Application of Derivative (Monotonicity)

1. Introduction

In this chapter, we shall study the nature of a function which is governed by the sign of its derivative. If the graph of a function is in upward going direction or in downward coming direction then it is called as monotonic function, and this property of the function is called **Monotonicity**. If a function is defined in any interval, and if in some part of the interval, graph moves upwards and in the remaining part moves downward then function is not monotonic in that interval.

2. Monotonic Function

These are of two types –

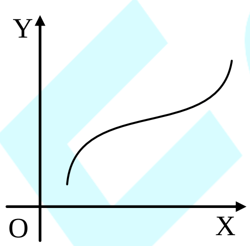
2.1 Monotonic Increasing :

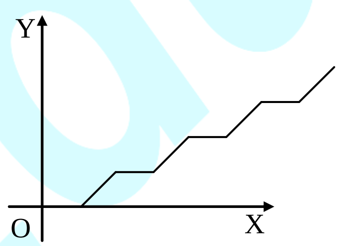
A function f(x) defined in a domain D is said to be monotonic increasing function if the value of f(x) does not decrease (increase) by increasing (decreasing) the value of x or

We can say that the value of f(x) should increase (decrease) or remain equal by increasing (Decreasing) the value of x.

$$\text{If } \begin{cases} x_1 < x_2 \Rightarrow f(x_1) \le f(x_2) \\ \text{or } x_1 < x_2 \Rightarrow f(x_1) \not> f(x_2), \forall x_1, x_2 \in D \end{cases}$$

$$\text{or } \begin{cases} x_1 > x_2 \Rightarrow f(x_1) \ge f(x_2) \\ \text{or } x_1 > x_2 \Rightarrow f(x_1) \not< f(x_2), \forall x_1, x_2 \in D \end{cases}$$





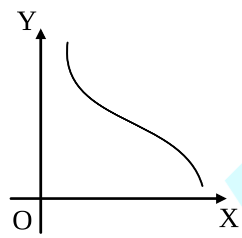
2.2 Monotonic Decreasing :

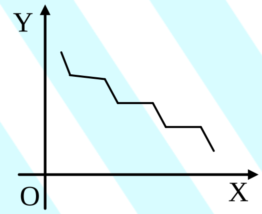
A function f(x) defined in a domain D is said to be monotonic decreasing function if the value of f(x) does not increase (decrease) by increasing (decreasing) the value of x or

We can say that the value of f(x) should decrease (increase) or remain equal by increasing (Decreasing) the value of x.

$$\text{If } \begin{cases} x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2) \\ \text{or } x_1 < x_2 \Rightarrow f(x_1) \not< f(x_2), \forall x_1, x_2 \in D \end{cases}$$

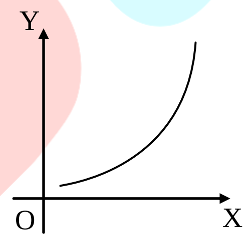
$$\text{or } \begin{cases} x_1 > x_2 \Rightarrow f(x_1) \le f(x_2) \\ \text{or } x_1 > x_2 \Rightarrow f(x_1) \not> f(x_2), \forall x_1, x_2 \in D \end{cases}$$



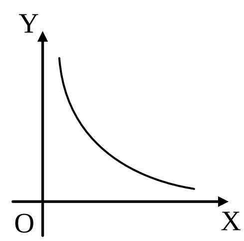


A function is said to be monotonic function in a domain if it is either monotonic increasing or monotonic decreasing in that domain.

NOTE : If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in D$, then f(x) is called strictly increasing in domain D.



Similarly if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2),\ \forall\ x_1, x_2 \in D$ then it is called strictly decreasing in domain D.



For Example

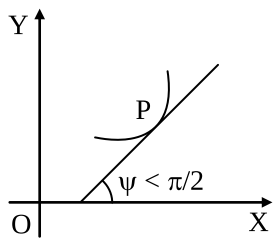
(i) $f(x) = e^x$ is a monotonic increasing function where as g(x) = 1/x is monotonic decreasing function.

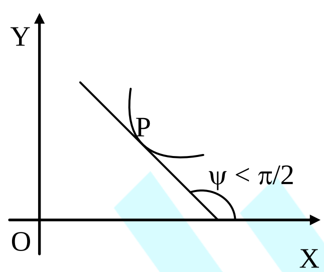
(ii) $f(x) = x^2$ and g(x) = | x | are monotonic increasing for x > 0 and monotonic decreasing for x < 0. In general they are not monotonic functions.

(iii) Sin x, cos x are not monotonic function whereas tan x, cot x are monotonic.

3. Method of Testing Monotonicity

(i) At a Point : A function f(x) is said to be monotonic increasing (decreasing) at a point x = a of its domain if it is monotonic increasing (decreasing) in the interval (a – h, a + h) where h is a small positive number. Hence we may observe that if f(x) is monotonic increasing at x = a, then at this point tangent to its graph will make an acute angle with the x–axis where as if the function is monotonic decreasing these tangent will make an obtuse angle with x–axis. Consequently $f'(a)$ will be positive or negative according as f(x) is monotonic increasing or decreasing at x = a.





So at x = a , function f(x) is

Monotonic increasing $\Rightarrow f'(a) > 0$

Monotonic deacreasing $\Rightarrow f'(a) < 0$

(ii) In an interval : A function f(x) defined in the interval [a, b] will be

$$\left.\begin{array}{ll} \text{Monotonic increasing} & \Rightarrow f'(x) \geq 0 \\ \text{Monotonic decreasing} & \Rightarrow f'(x) \leq 0 \\ \text{Constant} & \Rightarrow f'(x) = 0 \\ \text{Strictly increasing} & \Rightarrow f'(x) > 0 \\ \text{Strictly decreasing} & \Rightarrow f''(x) < 0 \end{array}\right\} \forall x \in (a,b)$$

NOTE :

(i) In the above result f'(x) should not be zero for all value of x otherwise f(x) will be a constant function.

(ii) If in [a, b], f'(x) < 0, for atleast one value of x and f'(x) > 0 for atleast one value of x then f(x) will not be monotonic in [a, b].

4. Examples of Monotonic Function

If a function is monotonic increasing (decreasing) at every point of its domain, then it is said to be monotonic increasing(decreasing) function.

In the following table we have examples of some monotonic / not monotonic functions.

Monotonic Increasing	**Monotonic Decreasing**	**Not Monotonic**
x^3	1/x	x^2
x | x |	1 – 2x	| x |
e^x	e^{-x}	$e^x + e^{-x}$
$\log_a x, a > 1$	$\log_a x, a < 1$	sin x
tan x	cot x	cos x
sinh x	cosech x	cosh x
[x]	coth x	sech x

5. Properties of Monotonic Functions

(i) If f(x) is strictly increasing in some interval, then in that interval, f' exists and that is also strictly increasing function.

(ii) If f(x) is continuous in [a, b] and differentiable in (a, b), then

$f'(c) \geq 0 \ \forall\ c \in (a, b) \quad \Rightarrow$ f(x) is monotonic increasing in [a, b]

$f'(c) \leq 0 \ \forall\ c \in (a, b) \quad \Rightarrow$ f(x) is monotonic decreasing in [a, b]

(iii) If both f(x) and g(x) are increasing (or decreasing) in [a,b] and gof is defined in [a, b] then gof is increasing.

(iv) If f(x) and g(x) are two monotonic functions in [a, b] such that one is increasing and other is decreasing then gof, if it is defined, is decreasing function.