} = C_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$$

6.1 Properties of Matrix Multiplication :

If A, B and C are three matrices such that their product is defined , then

(i) $AB \neq BA$ (Generally not commutative)

(ii) $(AB) C = A (BC)$ (Associative Law)

(iii) $IA = A = AI$

I is identity matrix for matrix multiplication

(iv) $A (B + C) = AB + AC$ (Distributive Law)

(v) If $AB = AC \not\Rightarrow B = C$

(Cancellation Law is not applicable)

(vi) If $AB = 0$. It does not mean that $A = 0$ or $B = 0$, again product of two non- zero matrix may be zero matrix.

(vii) tr (AB) = tr (BA)

Note :

(i) The multiplication of two diagonal matrices is again a diagonal matrix.

(ii) The multiplication of two triangular matrices is again a triangular matrix.

(iii) The multiplication of two scalar matrices is also a scalar matrix.

(iv) If A and B are two matrices of the same order, then

(a) $(A + B)^2 = A^2 + B^2 + AB + BA$

(b) $(A - B)^2 = A^2 + B^2 - AB - BA$

(c) $(A - B) (A + B) = A^2 - B^2 + AB - BA$

(d) $(A+ B) (A - B) = A^2 - B^2 - AB + BA$

(e) $A (-B) = (-A) B = -(AB)$

6.2 Positive Integral powers of a Matrix :

The positive integral powers of a matrix A are defined only when A is a square matrix. Also then

$A^2 = A.A \quad A^3 = A.A.A = A^2A$

Also for any positive integers m,n

(i) $A^m A^n = A^{m+n}$

(ii) $(A^m)^n = A^{mn} = (A^n)^m$

(iii) $I^n = I, I^m = I$

(iv) $A^o = I_n$ where A is a square matrices of order n.

7. Transpose of a Matrix

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called transpose of Matrix A and is denoted by A^T or A'.

From the definition it is obvious that

If order of A is $m \times n$, then order of A^T is $n \times m$.

7.1 Properties of Transpose :

(i) $(A^T)^T = A$

(ii) $(A \pm B)^T = A^T \pm B^T$

(iii) $(AB)^T = B^T A^T$

(iv) $(kA)^T = k(A)^T$

(v) $(A_1A_2A_3 ......A_{n-1}A_n)^T$

$= A_n^T A_{n-1}^T.....A_3^TA_2^TA_1^T$

(vi) $I^T = I$

(vii) tr (A) = tr (A^T)

8. Symmetric & Skew-Symmetric Matrix

(a) Symmetric Matrix : A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all i,j or $A^T = A$

Note :

(i) Every unit matrix and square zero matrix are symmetric matrices.

(ii) Maximum number of different element in a symmetric matrix is $\frac{n(n+1)}{2}$.

(b) Skew - Symmetric Matrix : A square matrix $A = [a_{ij}]$ is called

skew - symmetric matrix if

$a_{ij} = -a_{ji}$ for all i, j

or $A^T = -A$

Note :

(i) All Principal diagonal elements of a skew - symmetric matrix are always zero because for any diagonal element –

$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

(ii) Trace of a skew symmetric matrix is always 0

8.1 Properties of Symmetric and skew- symmetric matrices :

(i) If A is a square matrix, then $A + A^T$, AA^T, A^TA are symmetric matrices while $A - A^T$ is Skew-Symmetric Matrices.

(ii) If A is a Symmetric Matrix, then $-A$, KA, A^T, A^n, A^{-1}, B^TAB are also symmetric matrices where $n \in N$, $K \in R$ and B is a square matrix of order that of A.

(iii) If A is a skew symmetric matrix, then-

(a) A^{2n} is a symmetric matrix for $n \in N$

(b) A^{2n+1} is a skew-symmetric matrices for $n \in N$

(c) kA is also skew-symmetric matrix where $k \in R$.

(d) $B^T AB$ is also skew-symmetric matrix where B is a square matrix of order that of A

(iv) If A, B are two symmetric matrices, then-

(a) $A \pm B$, $AB + BA$ are also symmetric matrices.

(b) $AB - BA$ is a skew - symmetric matrix.

(c) AB is a symmetric matrix when AB = BA.

(v) If A, B are two skew-symmetric matrices, then-

(a) $A \pm B$, $AB - BA$ are skew-symmetric matrices.

(b) $AB + BA$ is a symmetric matrix.

(vi) If A is a skew - symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.

