MATRICES

1.1 Basic Concepts

A set of mn numbers arranged in the form of an ordered set of m rows and n columns is called m × n matrix (to be read as m by n matrix.)

Thus m × n matrix (to be read as m by n matrix.)

Thus m × n matrix A is written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

or $A = [a_{ij}]$; i = 1, 2, ...m
j = 1, 2, ...n

or $A = [a_{ij}]_{m \times n}$

where a_{ij} represents the element at the intersection of ith row and jth column.

In case the order of a matrix is established or known then we shall simply write

$A = [a_{ij}]$ of type m × n.

1.2 Various Types of Matrices

1. **Square Matrices :** A matrix in which the number of rows is equal to the number of columns is called a Square Matrix. Thus m × n matrix A will be a square matrix. if m = n, and it will be tenned as a square matrix of order n or n-rowed square matrix.

2. **Diagonal Matrices :** In a square matrix all those elements a_{ij} for which i = j i.e. all those elements which occur in the same row and same column namely a_{11}, a_{22}, a_{33} are called the diagonal elements and the line along which they lie is called the principal diagonal. Also the sum of the diagonal elements of a square matrix A is called trace of A.

   i.e. $a_{11}, + a_{22}, + a_{33}$ = Trace of A.

   In general $a_{11}, a_{22}, \dots\dots\dots a_{nn}$ are the diagonal elements of n-rowed square matrix and

   $a_{11} + a_{22} + \dots\dots\dots\dots a_{nn}$ = Trace of A.

   A square matrix A is said to be a diagonal matrix if all its non-diagonal elements be zero.

   Thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ or $\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$

   Above are diagonal matrices of the type 3 × 3. These are in short written as

   Diag [1, 4, 8] or Diag $[d_1, d_2, d_3]$

3. **Scalar Matrix :** A diagonal matrix [i.e. all non-diagonal elements being zero] whose all the diagonal elements are equal is called a scalar matrix.

   Thus, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ or $\begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$

   are both scalar matrices of type 3 × 3.

   In general for a scalar matrix.

   $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = d$ for $i = j$

4. **Unit Matrix :** A square matrix A all of whose non-diagonal elements are zero (i.e. it is a diagonal matrix) and also all the diagonal elements are unity (i.e. it is a diagonal matrix) and also all the diagonal elements are unity is called a unit matrix or an identity matrix.

Thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are unit matrices of order 3

In general for a unit matrix,

$a_{ij} = 0$ for $i \neq j$ and $a_{ij} = 1$ for $i = j$.

They are generally denoted by I_3, I_4 or I_n where 3, 4, n denote the order of the square matrix. In case the order be known then we may simply denote it by I.

5. **Zero matrix or Null Matrix :** Any m × n matrix in which all the elements are zero matrix is called a zero matrix or null matrix of the type m × n and is denoted by $O_{m \times n}$.

Thus $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

All the above are zero or null matrices of the type 3 × 2, 3 × 3 and 2 × 4 respectively.

6. **Determinant of a Square Matrix :** If we have a square matrix having same number of rows and columns it will have $n \times n = n^2$ arrays of numbers. These n^2 numbers also determine a determinant having n rows and n columns and is denoted by Det A or | A |.

7. **Equality of Matrices :** Two matrices $A = [a_{ij}]_{m \times x}$, $B = [b_{ij}]_{m \times n}$ are said to be equal and written as A = B if and only if they have the same order or are of the same type i.e. each has as many rows and column as the other [In this case they are said to be comparable and also each element of one is equal to the corresponding element of the other i.e. $a_{ij} = b_{ij}$ for each pair of subscripts i and j where i = 1, 2, ...m and j = 1, 2, ...n.]

8. **Idempotent Matrix**

A matrix A such that $A^2 = A$ is called Idempotent matrix.

9. **Periodic Matrix**

A matrix A will be called a periodic matrix if $A^{k+1} = A$ where k is +ive integer. If, however, k is the least +ive integer for which $A^{k+1} = A$, then k is said to be the period of A.

