GRAVITATION

1. GRAVITATION

The discovery of the law of gravitation

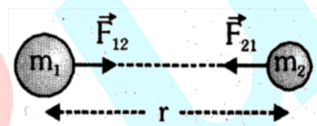
The way the law of universal gravitation was discovered is often considered as the paradigm of modern scientific technique. The major steps involved were :

- The hypothesis about planetary motion given by Nicolaus Copernicus (1473-1543).
- The careful experimental measurements of the positions of the planets and the Sun by Tycho Brahe (1546-1601).
- Analysis of the data and the formulation of empirical laws by Johannes Kepler (1571-1630).
- The development of a general theory by Isaac Newton (1642-1727).

1.1 Newton's law of Gravitation

It states that every particle in the universe attracts all other particle with a force which is directly proportional to the product of their masses and is inversely proportional to the square of the distance between them.

If $|\vec{F}_{12}| = |\vec{F}_{21}| = F$, then



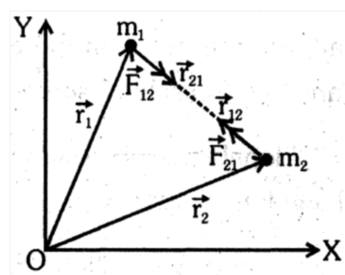
$F \propto m_1 m_2$ and $F \propto \dfrac{1}{r^2}$

So $F \propto \dfrac{m_1 m_2}{r^2}$

$\therefore F = \dfrac{G m_1 m_2}{r^2}$ [G = Universal gravitational constant]

Note: This formula is applicable only for spherically symmetric masses or point masses.

1.2 Vector form of Newton's law of Gravitation:



Let $\vec{r}_{12}$ = Position vector of m_1 w.r.t. $m_2 = \vec{r}_1 - \vec{r}_2$

$\vec{r}_{21}$ = Position vector of m_2 w.r.t. $m_1 = \vec{r}_2 - \vec{r}_1$

$\vec{r}_{21}$ = Gravitational force exerted on m_2 by m_1

$\vec{r}_{12}$ = Gravitational force exerted on m_1 by m_2

$$\vec{F}_{12} = -\frac{G m_1 m_2}{r_{21}^2}\hat{r}_{12} = -\frac{G m_1 m_2}{r_{21}^3}\vec{r}_{12}$$

Negative sign shown that :

(i) The direction of $\vec{F}_{12}$ is opposite to that of $\hat{r}_{12}$

(ii) The gravitational force is attractive in nature

Similarly $\vec{F}_{21} = -\dfrac{G m_1 m_2}{r_{12}^2}\hat{r}_{21}$ or $\vec{F}_{21} = -\dfrac{G m_1 m_2}{r_{12}^3}\hat{r}_{21}$ $\Rightarrow \vec{F}_{12} = -\vec{F}_{21}$, $\vec{r}_{12} = -\vec{r}_{21}$

The gravitational force between two bodies are equal in magnitude and opposite in directions.

1.3 Universal Gravitational Constant "G"

- Universal Gravitational constant is a scalar quantity.
- **Value of G : SI :** $G = 6.67 \times 10^{-11}$ N-m^2/kg^2;

  CGS : $G = 6.67 \times 10^{-8}$ dyne-cm^{2}g^2 **Dimensions :** $[M^{-1}L^3T^{-2}]$
- Its value is same throughout the universe; G does not depend on the nature and size of the bodies; it does not depend even upon the nature of the medium between the bodies.

- Its value was first found out by the scientist **"Henry Cavendish"** with the help of "Torsion Balance" experiment.

GOLDEN KEY POINTS

- Gravitational force is always attractive.
- Gravitational forces are developed in the form of action and reaction pair. Hence they obey Newton's third law of motion.
- It is independent of the nature of medium between two masses.
- Gravitational forces are central forces as they act along the line joining the centre of gravity of the two bodies.
- Gravitational forces are conservative forces so work done by gravitational force does not depends on path.
- If any particle moves along a closed path under the action of gravitational force then the work done by this force is always zero for round the trip.
- Gravitational force is weaker than the electromagnetic and nuclear forces.
- Force developed between any two masses is called gravitational force and force between Earth and any body is called force of gravity.
- The total gravitational force on a particle due to a number of particles is the resultant of the forces of attraction exerted on the given particle due to the individual particles i.e. $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...$ It means the principle of superposition is valid.
- Gravitational force holds good over a wide range of distances. It is found true from interplanetary distances to interatomic distances.
- It is a two body interaction i.e. gravitational force between the two particles is independent of the presence or absence of other bodies or particles.
- A uniform spherical shell of matter attracts a particle that is outside the shell as if all its mass were concentrated at its centre.

Illustrations

Illustration 1.

Two spherical balls of mass 10 kg each are placed 100 m apart. Find the gravitational force of attraction between them.

Solution:

$$F = \frac{Gm_1m_2}{r^2} = \frac{6.67\times10^{-11}\times10\times10}{(100)^2} = 6.67 \times 10^{-13}\text{ N.}$$

Illustration 2.

Two particles of masses 1 kg and 2 kg are placed at a separation of 50 cm. Assuming that the only forces acting on the particles are their mutual gravitation, find the initial acceleration of the heavier particle.

Solution:

Force exerted by one particle on another is $F = \frac{Gm_1m_2}{r^2} = \frac{6.67\times10^{-11}\times1\times2}{(0.5)^2} = 5.34 \times 10^{-10}$ N

Acceleration of heavier particle $= \frac{F}{m_2} = \frac{5.3\times10^{-10}}{2} = 2.67 \times 10^{-10}$ m/s^2.

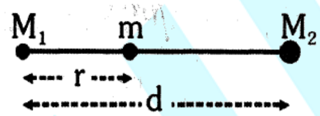
Note : This example shows that gravitational force is quite weak but this is the only force keep binds our solar system and also the universe comprising of all galaxies and other interstellar system.

Illustration 3.

Two stationary particles of masses M_1 and M_2 are 'd' distance apart. A third particle lying on the line joining the particles, experiences no resultant gravitational force. What is the distance of this particle from M_1 ?

Solution:

Let m be the mass of the third particle



Force on m toward M_1 is $F_1 = \frac{GM_1m}{r^2}$

Force on m towards M_2 is $F_2 = \frac{GM_2m}{(d-r)^2}$

Since net force on m is zero $\therefore F_1 = F_2$

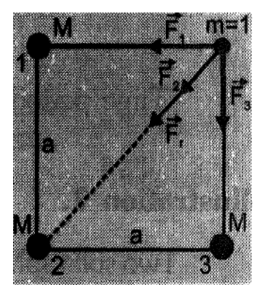
$$\Rightarrow \frac{GM_1m}{r^2} = \frac{GM_2m}{(d-r)^2} \Rightarrow \left(\frac{d-r}{r}\right)^2 = \frac{M_2}{M_1} \Rightarrow \frac{d}{r} - 1 = \frac{\sqrt{M_2}}{\sqrt{M_1}} \Rightarrow r = d\left[\frac{\sqrt{M_1}}{\sqrt{M_1}+\sqrt{M_2}}\right]$$

Illustration 4.

Three masses, each equal to M are placed at the three corners of a square of side a Calculate the force of attraction on unit mass placed at the fourth corner.

Solution:

Force on m = 1 due to masses at corners 1 and 3 are $\vec{F}_1$ and $\vec{F}_3$ with $F_1 = F_3 = \frac{GM}{a^2}$



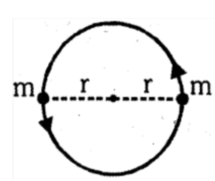
Resultant of $\vec{F}_1$ and $\vec{F}_3$ is $F_r = \sqrt{2}\frac{GM}{a^2}$ and its direction is along the diagonal i.e. toward corner 2

Force on m due to mass M at 2 is $F_2 = \frac{GM}{(\sqrt{2}a)^2} = \frac{GM}{2a^2}$; F_r and F_2 act in the same direction.

Resultant of these two is the net force :

$F_{net} = \frac{\sqrt{2}GM}{a^2} + \frac{GM}{2a^2} = \frac{GM}{a^2}\left[\sqrt{2}+\frac{1}{2}\right]$; it is directed along the diagonal as shown in the figure.

Illustration 5.



Two particles each of equal mass (m) move along a circle of radius (r) under the action of their mutual gravitational attraction. Find the speed of each particle.

Solution:

For the circular motion of each particle, $\frac{mv^2}{r} = \frac{Gmm}{(2r)^2} \Rightarrow v^2 = \frac{Gm}{4r} \Rightarrow v = \frac{1}{2}\sqrt{\frac{Gm}{r}}$.

Illustration 6.

Three particles, each of mass m, are situated at the vertices of an equilateral triangle of side 'a'. The only forces acting on the particles are their mutual gravitational forces. It is intended that each particle moves along a circle while maintaining their original separation 'a'. Determine the initial velocity that should be given to each particle and the time period of the circular motion.

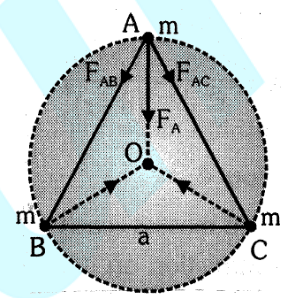
Solution:

The resultant force on particle at A due to other two particles is

$$F_A = \sqrt{F_{AB}^2 + F_{AC}^2 + 2F_{AB}F_{AC}\cos 60^\circ} = \sqrt{3}\frac{Gm^2}{a^2} \quad \text{.....(i)} \qquad \left[\because F_{AB} = F_{AC} = \frac{Gm^2}{a^2}\right]$$

Radius of the circle $r = \dfrac{a}{\sqrt{3}}$

If each particle is given a tangential velocity v, so that the resultant force acts as the centripetal force,



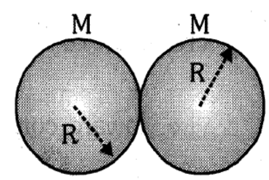
then $\dfrac{mv^2}{r} = \sqrt{3}\dfrac{mv^2}{a}$ .....(ii)

From (i) and (ii), $\sqrt{3}\dfrac{mv^2}{a} = \dfrac{Gm^2\sqrt{3}}{a^2} \Rightarrow v = \sqrt{\dfrac{Gm}{a}}$

Time period $T = \dfrac{2\pi r}{v} = \dfrac{2\pi a}{\sqrt{3}} = 2\pi\sqrt{\dfrac{a^3}{3Gm}}$.