(vii) Every square matrix A can uniquely be expressed as sum of a symmetric and skew symmetric matrix i.e.

$$A = \left[\frac{1}{2}(A + A^T)\right] + \left[\frac{1}{2}(A - A^T)\right]$$

9. Determinant of a Matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix, then its determinant, denoted by |A| or Det (A) is defined as

$$|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

9.1 Properties of the Determinant of a matrix :

(i) |A| exists $\Leftrightarrow$ A is a square matrix

(ii) $|AB| = |A|\,|B|$

(iii) $|A^T| = |A|$

(iv) $|kA| = k^n |A|$, if A is a square matrix of order n.

(v) If A and B are square matrices of same order then $|AB| = |BA|$

(vi) If A is a skew symmetric matrix of odd order then $|A| = 0$

(vii)If $A = \text{diag}(a_1, a_2 ....a_n)$ then $|A| = a_1a_2 ...a_n$

(viii) $|A|^n = |A^n|$, $n \in N$.

10. Adjoint of a Matrix

If every element of a square matrix A be replaced by its cofactor in |A|, then the transpose of the matrix so obtained is called the adjoint of matrix A and it is denoted by adj A

Thus if $A = [a_{ij}]$ be a square matrix and F^{ij} be the cofactor of a_{ij} in |A|, then

$$\text{Adj } A = [F^{ij}]^T$$

Hence if $A = \begin{bmatrix} a_{11} & a_{12} & ..a_{1n} \\ a_{21} & a_{22} & ..a_{2n} \\ .... & .... & .... \\ .... & .... & .... \\ a_{n1} & a_{n2} & ..a_{nn} \end{bmatrix}$, then

$$\text{Adj } A = \begin{bmatrix} F_{11} & F_{12} & ...F_{1n} \\ F_{21} & F_{22} & ...F_{2n} \\ .... & .... & .... \\ .... & .... & .... \\ F_{n1} & F_{n2} & ...F_{nn} \end{bmatrix}^T$$

10.1 Properties of adjoint matrix :

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

(i) $A (\text{adj } A) = |A| I_n = (\text{adj } A) A$

(Thus A (adj A) is always a scalar matrix)

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj} (\text{adj } A) = |A|^{n-2} A$

(iv) $|\text{adj} (\text{adj } A)| = |A|^{(n-1)^2}$

(v) $\text{adj} (A^T) = (\text{adj } A)^T$

(vi) $\text{adj} (AB) = (\text{adj } B) (\text{adj } A)$

(vii) $\text{adj} (A^m) = (\text{adj } A)^m$, $m \in N$

(viii) $\text{adj} (kA) = k^{n-1} (\text{adj } A)$, $k \in R$

(ix) $\text{adj} (I_n) = I_n$

(x) adj 0 = 0

(xi) A is symmetric ⇒ adj A is also symmetric

(xii) A is diagonal ⇒ adj A is also diagonal

(xiii) A is triangular ⇒ adj A is also triangular

(xiv) A is singular ⇒ |adj A| = 0

11. Inverse of a Matrix

If A and B are two matrices such that

$$AB = I = BA$$

then B is called the inverse of A and it is denoted by A^{-1}, thus

$$A^{-1} = B \Leftrightarrow AB = I = BA$$

To find inverse matrix of a given matrix A we use following formula

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

Thus A^{-1} exists $\Leftrightarrow |A| \neq 0$

Note :

(i) Matrix A is called invertible if A^{-1} exists.

(ii) Inverse of a matrix is unique.

11.1 Properties of Inverse Matrix :

Let A and B are two invertible matrices of the same order, then

(i) $(A^T)^{-1} = (A^{-1})^T$

(ii) $(AB)^{-1} = B^{-1} A^{-1}$

(iii) $(A^k)^{-1} = (A^{-1})^k$, $k \in N$

(iv) $\text{adj} (A^{-1}) = (\text{adj } A)^{-1}$

(v) $(A^{-1})^{-1} = A$

(vi) $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$

(vii) If $A = \text{diag} (a_1, a_2, ..., a_n)$, then

$A^{-1} = \text{diag} (a_1^{-1}, a_2^{-1}, ..., a_n^{-1})$

(viii) A is symmetric matrix ⇒ A^{-1} is symmetric matrix.

(ix) A is triangular matrix and $|A| \neq 0$ ⇒ A^{-1} is a triangular matrix.

(x) A is scalar matrix ⇒ A^{-1} is scalar matrix.

(xi) A is diagonal matrix ⇒ A^{-1} is diagonal matrix.

(xii) AB = AC ⇒ B = C, iff $|A| \neq 0$.