10. **Nilpotent Matrix**

A matrix A will be called a nilpotent matrix if $A^k = O$ (null matrix) where k is a +ive integer. If, however, k is the least +ive integer for which $A^k = O$, then k is the index of the nilpotent matrix A.

11. **Involuntary Matrix.**

A matrix A will be called an involuntary matrix if $A^2 = I$ (unit matrix).

Since $I^2 = I$ always.

$\therefore$ Unit Matrix I is involuntary.

Hence we can say that two matrices are equal if and only if one is duplicate of the other.

12. **Symmetric Matrices**

A square matrix $A = [a_{ij}]$ will be called symmetric if for all values of i and j, $a_{ij} = a_{ji}$.

i.e. every i-jth element = j-ith element.

e.g. $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3 \times 3}$ is a symmetric matrix.

13. Skew Symmetric Matrix

A square matrix $A = [a_{ij}]$ will be called skew symmetric if its i-jth element is –ive of j-ith element for all values of i and j i.e. $a_{ij} = -a_{ji}$ for all values of i and j.

Since diagonal elements will be of the type a_{11}, a_{22}, a_{33} .... a_{ii} and by given condition $a_{ii} = -a_{ii}$ for all values of i

or $2a_{ii} = 0$

$\therefore$ $a_{ii} = 0$.

Hence the diagonal elements of skew symmetric matrix are zero.

e.g. $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$ is a skew symmetric matrix.

Property : A' = – A.

14. Hermitian Matrix

A square matrix $A = [a_{ij}]$ is called Hermitian matrix if every i-jth element of A is equal to conjugate complex of j-ith element of A. In other words if for all value sof i and j, $a_{ij} = \overline{a_{ji}}$ then the matrix A is Hermitian

Property :

$A = A^{\theta} = (\overline{A}')$

15. Skew Hermitian matrix

A square matrix $A = [a_{ij}]$ will be called a skew Hermitian matrix if every i-jth element of A is equal to negative conjugate complex of j-ith element of A. In other words for all values of i-j

$a_{ij} = -\overline{a_{ji}}$.

All the elements in the principal diagonal will be of the type a_{11}, a_{22},........a_{ii} and by definition $a_{ii} = -\overline{a_{ii}}$.

or $a_{ii} + \overline{a_{ii}} = 0$. If $a_{ii} = a + ib$ then $\overline{a_{ii}} = a - ib$ and their sum 2a is zero if a = 0 i.e. a_{ii} is pure imaginary or else it could be possible if $a_{ii} = 0$. Hence all the diagonal elements of a skew Hermitian Matrices are either zeroes or pure imaginary.

1.3 Properties of Matrix Addition and matrix multiplication

A + B = B + A.

(a) Matrix addition is commutative

A + (B + C) = (A + B) + C.

(b) Matrix addition is associative.

(c) Multiplication of Matrices is distributive with respect to addition of matrices i.e.

A (B + C) = AB + AC.

(d) Matrix Multiplication is associative

i.e. A (BC) = (AB) C.

(e) The multiplication of Matrices is not always commutative. i.e. AB is not always equal to BA.

(f) Multiplication of a Matrix A by a null matrix conformable with A for multiplication is a null matrix i.e. AO = O.

In particular if A be a square matrix and O be square null matrix of the same order, then OA = OA = O.

(g) If AB = 0 then it does not necessarily mean that A = O or B = O or both are O as shown below.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

None of the matrices on the left is a null matrix whereas their product is a null matrix.

(h) **Multiplication of Matrix A by a Unit Matrix I :** Let A be a m × n matrix and I be a square unit matrix of order n, so that A and I are conformable for multiplication then

$AI_n = A$.

Similarly for IA to exist I should be square unit matrix of order m and in that case $I_mA = A$.