Illustration 7.

Two solid spheres of same size of a certain metal are placed in contact with each other. Prove that the gravitational force acting between them is directly proportional to the fourth power of their radius.



Solution:

The weights of the spheres may be assumed to be concentrated at their centres.

So $F = \dfrac{G\left[\frac{4}{3}\pi R^3\rho\right]\times\left[\frac{4}{3}\pi R^3\rho\right]}{(2R)^2} = \dfrac{4}{9}(G\pi^2\rho^2)R^4$

$\therefore$ $F \propto R^4$

Illustration 8.

A mass (M) is split into two parts (m) and (M–m), which are then separated by a certain distance. What ratio $\dfrac{m}{M}$ will maximise the gravitational force between them ?

Solution:

If r is the distance between m and (M–m) the gravitational force will be

$$F = G\frac{m(M-m)}{r^2} = \frac{G}{r^2}(mM - m^2)$$

For F to be maximum $\dfrac{dF}{dm} = 0$ and $\dfrac{d^2F}{dm^2} < 0$ as M and r constants, i.e. $\dfrac{d}{dm}\left[\dfrac{G}{r^2}(mM-m^2)\right] = 0$

$\Rightarrow$ $\dfrac{G}{r^2}(M-2m) = 0$ i.e. $M - 2m = 0$ $\left[\because \dfrac{G}{r^2} \neq 0\right]$

or $\quad \frac{m}{M} = \frac{1}{2}$, i.e., the force will be maximum when the two parts are identical.

BEGINNER'S BOX - 1

1. Four identical point masses, each equal to M are placed at the four corners of a square of side a. Calculate the force of attraction on another point mass m_1 kept at the centre of the square.

2. Three identical particles each of mass m are placed at the three corners of an equilateral triangle of side "a". Find the gravitational force exerted on one body due to the other two.

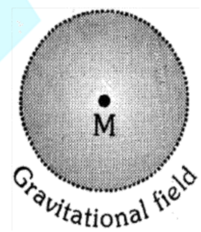
3. Three identical point masses, each of mass 1 kg lie in the x–y plane at points (0, 0) (0, 0.2m) and (0.2m, 0) respectively.
The gravitational force on the mass at the origin is :-

(A) $1.67 \times 10^{-11}\ (\hat{i} + \hat{j})$ N
(B) $3.34 \times 10^{-10}\ (\hat{i} + \hat{j})$ N
(C) $1.67 \times 10^{-9}\ (\hat{i} + \hat{j})$ N
(D) $3.34 \times 10^{-10}\ (\hat{i} - \hat{j})$ N

2. GRAVITATIONAL FIELD AND IT'S INTENSITY

2.1 Gravitational Field

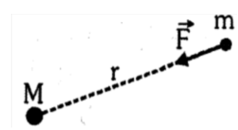
The gravitational field is the space around a mass or an assembly of masses within which it can exert gravitational forces on other masses.
Theoretically speaking, the gravitational field extends up to infinity. However, in actual practice, the gravitational field may become too weak to be measured beyond a particular distance.



2.2 Gravitational Field Intensity $(\vec{I})$

The gravitational field intensity at a point within a gravitational field is defined as the gravitational force exerted on unit mass placed at that point.

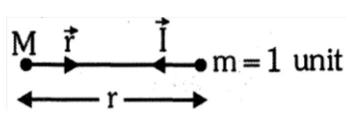
$$\vec{I} = \frac{\vec{F}}{m}$$



Gravitational field intensity is a vector quantity whose direction is same as that of the gravitational force.
Its **SI** unit is 'N/kg'.

Dimensions of intensity $= \frac{[F]}{[m]} = \frac{[M^1L^1T^{-2}]}{[M^1]} = [M^0L^1T^{-2}]$.

2.3 Gravitational Field Intensity Due to a Particle (Point – Mass) :



Gravitational field intensity = gravitational force exerted on unit mass

$$\Rightarrow \quad \vec{I} = \frac{GM}{r^2}(-\hat{r}) = \frac{GM}{r^3}(-\vec{r}) \qquad \vec{I} = \frac{GM}{r^2}(-\hat{r})$$

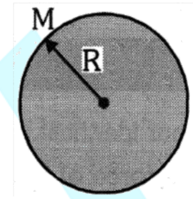
where 'M' is the mass of that particle due to which intensity is to be found.

2.4 Gravitational field intensity due to spherical mass distribution

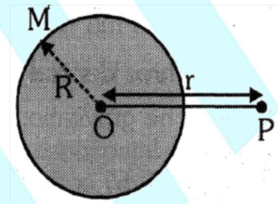
If the observation point is located on the surface or outside the surface then the spherical mass can be taken as a particle which is situated at the centre of the sphere. i.e. point mass.

(I) For solid sphere

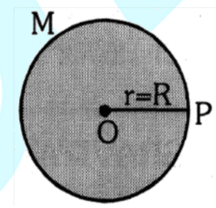
Let 'M' be the mass of sphere, 'R' the radius of sphere and 'r' the distance of the point under consideration from the centre of sphere.



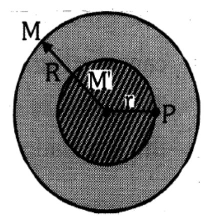
$\Rightarrow$ **Case I :** When r > R, i.e. outside the sphere that $\vec{I}_{out} = \frac{GM}{r^2}(-\hat{r})$



$\Rightarrow$ **Case II :** When r = R, i.e. at the surface that $\vec{I}_{surface} = \frac{GM}{R^2}(-\hat{r})$



$\Rightarrow$ **Case III :** When r < R, i.e., inside the sphere then



$$\vec{I}_{in} = \frac{GM'}{r^2}(-\hat{r}) \qquad \text{........(1)}$$

We know,

$$\frac{M'}{M} = \frac{V'\times\rho}{V\times\rho} = \frac{\frac{4}{3}\pi r^3\times\rho}{\frac{4}{3}\pi R^3\times\rho} = \frac{r^3}{R^3} \Rightarrow M'\frac{Mr^3}{R^3}$$

Putting the expression for M' in eq. (1), we get

$$\vec{I}_{in} = \frac{GMr^3}{r^2R^3}(-\hat{r}) \qquad \therefore \qquad \vec{I}_{in} = \frac{GMr}{R^3}(-\hat{r})$$

Important conclusions :-

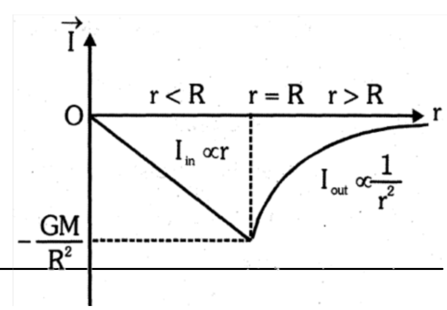
(1) $I_{out} = \frac{GM}{r^2}$ $\quad\therefore\quad$ $I_{out} \propto \frac{1}{r^2}$

(2) $I_{sur} = \frac{GM}{R^2}$

(3) $I_{in} = \frac{GMr}{R^3}$ $\quad\therefore\quad$ $I_{in} = \propto r$

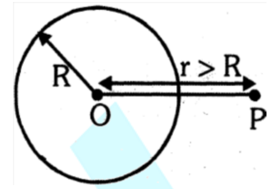
(4) So, $I_{max} = I_{sur} = \frac{GM}{R^2}$

(5) $I_{centre} = \frac{GM(0)}{R^3} = 0$ $\quad\therefore\quad$ $I_{min} = I_{centre} = 0$

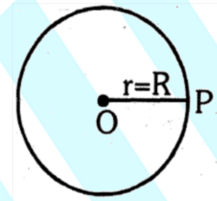


⇒ Graph between '$\vec{I}$' and 'r' for a solid sphere:

(II) Gravitational field intensity due to a Spherical Shall



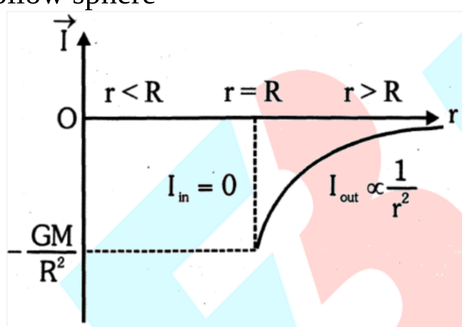
Case I : If r > R, the point is outside the shell then $\vec{I}_{out} = \frac{GM}{r^2}(-\hat{r})$



Case II : If r = R, the point is on the surface then $\vec{I}_{surface} = \frac{GM}{R^2}(-\hat{r})$

Case III : If r < R, the point is inside the shell then I = 0

⇒ $\vec{I}$ v/s r graph for hollow sphere



3. ACCELERATION DUE TO GRAVITY

3.1 Gravity

In Newton's law of gravitation, the force of attraction between any two bodies is gravitation. If one of the bodies is Earth then the gravitation is called 'gravity'. Hence, gravity is the force by which Earth attracts a body towards its centre. It is a special case of gravitation.

3.2 Acceleration due to gravity near Earth's surface

Let us assume that Earth is a uniform sphere of mass M and radius R. The magnitude of the gravitational force of Earth on a particle of mass m, located outside the Earth at a distance r from its is centre, is $F = \frac{GMm}{r^2}$

Now according to Newton's second law $F = ma_g$

Therefore $a_g = \frac{GM}{r^2}$ .....(i)

At the surface of Earth, acceleration due to gravity $g = \frac{GM}{R^2} = 9.8 \text{ m/s}^2$

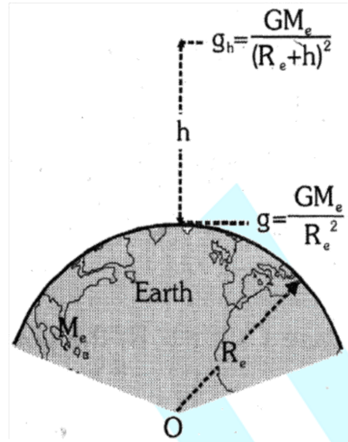
However any g value measured at a given location will differ from the g value calculated according to equation due to any three reasons :-

(i) Earth's mass is not distributed uniformly.

(ii) Earth is not a perfect sphere and

(iii) Earth rotates.

3.3 Variation in Acceleration due to gravity



- **Due to Altitude (height) :**

From diagram

$$\frac{g_h}{g} = \frac{R_e^2}{(R_e+h)^2} = \frac{R_e^2}{R_e^2\left[1+\frac{h}{R_e}\right]^2} = \left(1+\frac{h}{R_e}\right)^{-2}$$

By Binomial expansion $\left(1+\frac{h}{R_e}\right)^{-2} \propto \left(1-\frac{2h}{R_e}\right)$

[If h << R_e, then higher power terms become negligible]

$\therefore g_h = g\left[1-\frac{2h}{R_e}\right]$

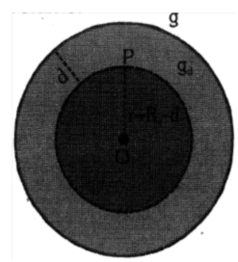
Note :

(1) This formula is valid if h is upto 5% of earth's radius. (320 km from earth's surface)

(2) If h is greater than 5% of the earth's radius we use $g_h = \frac{GM_e}{(R_e+h)^2}$

- **Due to depth :**

Assuming that the density of Earth remains same throughout the volume.



At Earth's surface : $g = \frac{4}{3}\pi G R_e \rho$ ……..(i)

At a depth d inside the Earth :

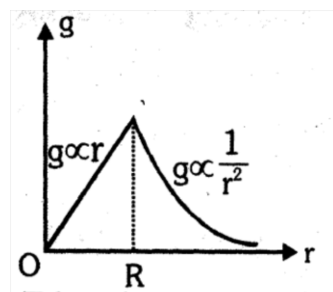
For point P only mass of the inner sphere is effective $g_d = \frac{GM'}{r^2}$

Mass of sphere of radius r = M'

$$M' = \frac{4}{3}\pi r^3 \rho = \frac{4}{3}\pi r^3 \times \frac{M_e}{4/3\pi R^3} = M' = \frac{M_e}{R^3} r^3$$

$$g_d = \frac{G}{r^2} \times \frac{M_e r^3}{R_e^3} = \frac{GM_e}{R_e^2} \times \frac{r}{R_e} = \frac{GM_e}{R_e^2} \times \frac{R_e - d}{R_e}$$

$g_d = g\left[1-\frac{d}{R_e}\right]$ valid for any depth



(Taking direction towards centre of earth as positive)

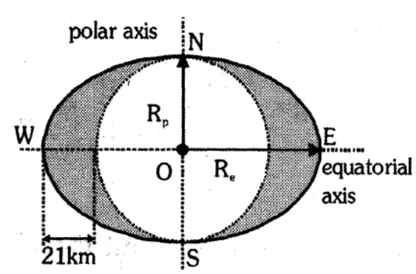
- **Due to shape of the Earth :**

From the diagram

$R_p < R_e$ ($R_e = R_p$ + 21 km) $g_p = \frac{GM_e}{R_p^2}$ & $g_e = \frac{GM_e}{(R_p+21000)^2}$

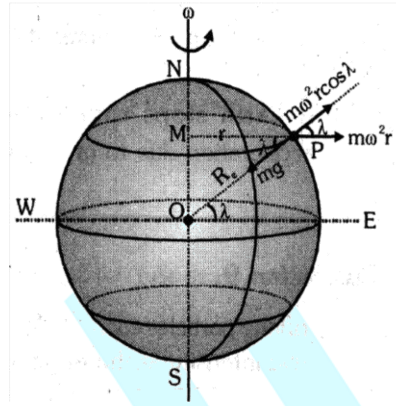
$\therefore g_e < g_p$

By putting the values $g_p - g_e = 0.02\ m/s^2$



(not to scale)

- **Due to Rotation of the Earth :**



Net force on particle at P $\quad mg' = mg - mr\omega^2\cos\lambda$

$g' = g - r\omega^2\cos\lambda$ from, ΔOMP $r = R_e\cos\lambda$

where λ = Latitude

Substituting for r, $\quad$ we have $g' = g - R_e\omega^2\cos\lambda$

- At the equator ($\lambda = 0°$) :

  $g_{eq} = g - \omega^2 R\cos^2(0°)$ $\qquad g_{min}$ or $g_{eq} = g - \omega^2 R$

- At the poles ($\lambda = 90°$) :

  $g_{pole} = g - \omega^2 R\cos^2(90°) = g - \omega^2 R(0)$: g_{max} or $g_{pole} = g$

It means that acceleration due to gravity at the poles does not depend upon the angular velocity or rotation of earth.

Condition of weightlessness on Earth's surface

If apparent weight of body is zero then angular speed of Earth can be calculated as-

$$mg' = mg - mR_e\omega^2\cos^2\lambda.$$

$$0 = mg - mR_e\omega^2\cos^2\lambda \Rightarrow \omega = \frac{1}{\cos\lambda}\sqrt{\frac{g}{R_e}}$$

But at equator $\lambda = 0°$ $\therefore$ $\omega = \sqrt{\frac{g}{R_e}} = \frac{1}{800}$ rad/s = 0.00125 rad/s = 1.25×10^{-3} rad/s.

Note : If Earth will to rotate with 17 times of its present angular speed then bodies lying on equator would fly off into the space. Time period of Earth's rotation in this case would be 1.4 h.

GOLDEN KEY POINTS

- In terms of density $\quad g = \frac{GM}{R^2} = \frac{G}{R^2}\times\frac{4}{3}\pi R^3 \times \rho \quad \therefore \quad g = \frac{4}{3}\pi G R\rho$

  If ρ is constant then $g \propto r$

- If M is constant then $g \propto \frac{1}{R^2}$; For % variation in 'g' upto 5%, $\frac{\Delta g}{g} = -2\frac{\Delta R}{R}$

- If mass (M) and radius (R) correspond to a planet and if small changes ΔM and ΔR occur in (M) and (R) respectively then

  by $\quad g = \frac{GM}{R^2} \Rightarrow \frac{\Delta g}{g} = \frac{\Delta M}{M} - 2\frac{\Delta R}{R}$

  If R is constant then $\frac{\Delta g}{g} = \frac{\Delta M}{M}$; $\qquad$ If M is constant then $\frac{\Delta g}{g} = -2\frac{\Delta R}{R}$

- $\Delta g_h = g - g_h$ = decrement in g with small height $= g - g\left[1 - \frac{2h}{R_e}\right] \Rightarrow \frac{\Delta g_h}{g} = \frac{2h}{R_e}$

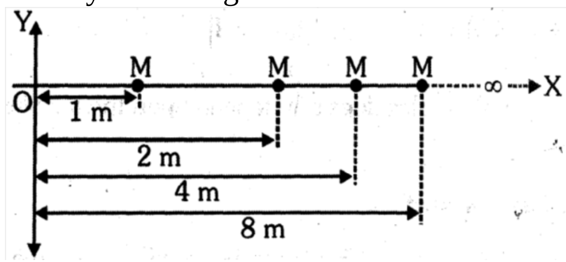
- $\Delta g_d = g - g_d$ = decrement in g with depth $= g - g\left[1 - \frac{d}{R_e}\right] \Rightarrow \frac{\Delta g_d}{g} = \frac{d}{R_e}$

- If Earth stops rotating about its own axis, then the apparent weight of bodies or effective acceleration due to gravity will increase at all the places except poles.

Illustrations

Illustration 9.

Infinite particles each of mass 'M' are placed at position x = 1 m, x = 2 m, x = 4 m ….. ∝. Find the gravitational field intensity at the origin.



Solution:

$$\vec{I}_{net} = \vec{I}_1 + \vec{I}_2 + \vec{I}_3 + \vec{I}_4 + ...\infty \text{ terms}$$

$$= \frac{GM}{(1)^2}\hat{i} + \frac{GM}{(2)^2}\hat{i} + \frac{GM\hat{i}}{(4)^2} + ...\infty \text{ terms} = GM\hat{i}\left(1 + \frac{1}{4} + \frac{1}{16} + ...\infty\right)$$ [Here in the GP a = 1 and r = $\frac{1}{4}$]

So, $$\vec{I}_{net} = GM\hat{i}\left[\frac{1}{1-\frac{1}{4}}\right] = GM\hat{i}\left[\frac{1}{\left(\frac{3}{4}\right)}\right] \Rightarrow \vec{I}_{net} = \frac{4}{3}GM\hat{i}.$$

Illustration 10.

At what depth below the Earth's surface the acceleration due to gravity is decreased by 1% ?

Solution:

$$\frac{\Delta g_d}{g} = \frac{d}{R_e} \Rightarrow \frac{1}{100} = \frac{d}{6400} \qquad \therefore d = 64 \text{ km}.$$

Illustration 11.

Which of the following statements are true about acceleration due to gravity?

(A) 'g' decreases in moving away from the centre of earth if r > R

(B) 'g' decreases in moving away from the centre of earth if r < R

(C) 'g' is zero at the centre of earth

(D) 'g' decreases if earth stops rotating on its axis

Solution:

Variation of g with distance : If r > R then $g \propto \frac{1}{r^2}$ $\therefore$ (A) is correct

If r < R then $g \propto r$ $\therefore$ (B) is incorrect & (C) is correct

variation of g with ω : $g' = g - \omega^2 R\cos^2\lambda$

If ω = 0 then g will not change at poles where cosλ = 0. while at other points g increases

$\therefore$ (D) is incorrect.

Illustration 12.

At what height above the Earth's surface the acceleration due to gravity will be $1/9^{th}$ of its value at the Earth's surface ? (Radius of Earth is 6400 km)

Solution:

Acceleration due to gravity at height h is $g' = \dfrac{g}{\left(1+\dfrac{h}{R_e}\right)^2} = \dfrac{g}{9} \Rightarrow \left(1+\dfrac{h}{R_e}\right) = 3 \Rightarrow h = 2R_e = 12800$ km.

Illustration 13.

Determine the speed with which Earth would have to rotate about its axis so that a person on the equator weighs $\dfrac{3}{5}$th of its present value. Write your answer in terms of g and R.