1.4 The Transpose of a Matrix and its properties

If A be a given matrix of the type m × n then the matrix obtained by changing the rows of A into columns and columns of A into rows is called Transpose of matrix A and is denoted by A'. As there are m rows in A therefore there will be m columns in A' and similarly as there are n columns in A there will be n rows in A'.

Hence the matrix A' is n × m type.

$$\text{e.g. } A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 5 & 9 \end{bmatrix}_{3 \times 2}$$

$$\text{then } A' = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 9 \end{bmatrix}_{2 \times 3}$$

(1) (A')' = A.

(2) (KA)' = KA' . K being a scalar.

(3) (A + B)' = A' + B'.

(4) (AB)' = B'A'.

(5) (ABC)' = C'B'A'

1.5 Conjugate of a Matrix. and Properties

Definition : Let $A = [a_{ij}]$ be a given matrix then the matrix obtained by replacing all the elements by their conjugate complex is called the conjugate of matrix A and is denoted by $\overline{A} = [\overline{a}_{ij}]$ where $\overline{a}_{ij}$ is conjugate of corresponding element a_{ij}.

(a) $\overline{\overline{A}} = A$

(b) $\overline{A + B} = \overline{A} + \overline{B}$

(c) $(\overline{KA}) = \overline{K}\,\overline{A}$, K being any complex number.

(d) $(\overline{AB}) = \overline{A}\,\overline{B}$

(e) $(\overline{A})' = (\overline{A'}) = A^{\theta}$.

1.6 Adjoint of a Matrix and Properties

If $A = [a_{ij}]$ be a n-squared matrix then the matrix $B = [b_{ij}]$ such that b_{ij} is the co-factor of the element a_{ji} in the determinant | A | is called the adjoint of matrix A and is written as adj. A.

In simple language we can say that adj. A is the transpose of the matrix formed by the co-factors of elements of |A|.

Working rule for finding the adjoint of A. Write down the determinant | A | and the co-factors of various rows which will be columns of adj A or replace each element in A by its co-factors and then take transpose to get adj. A.

The product of a matrix and its adjoint is commutative.

(a) If A be n-rowed square matrix then

$(\text{adj. } A)\, A = A\, (\text{adj. } A) = |A| \,.\, I_n$

where | A | is determinate A and I_n is the n-rowed unit matrix.

Deduction (a). If A is a n-squared singular matrix then

A (adj. A) = (adj. A) A = O (null matrix) A matrix is said to be Singular if its determinate is zero i.e. | A | = 0

Deduction (b) . $|\text{adj. } A| = |A|^{n-1}$ If |A| is not zero.

If clearly follows from above on taking determinants in (a) that

$|A| . |\text{adj. } A| = |A|^n = |\text{adj. } A| . |A|$

$|\text{adj. } A| = |A|^{n-1}$ provided | A |

is not zero.

If | A | is not zero then A is said to be non-singular matrix.

(c) Adj. (AB) = (Adj. B).(Adj.A).

1.7 The inverse of a Matrix and properties

Definition : If A and B be two n-squared matrices such that AB = BA = I then we shall say that $B = A^{-1}$. i.e. B is equal to inverse of A. Also the matrix B has an inverse. We shall say that $A = B^{-i}$ i.e. A is equal to inverse of B. It will be seen that every square matrix does not possess an inverse.

Properties

Inverse of a matrix is unique.

(a) We shall show below that if a matrix A has an inverse, then it is unique.

(b) Condition for a square matrix A to possess an inverse is that A is non-singular.

i.e. $|A| \neq 0$.

(c) Inverse by the help of adjoint.

$$A^{-1} = \frac{1}{|A|} (\text{adj. } A)$$

(d) If A be non-singular and AB = AC then B = C, where B and C are square matrices of the same order.

(e) Reversal Law for the inverse of product.

i.e. $(AB)^{-1} = B^{-1} A^{-1}$.

In other words it means that inverse of the product is the product of the inverses in the reverse order.