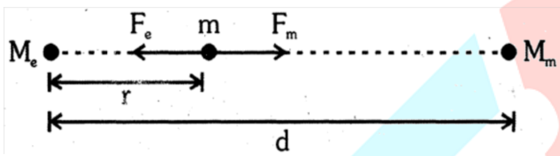
Solution:

Weight on the equator $\quad W' = \dfrac{3}{5}W \Rightarrow \dfrac{3}{5}mg = mg - m\omega^2 R \quad \Rightarrow \quad \omega = \sqrt{\dfrac{2g}{5R}}$

Illustration 14.

Draw a rough sketch of the variation in weight of a spacecraft which moves from earth to moon.

Solution:



Net weight = $E_e \sim F_m = \left|\dfrac{GM_e}{r^2} - \dfrac{GM_m}{(d-r)^2}\right| m$

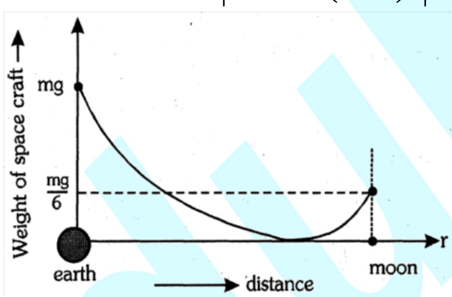
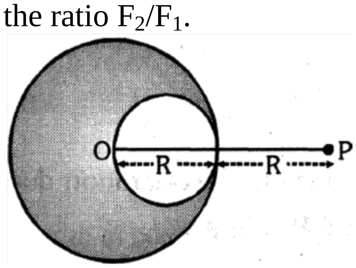


Illustration 15.

A solid sphere of uniform density and radius R exerts a gravitational force of attraction F_1 on a particle P, distant 2R from the centre of the sphere. A spherical cavity of radius R/2 is now formed in the sphere as shown in figure. The sphere with cavity now applies a gravitational force F_2 on the same particle P. Find the ratio F_2/F_1.



Solution:

$F_1 = \dfrac{GMm}{4R^2}$, F_2 = force due to whole sphere – force due to the sphere forming the cavity

$$= \frac{GMm}{4R^2} - \frac{GMm}{18R^2} \Rightarrow \frac{7GMm}{36R^2} \qquad \therefore \frac{F_2}{F_1} = \frac{7}{9}$$

Illustration 16.

The maximum vertical distance through which an astronaut can jump on the earth is 0.5 m. Estimate the corresponding distance on the moon.

Solution:

Θ mgh = constant $\quad \therefore h \propto \frac{1}{g} \Rightarrow h_m = \frac{h_e g_e}{g_m} = \frac{0.5 \times g}{g/6} = 3$ m.

BEGINNER'S BOX - 2

1. The magnitudes of the gravitational field at distances r_1 and r_2 from the centre of a uniform sphere of radius R and mass M are F_1 and F_2 respectively. then :-

(A) $\frac{F_1}{F_2} = \frac{r_1}{r_2}$ if $r_1 < R$ and $r_2 < R$

(B) $\frac{F_1}{F_2} = \frac{r_2^2}{r_1^2}$ if $r_1 > R$ and $r_2 > R$

(C) $\frac{F_1}{F_2} = \frac{r_1^3}{r_2^3}$ if $r_1 < R$ and $r_2 < R$

(D) $\frac{F_1}{F_2} = \frac{r_1^2}{r_2^2}$ if $r_1 < R$ and $r_2 < R$

2. A stone dropped from a height 'h' reaches the Earth's surface in l s. If the same stone is taken to Moon and dropped freely from the same height then it will reach the surface of the Moon in a time (The 'g' of Moon is 1/6 times that of Earth) :-

(A) $\sqrt{6}$ seconds (B) 9 seconds (C) $\sqrt{3}$ seconds (D) 6 seconds

3. The radius of Earth is about 6400 km and that of Mars is 3200 km. The mass of Earth is 10 times that of Mars. An object weighs 200 N on the surface of Earth. Its weight on the surface of Mars will be :-

(A) 80 N (B) 40 N (C) 20 N (D) 8 N

4. Weight of a body decreases by 1% when it is raised to a height h above the Earth's surface. If the body is taken to a depth h in a mine, then its weight will :-

(A) decrease by 0.5% (B) decrease by 2%

(C) increase by 0.5% (D) increase by 1%

5. At which height from the earth's surface does the acceleration due to gravity decrease by 1% ?

6. Find the percentage decrement in the weight of a body when taken to a height of 16 km above the surface of earth. (radius of earth is 6400 km)

7. What is the value of acceleration due to gravity at a height equal to half the radius of earth, from surface of earth ? [take g = 10 m/s^2 on earth's surface]

8. At which height from the earth's surface does the acceleration due to gravity decrease by 75% of its value at earth's surface ?

9. At which height above earth's surface is the value of 'g' same as in a 100 km deep mine?

10. At what depth below the surface does the acceleration due to gravity becomes 70% of its value on the surface of earth?

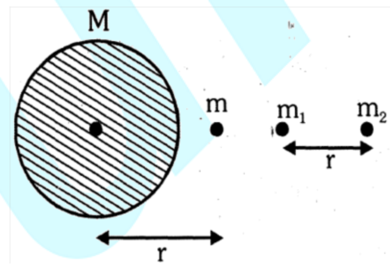
11. At what depth from earth's surface does the acceleration due to gravity becomes $\frac{1}{4}$ times that of its value at surface ?

12. If earth is assumed to be a sphere of uniform density then plot a graph between acceleration due to gravity (g) and distance from the centre of earth. **[AIPMT (Mains) 2006]**

4. GRAVITATIONAL POTENTIAL ENERGY

4.1 Gravitational Potential Energy (U)

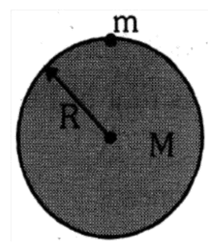
The gravitational potential energy of a particle situated at a point in some gravitational field is defined as the amount of work required to bring it from infinity to that point without changing its kinetic energy.



$$W = U = -\frac{GMm}{r} \text{ or } U = -\frac{Gm_1m_2}{r}$$

(Here negative sign shows the boundless of the two bodies)

- It is a scalar quantity.
- It’s SI unit is joule and Dimensions are $[M^1L^2T^{-2}]$
- The gravitational potential energy of a particle of mass 'm' placed on the surface of earth of mass 'M' and radius 'R' is given by :

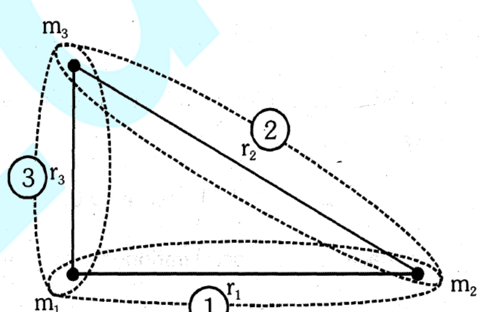


$$U = -\frac{GMm}{R}$$

4.2 Gravitational Potential Energy for three particle system

If there are more than two particles in a system, then the net gravitational potential energy of the whole system is the sum of gravitational potential energies of all the possible pairs in that system.

For eg. :



$$U_{system} = \left(-\frac{Gm_1m_2}{r_1}\right) + \left(-\frac{Gm_2m_3}{r_2}\right) + \left(-\frac{Gm_1m_3}{r_3}\right) \quad U_{system} = -\frac{Gm_1m_2}{r_1} - \frac{Gm_2m_3}{r_2} - \frac{Gm_1m_3}{r_3}$$

4.3 To find the change in potential energy of body or work done to raise a particle of mass 'm' to 'h' height above the surface of earth.

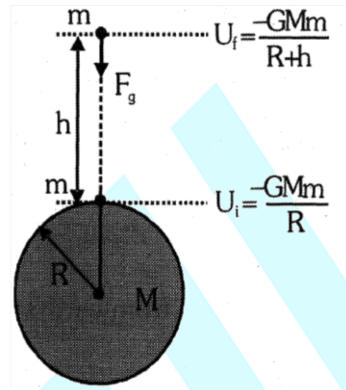
$$W = DU = U_f - U_i$$

$$\Rightarrow \quad W = \frac{GMm}{R+h} - \left(-\frac{GMm}{R}\right)$$

$$\Rightarrow \quad W = GMm\left(\frac{1}{R} - \frac{1}{R+h}\right) \Rightarrow W = GMm\left(\frac{R+h-R}{R(R+h)}\right)$$

$$\Rightarrow \quad W = gR^2m\left(\frac{h}{R^2\left(1+\frac{h}{R}\right)}\right) \qquad [\Theta\ GM = gR^2]$$

$$W = \frac{mgh}{\left(1+\frac{h}{R}\right)}$$



Special cases:

(i) If $h << R$, then $\frac{h}{R}$; 0 $\therefore W \propto \frac{mgh}{1+0} = mgh$

(ii) If $h = R$, then $W = \frac{mgh}{\left(1+\frac{R}{R}\right)} = \frac{mgR}{2}$.

4.4 The velocity required to project a particle to a height 'h' from the surface of earth.

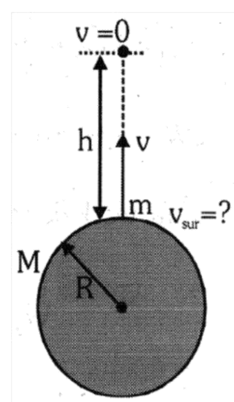
Applying 'COME' on the surface and at a height 'h'.

$$(K.E. + U)_{surface} = (K.E. + U)_{final}$$

$$\Rightarrow \quad \frac{1}{2}mv^2 - \frac{GMm}{R} = 0 - \frac{GMm}{R+h}$$

$$\Rightarrow \quad \frac{1}{2}mv^2 - \frac{GMm}{R} = -\left[\frac{GMm}{R+h}\right] \Rightarrow \frac{1}{2}mv^2 = \frac{mgh}{1+\frac{h}{R}}$$

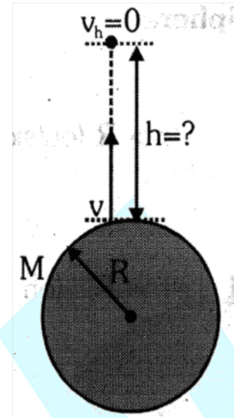
$$\Rightarrow \quad v^2 = \frac{2gh}{1+\frac{h}{R}} \qquad \Rightarrow \qquad v = \sqrt{\frac{2gh}{1+\frac{h}{R}}}$$



Note : If a body is released from a height 'h' above the surface of earth, then its velocity on reaching the earth's surface is also given by :

$$v = \sqrt{\frac{2gh}{1+\frac{h}{R}}}$$

4.5 To find the maximum height attained by a body when it is projected with velocity 'v' from the surface of earth.



From $v^2 = \dfrac{2gh}{1+\dfrac{h}{R}}$

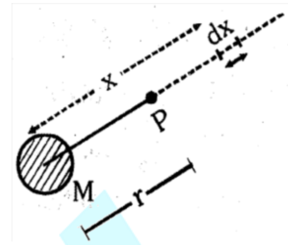
$$\Rightarrow \quad v^2 + \frac{v^2h}{R} = 2gh$$

$$\Rightarrow \quad v^2 = 2gh - \frac{v^2h}{R}$$

$$\Rightarrow \quad v^2 = h\left(2g - \frac{v^2}{R}\right) \quad \Rightarrow \quad h = \frac{v^2}{2g - \dfrac{v^2}{R}} = \frac{v^2R}{2gR - v^2}$$

$$h = \frac{v^2R}{2gR - v^2}$$

5. GRAVITATIONAL POTENTIAL



Gravitational field around a material body can be described not only by gravitational intensity $\vec{I}_g$, but also by a scalar function, the gravitational potential V. Gravitational potential is the amount of work done by external agent in bringing a body of unit mass from infinity to that point without changing its kinetic energy. $V = \dfrac{W_{ext}}{m}$

Gravitational force on unit mass at (P) will be $= \dfrac{GM(1)}{x^2} = \dfrac{GM}{x^2}$

Work done by this force when the unit mass is displaced through the distance dx is

$dW_{ext} = Fdx = \dfrac{GM}{x^2}.\,dx$

Total work done in brining the body of unit mass from infinity to point (P) is

$$W_{ext} = \int_{\infty}^{r} \frac{GM}{x^2}dx = -\left(\frac{GM}{x}\right)_{\infty}^{r} = -\frac{GM}{r}.$$

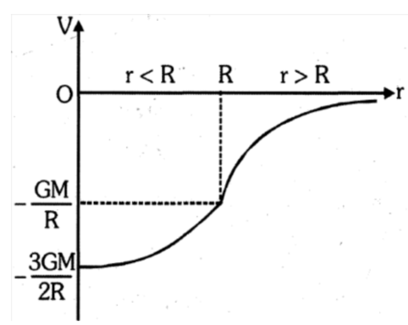
This work done is the measure of gravitational potential at point (P) $\therefore V_P = -\dfrac{GM}{r}$

- If $r = \infty$ then $V_\infty = 0$. Hence gravitational potential is maximum at infinity (as it is a negative quantity at point P)
- If $r = R_e$ (on the surface of Earth) $V_s = \dfrac{GM_e}{R_e}$
- **Relation between intensity and potential gradient**

  $V = -\int \vec{I}.d\vec{r} \Rightarrow dV = -\vec{I}.\,d\vec{r}$ $\therefore I = -\dfrac{dV}{dr}$ = –ve potential gradient.

- **Gravitational Potential due to solid sphere and spherical shell :**

Solid Sphere



Case I : r > R (outside the sphere) ; $V_{out} = -\dfrac{GM}{r}$

Case II : r = R (on the surface) ; $V_{surface} = -\dfrac{GM}{R}$

Case III : r < R (inside the sphere) ; $V_{in} = \dfrac{GM}{2R^3}[3R^2 - r^2]$

It is clear that the potential V will be minimum at the centre (r = 0) but maximum in magnitude.

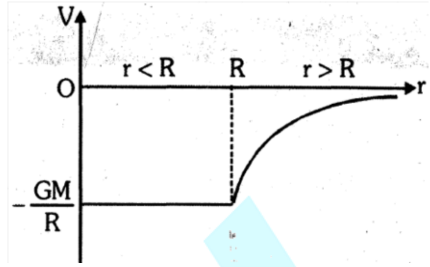
$$V_{centre} = -\frac{3}{2}\frac{GM}{R},\ V_{centre} = \frac{3}{2}V_{surface}$$

Spherical shell

Case I : r > R (outside the sphere) ; $V_{out} = -\frac{GM}{r}$

Case II : r = R (on the surface) ; $V_{surface} = -\frac{GM}{R}$

Case III : r < R (inside the sphere) ;



Potential is same every where and is equal to its value at the surface

$$V_{in} = -\frac{GM}{R}$$

6. ESCAPE VELOCITY & ESCAPE ENERGY

6.1 Escape Velocity (v_e)

It is the minimum velocity required for an object located at the planet's surface so that it just escapes the planet's gravitational field.

Consider a projectile of mass m, leaving the surface of a planet (or some other astronomical body or system), of radius R and mass M with escape speed v_e.

When the projectile just escapes to infinity, it has neither kinetic energy nor potential energy.

From conservation of mechanical energy $\frac{1}{2}mv_e^2 + \left(-\frac{GMm}{R}\right) = 0 + 0 \Rightarrow v_e = \sqrt{\frac{2GM}{R}}$

The escape velocity of a body from a location which is at height 'h' above the surface of planet, we can use :-

$$v_{es} = \sqrt{\frac{2GM}{r}} = \sqrt{\frac{2GM}{R+h}} \qquad \{\because r = R + h\}$$

Where, r = Distance from the centre of the planet.
h = Height above the surface of the planet.

Escape speed depends on :

(i) Mass (M) and radius (R) of the planet

(ii) Position from where the particle is projected.

Escape speed does not depend on :

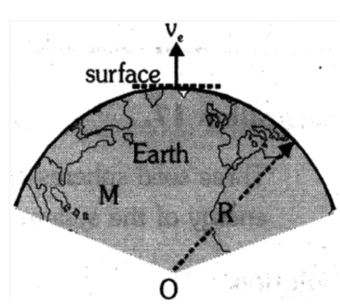
(i) Mass (m) of the body which is projected

(ii) Angle of projection.

If a body is thrown from the Earth's surface with escape speed, it goes out of earth's gravitational field and never returns back to the earth's surface.

6.2 Escape energy

Minimum energy given to a particle in the form of kinetic energy so that it can just escape the Earth's gravitational field.



Magnitude of escape energy = $\frac{GM}{R}$ (–ve of PE on the Earth's surface)

Escape energy = Kinetic Energy corresponding to the escape velocity $\Rightarrow \frac{GMm}{R} = \frac{1}{2}mv_e^2$

Note : In the above discussion it can be any. planet for that matter

GOLDEN KEY POINTS

- Gravitational potential energy or potential is a –ve quantity whose maximum value is zero at infinite separation.
- Relation between Force & Gravitational potential energy is

  $$F = \frac{dU}{dr}$$

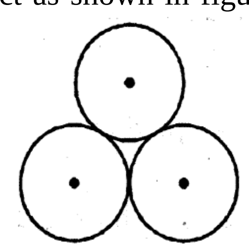
  Above relation is valid only for all conservative forces.
- $v_e = \sqrt{\frac{2GM}{R}}$ If M = constant then $v_e \propto \frac{1}{\sqrt{R}}$
- $v_e = \sqrt{2gR}$ If g = constant then $v_e \propto \sqrt{R}$
- $v_e = R\sqrt{\frac{8\pi G\rho}{3}}$ If ρ =constant then $v_e \propto R$
- Escape velocity does not depend on the mass of the body being projected, angle of projection or direction of projection.

  $V_e \propto= m^0$ and $v_e \propto \theta^0$
- Escape velocity at : Earth's surface v_e = 11.2 km/s, Moon surface v_e = 2.31 km/s.
- Atmosphere. on Moon is missing because root mean square velocity of gas particles is greater than escape velocity. i.e., $v_{rms} > v_e$
- Due to absence of atmosphere on moon, atmospheric pressure is zero. Hence, reading of a Barometer is also zero.
- If a hydrogen balloon is released from the surface of earth, then it moves upward because the upward buoyant force due to surrounding air exceeds its downwards weight. But if the balloon is released the surface of moon, then it will fall with g/6 acceleration under the influence of gravitational attraction of moon (up thrust is zero due to absence of atmosphere).
- If a bomb blast occurs on moon then its sound cannot be heard because sound is a mechanical wave which requires medium for propagation, which is absent there on moon.

Illustrations

Illustration 17.

Three solid spheres of mass M and radius R are placed in contact as shown in figure. Find the potential energy of the system ?



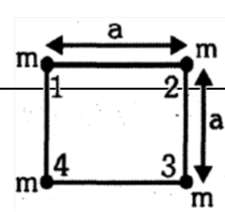
Solution:

$PE = PE_{12} + PE_{23} + PE_{31}$

$$= -\frac{GM^2}{2R} - \frac{GM^2}{2R} - \frac{GM^2}{2R} \Rightarrow PE = \frac{3GM^2}{2R}.$$

Illustration 18.

Four bodies each of mass m are placed at the different comers of a square of side a. Find the work done on the system to take any one body to infinity.



Solution:

Initial potential energy of the system

$PE_i = PE_{12} + P_{23} + PE_{34} + PE_{41} + PE_{13} + PE_{24}$

$$PE_i = -\frac{4GM^2}{a} - \frac{2GM^2}{a\sqrt{2}}$$

After taking any one body (say the mass placed at corner 4) to infinity only three bodies remain

$\therefore$ Final potential energy the system is

$$PE_i = P_{12} + PE_{13} + PE_{23} = -\frac{2GM^2}{a} - \frac{GM^2}{a\sqrt{2}}$$

$$W_{ext.} = PE_f - PE_i = \left(-\frac{2GM^2}{a} - \frac{GM^2}{a\sqrt{2}}\right) - \left(-\frac{4GM^2}{a} - \frac{2GM^2}{a\sqrt{2}}\right) = \frac{2GM^2}{a} + \frac{GM^2}{a\sqrt{2}}$$

Illustration 19.