(f) The operation of transposing and inverting are commutative

i.e. $(A')^{-1} = (A^{-1})'$

SOLVED PROBLEMS

Ex.1 ***If A = $\begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ and I is the identify matrix of order 2, show that I + A = (I – A) = $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.***

Sol. Let $\tan\frac{\alpha}{2}$ = p. Then $\sin\alpha = \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$, $\cos\alpha = \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$

Also, $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -p \\ p & 0 \end{bmatrix} = \begin{bmatrix} 1 & p \\ -p & 1 \end{bmatrix}$

$\therefore$ $(I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & p \\ -p & 1 \end{bmatrix}\begin{bmatrix} \frac{1-p^2}{1+p^2} & \frac{-2p}{1+p^2} \\ \frac{2p}{1+p^2} & \frac{1-p^2}{1+p^2} \end{bmatrix}$

$$= \begin{bmatrix} \frac{1-p^2}{1+p^2}+\frac{2p^2}{1+p^2} & \frac{-2p}{1+p^2}+\frac{p(1-p^2)}{1+p^2} \\ \frac{-p(1-p^2)}{1+p^2}+\frac{2p}{1+p^2} & \frac{2p^2}{1+p^2}+\frac{1-p^2}{1+p^2} \end{bmatrix} = \begin{bmatrix} \frac{1+p^2}{1+p^2}=1 & -p \\ p & \frac{1+p^2}{1+p^2}=1 \end{bmatrix} = \begin{bmatrix} 1 & -p \\ p & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

Also, $I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$

Hence, $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Ex.2 ***If F(x) = $\begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that F (x) . F(y) = F(x + y).***

Sol. Here. $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $F(y) = \begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

and $F(x+y) = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ....(1)

Now $F(x) . F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos x\cos y - \sin x\cos y & -\sin y\cos x - \sin x\cos y & 0 \\ \sin x\cos y + \cos x\sin y & -\sin x\sin y + \cos x\cos y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(x+y) \text{ [From (1)]}$$

Ex.3 ***Let A = $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, B = $\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, C = $\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$, Find a matrix D such that CD – AB = O.***

Sol. Since A, B, C are all square matrices of order 2, and CD – AB is well defined, D must be a square matrix of order 2.

Let D = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then CD – AB = O gives

$$\Rightarrow \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = O \quad \Rightarrow \quad \begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+5c-3 & 2b+5d \\ 3a+8c-43 & 3b+8d-22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ 2a + 5c – 3 = 0   2b + 5d = 0

3a + 8c – 43 = 0   3b + 8d – 22 = 0

On solving these equations, we have a = –191, b = –110, c = 77 and d = 44

Hence D = $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$

Ex.4 ***If A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, n ∈ N.***

Sol. We shall prove that result by using principle of mathematical induction.

We have P(n) : If A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

P(1) : If A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Therefore, the result is true for n = 1.

Let, the result be true for n = k. So

P (k) : If A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$

Now, we prove that the result is true for n = k + 1

Now, $A^{k+1} = A.A^k = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta\cos k\theta - \sin\theta\sin k\theta & \cos\theta\sin k\theta + \sin\theta\cos k\theta \\ -\sin\theta\cos k\theta + \cos\theta\sin k\theta & -\sin\theta\sin k\theta + \cos\theta\cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + k\theta) & \sin(\theta + k\theta) \\ -\sin(\theta + k\theta) & \cos(\theta + k\theta) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Therefore, the result is true for n = k + 1. Thus, by principle of mathematical induction, we have

$A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, holds for all n ∈ N.

Ex.5 ***If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix.***

Sol. If A and B are skew symmetric matrices, then A' = –A and B' = –B

We shall prove that (AB – BA)' = – (AB – BA)

Now, (AB – BA)' = (AB)' – (BA)'
= B'A' – A'B'
= (–B) (–A) – (–A) (–B)
= BA – AB
= – (AB – BA)

Hence, (AB – BA) is a skew symmetric matrix.