A body of mass m is placed on the surface of earth. Find the work required to lift this body by a height

(i) $h = \frac{R_e}{1000}$ (ii) $h = R_e$ (M_e = mass of earth, R_e = radius of earth)

Solution:

(i) $h = \frac{R_e}{1000}$, as $h << R_e$, so

we can apply $W_{ext} = mgh$; $W_{ext} = (m)\left(\frac{GM_e}{R_e^2}\right)\left(\frac{R_e}{1000}\right) = \frac{GM_e m}{1000 R_e}$

(ii) $h = R_e$, in this case h is not very less than R_e, so we cannot apply $\Delta U = mgh$

$$W_{ext} = U_f - U_i = m(V_f - V_i)\ ;\ W_{ext} = m\left[\left(-\frac{GM_e}{R_e + R_e}\right) - \left(-\frac{GM_e}{R_e}\right)\right]\ ;\ W_{ext} = \frac{GM_e m}{2R_e}.$$

Illustration 20.

If velocity given to an object from the surface of the Earth is n times the escape velocity then what will be its residual velocity at infinity ?

Solution:

Let the residual velocity be v, then from energy conservation $\frac{1}{2}m(nv_e)^2 - \frac{GMm}{R} = \frac{1}{2}mv^2 + 0$

$$\Rightarrow v^2 = n^2v_e^2 - \frac{2GM}{R} = n^2v_e^2 - v_e^2 = (n^2 - 1)v_e^2 \Rightarrow v = \left(\sqrt{n^2 - 1}\right)v_e.$$

Illustration 21.

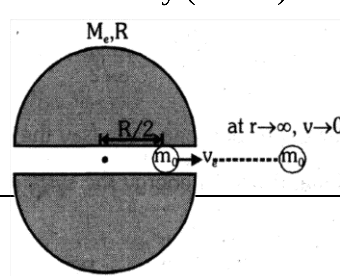
A narrow tunnel is dug along the diameter of the earth, and a particle of mass m_0 is placed at $\frac{R}{2}$ distance from the centre. Find the escape speed of the particle from that place.

Solution:

Suppose we project the particle with speed v_e, so that it just reaches infinity ($r \to \infty$).

Applying energy conservation principle

$$K_i + U_i = K_f + U_f$$



$$\frac{1}{2}m_0v_e^2 + m_0\left[-\frac{GM_e}{2R^3}\left\{3R^2 - \left(\frac{R}{2}\right)^2\right\}\right] = 0$$

$$\Rightarrow v_e = \sqrt{\frac{11GM_e}{4R}}.$$

Illustration 22.

The escape velocity for a planet is v_e. A particle starts from rest at a large distance from the planet, reaches the planet only under gravitational attraction, and passes through a smooth tunnel through its centre. Its speed at the centre of the planet will be :-

(A) $\sqrt{1.5}v_e$ (B) $\frac{v_e}{\sqrt{2}}$ (C) v_e (D) zero

Ans. (A)

Solution:

From mechanical energy conservation, $0 + 0 = \frac{1}{2}mv^2 - \frac{3GMm}{2R} \Rightarrow v = \sqrt{\frac{3GM}{R}} = \sqrt{1.5}\,v_e$.

Illustration 23.

A particle is projected vertically upwards from the surface of the earth (radius R_e) with a speed equal to one fourth of escape velocity. What is the maximum height attained by it ?

(A) $\frac{16}{15}R_e$ (B) $\frac{R_e}{15}$ (C) $\frac{4}{15}R_e$ (D) None of these

Ans. (B)

Solution:

From conservation of mechanical energy, $\frac{1}{2}mv^2 = \frac{GMm}{R_e} - \frac{GMm}{R}$

Where R = maximum distance from centre of the earth Also $v = \frac{1}{4}v_e = \frac{1}{4}\sqrt{\frac{2GM}{R_e}}$

$$\Rightarrow \frac{1}{2}m \times \frac{1}{16} \times \frac{2GM}{R_e} \times \frac{GMm}{R_e} - \frac{GMm}{R} \Rightarrow R = \frac{16}{15}R_e \Rightarrow h = R - R_e = \frac{R_e}{15}.$$

Illustration 24.

A mass of 6×10^{24} kg (= mass of earth) is to be compressed in a sphere in such a way that the escape velocity from its surface is 3×10^8 m/s (equal to the velocity of light). What should be the radius of the sphere?

(A) 9 mm (B) 8 mm (C) 7 mm (D) 6 mm

Ans. (B)

Solution:

As, $v_e = \sqrt{\left(\frac{2GM}{R}\right)}$, $R = \left(\frac{2GM}{v_e^2}\right)$, $\therefore R = \frac{2\times6.67\times10^{-11}\times6\times10^{24}}{(3\times10^8)^2} = 9 \times 10^{-3}$m = 9 mm.

Illustration 25.

Gravitational potential difference between a point on the surface of a planet and point 10 m above is 4 J/kg. Considering the gravitational field to be uniform, how much work is done in moving a mass of 2 kg from the surface to a point 5 m above the surface?

(A) 4 J (B) 5 J (C) 6 J (D) 7 J

Ans. **(A)**

Solution:

Gravitational field g = $\frac{\Delta V}{\Delta x} = -\left(\frac{-4}{10}\right) = \frac{4}{10}$ J / kg–m

Work done in moving a mass of 2 kg from the surface to a point 5 m above the surface.

W = mgh = (2 kg) $\left(\frac{4}{10}\frac{J}{kg-m}\right)$ (5 m) = 4 J

Illustration 26.

A body of mass m kg starts falling from a distance 2R above the earth's surface. What is its kinetic energy when it has fallen to a distance 'R' above the earth's surface ? (Where R is the radius of Earth)

Solution:

By conservation of mechanical energy,

$$-\frac{GMm}{3R} + 0 = -\frac{GMm}{2R} + \text{K.E.} \Rightarrow \text{K.E.} = \frac{GMm}{R}\left[\frac{1}{2}-\frac{1}{3}\right] = \frac{1}{6}\frac{GMm}{R} = \frac{1}{6}\frac{1(gR^2)m}{R} = \frac{1}{6}mgR.$$

Illustration 27.

With what velocity must a body be thrown from the earth's surface so that it may reach a height $4R_e$ above the Earth's surface ? (Radius of the Earth R_e = 6400 km, g = 9.8 m/s^2).

Solution:

By using conservation of mechanical energy $\frac{1}{2}m_0v^2 - \frac{GMm_0}{R_e} = 0 - \frac{GMm_0}{(R_e + 4R_e)}$

$$\frac{1}{2}m_0v^2 = -\frac{GMm_0}{5R_e} + \frac{GMm_0}{R_e} \Rightarrow m_0v^2 = \frac{4}{5}\frac{GMm_0}{R_e} \Rightarrow v^2 = \frac{8}{5}\frac{GM}{R_e} = \frac{8}{5}\frac{gR_e^2}{R_e}$$

$$v^2 = \frac{8}{5} \times 9.8 \times 6400 \times 10^3 = 10^8 \Rightarrow v = 10 \text{ km/s}.$$

BEGINNER'S BOX - 3

1. The gravitational acceleration on the surface of earth is g. Find the increase in potential energy in lifting an object of mass m to a height equal to the radius of earth.

2. In a certain region of space gravitational field is given by I = –(K/r) (Where r is the distance from a fixed point and K is constant). Taking the reference point to be at r = r_0 with V = V_0. Find the potential at a distance r.

3. Two masses of 10^2 kg and 10^3 kg are separated by 1 m distance. And the gravitational potential at the mid point of the line joining them.

4. The magnitude of intensity of gravitational field at a point situated at a distance 8000 km from the centre of Earth is 6.0 N-kg. The magnitude of gravitational potential at that point in N-m/kg will be :-

(A) 6 (B) 4.8 ×10^7 (C) 8 × 10^5 (D) 4.8 × 10^2

5. The gravitational field due to a certain mass distribution is $E = \frac{K}{x^3}$ in the x-direction (K is a constant). Taking the gravitational potential to be zero at infinity, its value corresponding to distance x is :-

6. Two bodies of respective masses m and M are placed d distance apart. The gravitational potential (V) at the position where the gravitational field due to them is zero is :-

(A) $V = -\frac{G}{d}(m+M)$ (B) $V = -\frac{G}{d}$

(C) $V = -\frac{GM}{d}$ (D) $V = -\frac{G}{d}(\sqrt{m}+\sqrt{M})^2$

7. A body of mass m is situated at a distance $4R_e$ above the Earth's surface, where R_e is the radius of Earth. What minimum energy should be given to the body so that it may escape ?

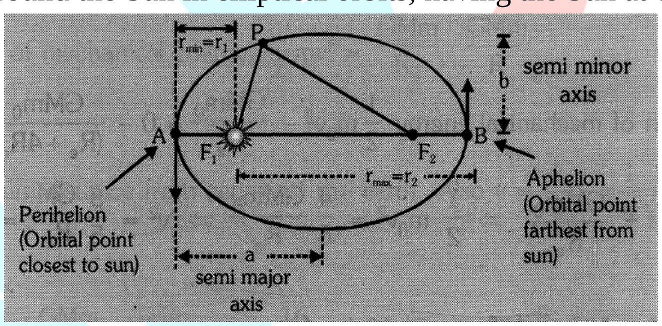
(A) mgR_e (B) $2mgR_e$ (C) $\frac{mgR_e}{5}$ (D) $\frac{mgR_e}{16}$

7. KEPLER'S LAWS OF PLANETARY MOTION

Kepler found important regularities in the motion of the planets. These regularities are known as 'Kepler's three laws of planetary motion'.

(a) First Law (Law of Orbits) :

All planets move around the Sun in elliptical orbits, having the Sun at one focus of the orbit.



When a particle moves with respect to two fixed points in such a way that the sum of the distances from these two points is always constant then the path of the particle is an ellipse and the two fixed points are called focal points.

<u>According to Figure</u> :-

$PF_1 + PF_2 = AF_1 + AF_2 = BF_1 + BF_2$ = constant

But in ellipse $AF_1 = BF_2$ (minimum distance from both focal is same)

$PF_1 + PF_2 = BF_2 + AF_2 = BF_1 + BF_1 = 2a$ = length of major axis

$r_1 + r_2 = r_{min} + r_{max} = 2a$

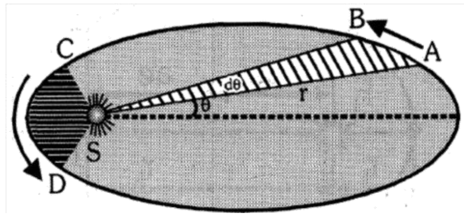
$\therefore \quad a = \frac{r_1 + r_2}{2} = \frac{r_{min} + r_{max}}{2}$ (Mean distance)

(b) Second Law (Law of Areas) :

A line joining any planet to the Sun sweeps out equal areas in equal intervals of time, i.e., the areal speed of the planet remains constant.

According to the second law, if a planet moves from A to B in a given time interval, and from C to D in the same time interval, then the areas ASB and CSD will be equal.

dA = area of the curved triangle SAB = $\frac{1}{2}$(AB × SA) = $\frac{1}{2}$(rdθ × r) = $\frac{1}{2}r^2\,d\theta$



Thus, the instantaneous areal speed of the planet is $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\omega = \frac{1}{2}rv$ ….(i)

where ω is the angular speed of the planet.

Let L be the angular momentum of the planet about the Sun S and m the mass of the planet.

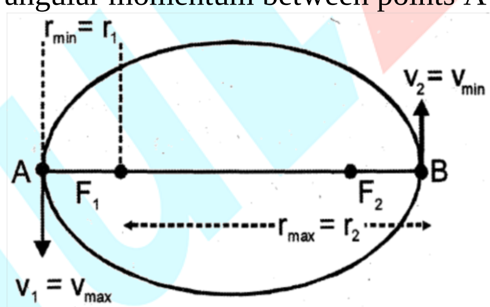
Then $L = I\omega = mr^2\omega = mvr$ ….(ii)

where I ($=mr^2$) is the instantaneous moment of inertia of the planet about the Sun S.

From eq. (i) and (ii), $\frac{dA}{dt} = \frac{L}{2m}$ …..(iii)

Now, the areal speed dA/dt of the planet is constant, according to Kepler's second law. Therefore, according to eq. (iii), the angular momentum L of the planet is also constant, that is, the angular momentum of the planet is conserved. Thus, Kepler's second law is equivalent to conservation of angular momentum.

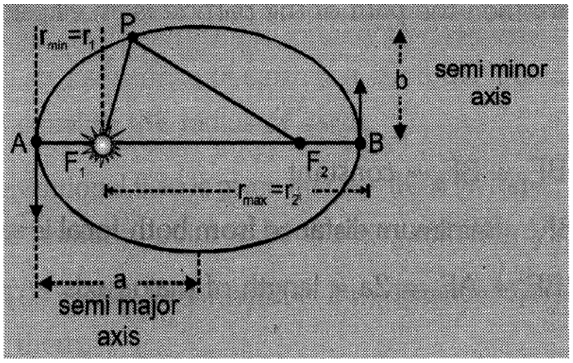
Applying conservation of angular momentum between points A and B



$L_A = L_B$ ⇒ $mv_{max}r_{min} = mv_{min}r_{max}$

⇒ $v_{max}\,r_{min} = v_{min}\,r_{max}$

A planet moves around the sun in an elliptical orbit of semi major axis a and eccentricity e



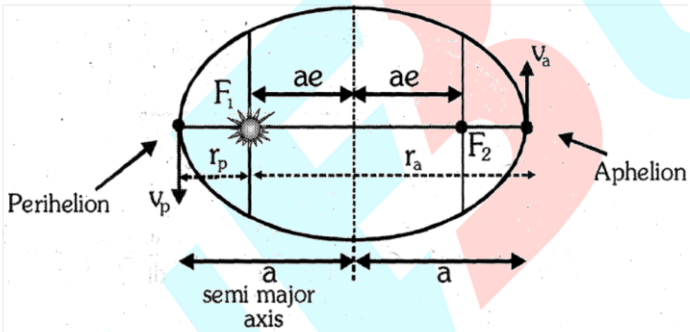
For an ellipse : its general equation is given as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If a > b then a is semi major axis, b is semi minor axis and e is eccentricity where

$$b^2 = a^2 (1 - e^2) \qquad \Rightarrow \qquad e = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

Applying the conservation of angular momentum (COAM) at the perihelion and aphelion $mv_pr_p = mv_ar_a$



$$r_{max.} = a(1 + e) \ ; \ r_{min.} = a(1 - e) \ \Rightarrow \ \frac{v_p}{v_a} = \frac{v_{max}}{v_{min}} = \frac{1+e}{1-e} \qquad \ldots\ldots(1)$$

By conservation of mechanical energy

$$\frac{1}{2}mv_p^2 - \frac{GMm}{r_p} = \frac{1}{2}mv_a^2 - \frac{GMm}{r_a} = -\frac{GMm}{2a} \qquad \ldots\ldots(2)$$

By solving eq^n (1) and (2),

$$\Rightarrow \qquad v_a = \sqrt{\frac{GM}{a}\left(\frac{1-e}{1+e}\right)} \ ; \ v_P = \sqrt{\frac{GM}{a}\left(\frac{1+e}{1-e}\right)}$$

(c) Third Law (Law of Periods) :

The square of the period of revolution of any planet around the Sun is directly proportional to the cube of the semi-major axis of its elliptical orbit.

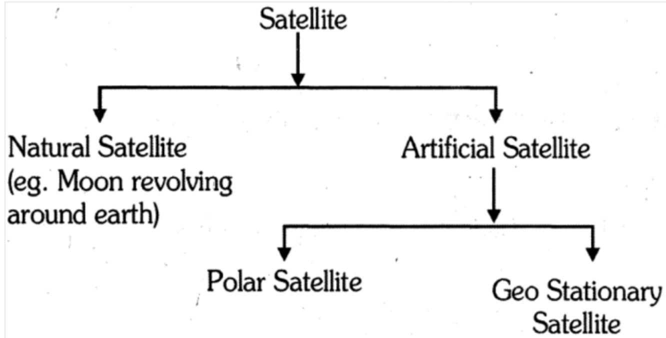
$$T^2 \propto a^3$$

Note : For a circular orbit semi major axis = Radius of the orbit

$$T^2 \propto R^3$$

8. SATELLITE MOTION

A light body revolving round a heavier planet due to gravitational attraction, is called a satellite. Moon is a natural satellite of Earth.



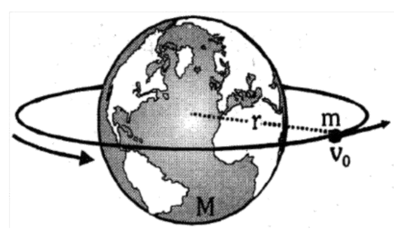
8.1 Essential Conditions for Satellite Motion

- The centre of satellite's orbit should coincide with the centre of Earth.
- Plane of the orbit of satellite should pass through the centre of Earth.

- It follows that a satellite can revolve round the earth only in those circular orbits whose centres coincide with the centre of earth. Circles drawn on globe with centres coincident with earth are known as 'great circles'. Therefore, a satellite revolves around the earth along circles concentric with great circles.

8.2 Orbital velocity (v_0)

A satellite of mass m moving in an orbit of radius r with speed v_0. The required centripetal force is provided by gravitation.



$$F_{cp} = F_g \Rightarrow \frac{mv_0^2}{r} = \frac{GMm}{r^2} \Rightarrow v_0 = \sqrt{\frac{GM}{r}} = \sqrt{\frac{GM}{(R_e + h)}}$$

$(r = R_e + h)$

For a satellite very close to the Earth's surface $h << R_e$ ∴ $\propto R_e$

$$\therefore \quad v_0 = \sqrt{\frac{GM}{R_e}} = \sqrt{gR_e}\ ; \quad 8 \text{ km/s}$$

- If a body is taken to some height (small) from Earth and given a horizontal velocity of magnitude 8 km/s then it becomes a satellite of Earth.
- V_0 depends upon : Mass of planet, Radius of the circular orbit of satellite.
- If orbital velocity of a satellite becomes $\sqrt{2}$ v_0 (or increased by 41.4%) or K.E. is doubled then it escapes from the gravitational field of Earth.

8.3 Time Period of a Satellite

$$T = \frac{2\pi r}{v_0} = \frac{2\pi r^{3/2}}{\sqrt{GM}} = \frac{2\pi r^{3/2}}{R\sqrt{g}} \Rightarrow T^2 = \frac{4\pi^2}{GM} r^3 \Rightarrow T^2 \; \mu \; r^3 (r = R + h)$$

For a satellite close to Earth's surface $\quad v_0 = \sqrt{\frac{GM_e}{r_e}} \propto 8$ km/s

$$T_0 = 2\pi\sqrt{\frac{R_e}{g}} = 84 \text{ minutes} = 1 \text{ hour } 24 \text{ minute} = 1.4 \text{ h} = 5063 \text{ s}$$

In terms of density $T_0 = \dfrac{2\pi(R_e)^{1/2}}{(G \times 4/3\pi R_e \times \rho)^{1/2}} = \sqrt{\dfrac{3\pi}{G\rho}}$.

Time period of a near by satellite only depends on the density of the planet.

For Moon $\quad h_m = 380{,}000$ km and $T_m = 27$ days

$$v_o = \frac{2\pi(R_e + h)}{T_m} = \frac{2\pi(386400 \times 10^3)}{27 \times 24 \times 60 \times 60} \propto 1.04 \text{ km/s.}$$

8.4 Energy of a satellite

Kinetic energy $\quad$ K.E. $= \dfrac{1}{2}mv_0^2 = \dfrac{GMm}{2r} = \dfrac{L^2}{2mr^2}$ $\;(L = mrv_0 = m\sqrt{GMr}\,)$

Potential energy $\quad$ P.E. $= \dfrac{GMm}{r} = -mv_0^2 = -\dfrac{L^2}{mr^2}$

Total mechanical energy $\quad$ T.E. = P.E. + K.E. $= -\dfrac{mv_0^2}{2} = -\dfrac{GMm}{2r} = -\dfrac{L^2}{2mr^2}$.

- **Binding energy :**

Total mechanical energy (potential + kinetic) of a closed system is negative. The modulus of this total mechanical energy is known as the binding energy of the system. This is the energy due to which system is bound or the different parts of the system are bonded to each other.

Binding energy of a satellite (system)

B.E. – T.E. $\quad$ B.E. $= \dfrac{1}{2}mv_0^2 = \dfrac{GMm}{2r} = \dfrac{L^2}{2mr^2}$. $\quad$ Hence B.E. = K.E. = –T.E. $= \dfrac{-\text{P.E.}}{2}$

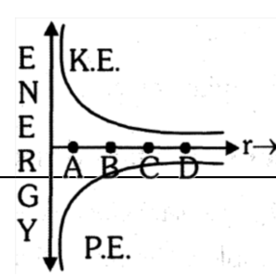
Escape energy and ionisation energy are the practical examples of binding energy.