Ex.6 ***Show that the matrix B'AB is symmetric or skew symmetric according as A is symmetric or skew symmetric.***

Sol. **Case 1 :** Let A be a symmetric matrix. Then A' = A

Now consider (B'AB)'
= B'A' (B')'
= B'A'B
= B'AB (∵ A' = A)

Hence, B'AB is a symmetric matrix.

Case 2 : Let A be a skew symmetric matrix. Then A' = –A

Now consider (B' AB')'
= B'A' (B')'
= B'A'B
= B' (–A) B
= – B'AB

Hence, B'AB is a skew symmetric matrix.

Ex.7 ***Find the values of x, y, z if the matrix A =*** $\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ ***satisfy the equation A'A = I.***

Sol. Here, $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ $\Rightarrow$ $A' = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$

Let AA' = I. Then

$$\begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}\begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4y^2+z^2 & 2y^2-z^2 & -2y^2+z^2 \\ 2y^2-z^2 & x^2+y^2+z^2 & x^2-y^2-z^2 \\ -2y^2+z^2 & x^2-y^2-z^2 & x^2+y^2+z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ $4y^2 + z^2 = 1,$ $x^2 + y^2 + z^2 = 1$

$2y^2 - z^2 = 0,$ $x^2 - y^2 - z^2 = 0$

$\Rightarrow$ $x^2 = \frac{1}{2},\ y^2 = \frac{1}{6}$ and $z^2 = \frac{1}{3}$ $\Rightarrow$ $x = \pm\frac{1}{\sqrt{2}},\ y = \pm\frac{1}{\sqrt{6}}$ and $z = \pm\frac{1}{\sqrt{3}}$

Ex.8 ***Using elementary transformations, find the inverse of each of the matrices, if it exists.***

$$\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$$

Sol. Consider $A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$

We write A = IA

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad \Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A\ (R_2 \to R_2 + 3A_1)$$

$$\Rightarrow \begin{bmatrix} 1 & -3 & -2 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A\ (R_3 \to R_3 + 2A_1)$$

$$\Rightarrow \begin{bmatrix} 1 & -3 & -2 \\ 0 & -1 & 4 \\ 0 & 9 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} A\ (R_2 \to R_3) \qquad \Rightarrow \begin{bmatrix} 1 & -3 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ -15 & 1 & 9 \end{bmatrix} A\ (R_3 \to R_3 + 9R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & -1 & 4 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -2 & 0 & 1 \\ -15 & 1 & 9 \end{bmatrix} A\ (R_1 \to R_1 + 3R_2) \Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 2 & 0 & -1 \\ -15 & 1 & 9 \end{bmatrix} A\ (R_2 \to (-1)\, R_2)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ 2 & 0 & -1 \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A\ (R_2 \to \frac{1}{25} R_3) \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ 2 & 0 & -1 \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A\ (R_1 \to R_1 - 10R_3)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A\ (R_2 \to R_2 + 4R_3)$$

Hence, $A^{-1} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$

EXERCISE – I | UNSOLVED PROBLEMS

Q.1 Construct a 3 × 2 matrix whose elements in the ith row and jth column are given by

$a_{ij} = \frac{3i+j}{2}$

Q.2 Find the values of x and y, if $\begin{bmatrix} x+10 & y^2+2y \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 3x+y & 3 \\ 0 & y^2-5y \end{bmatrix}$

Q.3 Find x, y, z and w if $\begin{bmatrix} 2x-3y & z-w & 3 \\ 1 & x+4y & 3x+4w \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 6 & 29 \end{bmatrix}$

Q.4 Simplify : $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\sin\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

Q.5 Find x and y, if $2x + 3y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ and $3x + 2y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}$

Q.6 Prove that : Elements in the main diagonal of a skew symmetric matrix are all zero.

Q.7 Prove that : A matrix which is both symmetric as well as skew-symmetric is a zero matrix.

Q.8 Prove that : for any square matrix A with real number entries, A + A' is a symmetric matrix and A – A' is a skew symmetric matrix.

Q.9 Show that Any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.

Q.10 If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then

(i) Show that A + A' is a symmetric matrix.

(ii) Find the value of θ satisfying the equation
$A' + A = I_2$

Q.11 Express as the sum of symmetric and skew symmetric matrices : $A = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$

Q.12 If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$. Find f(A), where $f(x) = x^2 - 5x - 14$.

Q.13 If $A = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$, then prove by the principle of mathematical induction that

$A^n = \begin{bmatrix} \cos n\theta & i\sin n\theta \\ i\sin n\theta & \cos n\theta \end{bmatrix}$, where $n \in N$

Q.14 By using elementary transformations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

Q.15 Prove that : If A is a square matrix, then adj A' = (adj A)'

Q.16 Show that If A is a symmetric matrix, then adj A is also symmetric.

Q.17 Show that $|\text{adj } A| = |A|^{n-1}$

Q.18 Prove that : If A and B are invertible square matrices of the same order, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Q.19 Prove that : If A and B are invertible square matrices of the same order, then

(adj AB) = (adj B) . (adj A)

Q.20 For the matrix $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, find λ and μ so that $A^2 + \lambda I = \mu A$. Hence, find A^{-1}.

Q.21 If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$, find AB and use this result to solve the following system of equations :

$2x - y + z = -1$

$-x + 2y - z = 4$

$x - y + 2z = -3$

Q.22 Solve the following system of equations by matrix method :

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4 \; ; \; \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1 \; ; \; \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Q.23 Solve the following system of equations :

$x - y + z = 3$

$2x + y - z = 2$

$-x - 2y + 2z = 1$

EXERCISE – II BOARD PROBLEMS

Q.1 If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1} and hence prove that $A^2 - 4A - 51 = 0$.

Q.2 If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, prove that $A - A^T$ is a skew symmetric matrix where A^T denotes he transpose of A.

Q.3 If $A = \begin{bmatrix} 4 & 1 \\ 5 & 8 \end{bmatrix}$, show that $A + A^T$ is a symmetric matrix where A^T denotes the transpose of matrix A.

Q.4 Find a matrix X such that 2A + B + X = 0, where $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$.

Q.5 If $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and B = [–2 –1 –4], verify that (AB)' = B'A'.

Q.6 Construct a 2 × 3 matrix A, whose elements are given by $a_{ij} = \frac{(i-2j)^2}{2}$.

Q.7 From the following equation, find the values of x and y :

Q.8 Solve using matrix method

(a) $x + 2y + z = 7$

(b) $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$

$x + 3z = 11$

$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$

$2x - 3y = 1$

$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$

Q.9 Find X such that X. $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$.

Q.10 Solve for x and y given that $\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Q.11 Find the values of a and b for which the following holds $\begin{bmatrix} a & b \\ -a & 2b \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

Q.12 Given that A = $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$, find AB.

Use this to solve the following system of equations :

$x - y = 3$
$2x + 3y + 4z = 17$
$y + 2z = 7$

Q.13 Express A = $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Q.14 Solve $2x - 3y + z = -1$, $x - 2y + 3z = 6$, $-3y + 2z = 0$

Q.15 If A = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, prove that $A^3 - 4A^2 + A = 0$

Q.16 If A = $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, B = [–2 –1 –4], verify that (AB)' = B'A'.

Q.17 If A = $\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$; find k so that $A^2 = 8A + kI$.

Q.18 If A = $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Use it to solve the following system of equations :

$2x - 3y + 5z = 16$, $3x + 2y - 4z = -4$, $x + y - 2z = -3$

Q.19 Solve $3x - y + z = 5$, $2x - 2y + 3z = 7$, $x + y - z = -1$

Q.20 Solve $x + y + z = 6$, $x - y + z = 2$, $2x + y - z = 1$

Q.21 Let A = $\begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}$. Express A as sum of two matrices such that one is symmetric and the other is skew symmetric.

Q.22 If A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, verify $A^2 - 4A - 51 = 0$.

Q.23 Find inverse $\begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix}$

Q.24 If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the system of equations

$2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$, $x + y - 2z = -3$

Q.25 Solve $3x - 2y + 3z = 8$, $2x + y - z = 1$, $4x - 3y + 2z = 4$

Q.26 Obtain the inverse of the following matrix using elementary operations :

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Q.27 If $A = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, find A^{-1}. Using A^{-1} solve the following system of equations :

$2x - 3y + 5z = 16$,
$3x + 2y - 4z = -4$,
$x + y - 2z = -3$

Q.28 Express the following matrix as the sum of a symmetric and skew symmetric matrix, and verify your result:

$$\begin{pmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{pmatrix}$$

Q.29 Solve $4x + 3y + 2z = 60$, $x + 2y + 3z = 45$, $6x + 2y + 3z = 70$

Q.30 Using matrices, slove the following system of equations : $2x + 3y + 3z = 5$, $x - 2y + z = 4$, $3x - y - 2z = 3$

Q.31 The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some others (say x) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these values, namely, honesty cooperation and supervision, suggest one more value which the management of the colony must include for awards.

Answers

EXERCISE – 1 (UNSOLVED PROBLEMS)

1. $A = \begin{bmatrix} 2 & 5/2 \\ 7/2 & 4 \\ 5 & 1/2 \end{bmatrix}$

2. x = 3, y = 1

3. x = 2, y = 1, z = 3, w = 5

4. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5. $x = \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}$ $y = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$

10. (ii) $\theta = 2n\pi \pm \frac{\pi}{3}, n \in z$

11. $\underset{\text{S.M}}{\begin{bmatrix} 6 & -1/2 & -4 \\ -1/2 & -5 & 7/2 \\ -4 & 7/2 & -1 \end{bmatrix}} + \underset{\text{S.S.M}}{\begin{bmatrix} 0 & 3/2 & -1 \\ -3/2 & 0 & 1/2 \\ 1 & -1/2 & 0 \end{bmatrix}}$

12. f(A) = 0

14. $A^{-1} = \begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$

20. $\lambda = 8, \mu = 8$ $A^{-1} = \begin{bmatrix} 5/8 & -1/8 \\ -7/8 & 3/8 \end{bmatrix}$

21. x = 1; y = 2; z = –1

22. x = 2; y = 3 and z = 5

23. $x = \frac{5}{3}; y = -\frac{4}{3} + k \;\; z = k$

EXERCISE – 2 (BOARD PROBLEMS)

1. $\frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$

4. $\begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}$

6. $\begin{bmatrix} 1/2 & 9/2 & 25/2 \\ 0 & 2 & 8 \end{bmatrix}$

7. x = 2 ; y = 9

8. (a) x = 2, y = 1, z = 3   (b) x = 2, y = 3, z = 5

9. $\begin{bmatrix} -60 & -142 \\ 25 & 59 \end{bmatrix}$

10. x = 2; y = 1

11. a = 1 ; b = –3

12. x = 2, y = –1, z = 4

13. $\frac{1}{2}\begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$

14. x = 1, y = 2, z = 3

17. –7

18. $A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$; x =2, y = 1, z = 3

19. x = 1, y = –1, z = 1

20. x = 1, y = 2, z = 3

21. $\begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 1 & 9/2 \\ 5/2 & 9/2 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 5/2 \\ 1 & 0 & -3/2 \\ -5/2 & 3/2 & 0 \end{bmatrix}$

23. $\begin{bmatrix} -2 & 1/2 & 1 \\ 11 & -1 & -6 \\ 4 & -1/2 & -2 \end{bmatrix}$

24. $A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$; x = 1, y = 2, z = 3

25. x = 1, y = 2, z = 3

26. $\begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & 12 & 9 \end{bmatrix}$

30. x = 1, y = 2, z = –1