Work done in Changing the Orbit of a Satellite

W = Change in mechanical energy of the system but $E = \dfrac{-GMm}{2r}$

so $\quad W = E_2 - E_1 = \dfrac{GMm}{2}\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right)$

In the given graph of energy v/s position



At point A :

Θ $\quad$ |(P.E.)| > K.E. $\quad$ ∴ $\quad$ LE + PE = –ve

At A, B & C System is bounded.

At point D :

Θ $|PE| = K.E.$ $\therefore$ $TE = KE + PE = 0$

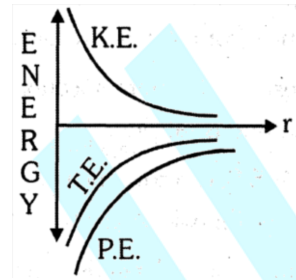
So, the system is unbounded.

Graphs :

$$KE = \frac{GMm}{2r}$$

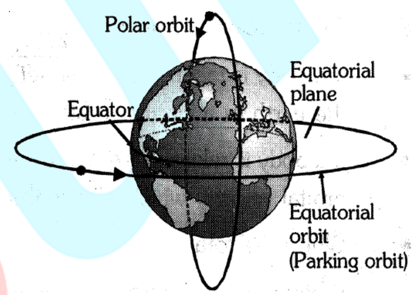
$$TE = -\frac{GMm}{2r}$$

$$PE = -\frac{GMm}{r}$$



9. GEO-STATIONARY SATELUTE & POLAR SATELUTE

Geo-Stationary Satellite



- It rotates in an equatorial plane.
- Its height from the Earth's surface is 36000 km. ($\approx 6R_e$)
- Its angular velocity and time period should be same as that of Earth.
- Its rotating sense should be same as that of Earth (West to East).
- Geo Stationary/Telecommunication/Parking/Synchronous/Satellite are always projected from equator (for example Singapore).
- Its orbit is called parking orbit and its orbital velocity is 3.1 km/s.

Polar Satellite

It is that satellite which revolves in the polar orbit around Earth. A polar orbit is one whose angle of inclination with the equatorial plane of Earth is 90° and a satellite in polar orbit will pass over both the north and south geographical poles once per revolution. Its time period is 190 min. and height is between 500 Km to 800 Km. Polar satellites are employed to obtain the cloud images, atmospheric data, information regarding ozone layer in the atmosphere and it detected the ozone hole over Antarctica etc.

10. WEIGHTLESSNESS

When the apparent weight of a body becomes zero, the body is said to be in a state of weightlessness. In a satellite around the earth, every part and parcel of the satellite has an acceleration towards the centre of the earth which is exactly the value of earth's acceleration due to gravity at that position. Thus in the satellite everything inside it is in a state of free fall. If a body is in a satellite (which does not produce its own gravity) orbiting the Earth at a height h above its surface then

True weight $= mg_h = \frac{mGM}{(R+h)^2} = \frac{mg}{\left(1+\frac{h}{R}\right)^2}$ and Apparent weight $= m(g_h - a)$

But a = $\frac{v_0^2}{r}\frac{GM}{r^2}=\frac{GM}{(R+h)^2}$ = g_h ⇒ Apparent weight = m(g_h – g_h) = 0.

Note : Condition of weightlessness can be overcome by creating artificial gravity by rotating the satellite in addition to its revolution.

GOLDEN KEY POINTS

- The time period of the longest pendulum on the surface of earth is given by T = $2\pi\sqrt{\frac{R}{g}}$ = 84.6 minutes .
  Note : The time period of a satellite orbiting close to the earth's surface is also 84.6 minutes.
- The angular velocity and time period of revolution of a G.S.S. is same as that of earth. It means that a G.S.S. completes its revolution around the earth once in 24 hours.
- Height of a G.S.S. (Geo-stationary satellite) from the surface of earth is about 36,000 km. Therefore, its distance from the centre of earth is about R+H = 36,000+6400 = 42,400 km $\propto$ 7R_e
- It is used as a communication satellite. It is also known as parking satellite, telecommunication satellite or synchronous satellite.
- One G.S.S. can cover nearly one-third surface area of earth. Therefore a minimum of three G.S.S. are required to cover the whole earth.
- Orbital velocity depends upon the mass of the central body and orbital radius (distance of satellite from the centre of the central body). If the distance of satellite increases, then the orbital velocity (v_0) decreases.
- Orbital velocity does not depend on the mass of satellite.
- If a body is taken to a small height and given a horizontal velocity of 8 km/s, it will start revolving around the earth in a circular orbit which means that it will become a satellite close to the earth's surface.
- If a body is released from a revolving satellite, then it will continue to move in the same orbit with the same orbital velocity which means that it will also become a satellite close to the earth.
- When the total energy of a satellite is negative, it will be moving in either a circular or an elliptical orbit.
- When the total energy of a satellite is zero, it will escape away from its orbit and its path becomes parabolic.
- If the gravitational force is inversely proportional to the n^{th} power of distance r, then the orbital velocity of a satellite $v_0 \propto r^{\frac{1-n}{2}}$ and time period $T \propto r^{\frac{n+1}{2}}$
- The total energy of any planet revolving around the sun is negative (Θ it is bounded) .
- First Satellite of Earth is Sputnik I. First Geo-satellite of India is Aryabhatt I.
  First Geo-stationary satellite of India is Apple I. Example of other Satellites of India are : Bhaskar-I, Rohini-I, Bhaskar-II (Geo Satellite); Insat-I (A), Insat-I(B) (Geo Stationary Satellite)

Illustrations

Illustration 28.

Two satellites S_1 and S_2 are revolving round a planet in coplanar and concentric circular orbits of radii R_1 and R_2 in the same sense respectively. Their respective periods of revolution are 1 h and 8 h. The radius of the orbit of satellite S_1 is equal to 10^4 km. Find the relative speed in km/h when they are closest.

Solution:

By Kepler's 3rd law, $\frac{T^2}{R^3}$ = constant $\therefore \frac{T_1^2}{R_1^3} = \frac{T_2^2}{R_2^3}$ or $\frac{1}{(10^4)^3} = \frac{64}{R_2^3}$ or $R_2 = 4 \times 10^4$ km

Distance travelled in one revolution, $S_1 = 3\pi R_1 = 2\pi \times 10^4$ and $S_2 = 2\pi R_2 = 2\pi \times 4 \times 10^4$

$v_1 = \frac{S_1}{t_1} = \frac{2\pi \times 10^4}{1} = 2\pi \times 10^4$ km/h and $v_2 = \frac{S_2}{t_2} = \frac{2\pi \times 4 \times 10^4}{8} = \pi \times 10^4$ km/h

$\therefore$ Relative velocity = $v_1 - v_2 = 2\pi \times 10^4 - \pi \times 10^4 = \pi \times 10^4$ km/h

Illustration 29.

A space-ship is launched into a circular orbit close to the Earth's surface. What additional speed should now be imparted to the spaceship so that it overcomes the gravitational pull of the Earth.

Solution:

Let ΔK be the additional kinetic energy imparted to the spaceship to overcome the gravitation pull then

ΔK = – (total energy of spaceship) = $\frac{GMm}{2R}$

Total kinetic energy $= \frac{GMm}{2R} + \Delta K = \frac{GMm}{2R} + \frac{GMm}{2R} = \frac{GMm}{R}$, then $\frac{1}{2}mv_2^2 = \frac{GMm}{R} \Rightarrow v_2 = \sqrt{\frac{2GM}{R}}$

But $v_1 = \sqrt{\frac{GM}{R}}$. So additional velocity required = $v_2 - v_1 = \sqrt{\frac{2GM}{R}} - \sqrt{\frac{GM}{R}} = \left(\sqrt{2}-1\right)\sqrt{\frac{GM}{R}}$

Alternate Solution:

Additional velocity = Escape velocity – Orbital velocity

$= v_{es} - v_0$

$= \sqrt{\frac{2GM}{R}} - \sqrt{\frac{GM}{R}}$

$= \left(\sqrt{2}-1\right)\sqrt{\frac{GM}{R}}$

Illustration 30.

An astronaut, inside an earth's satellite experiences weightlessness because:

(A) he is falling freely

(B) no external force is acting on him

(C) no reaction is exerted by the floor of the satellite

(D) he is far away from the earth's surface

Ans. (A, C)

Solution:

As astronaut's acceleration = g; so he is falling freely. Also no reaction is exerted by the floor of the satellite.

Illustration 31.

If a satellite orbits as close to the earth's surface as possible

(A) its speed is maximum

(B) time period of its revolution is minimum

(C) the total energy of the 'earth plus satellite' system is minimum

(D) the total energy of the 'earth plus satellite' system is maximum

Ans. **(A, B, C)**

Solution:

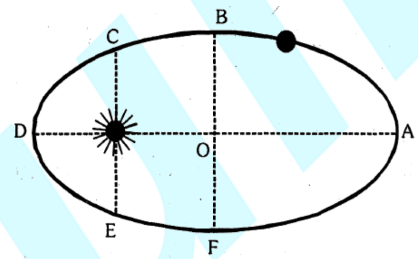
For (A) : orbital speed $v_0 = \sqrt{\frac{GM}{r}}$, $r_{min} = R$ so v_0 = miximum

For (B) : Time period of revolution $T^2 \propto r^3$

For (C/D) : Total energy = $\frac{GMm}{2r}$

Illustration 32.

A planet is revolving round the sun in an elliptical orbit as shown in figure. Select correct alternative(s)



(A) Its total energy is negative at D.

(B) Its angular momentum is constant

(C) Net torque on the planet about sun is zero

(D) Linear momentum of the planet is conserved

Ans. **(A, B, C)**

Solution:

For (A): For a bound system, the total energy is always negative.

For (B): For central force field, angular momentum is always conserved.

For (C): For central force field, torque = 0.

For (D): In presence of external force, linear momentum is not conserved.

Illustration 33.

An artificial satellite is moving in a circular orbit around the earth with a speed equal to half the magnitude of escape velocity from the earth.

(a) Determine the height of the satellite above the earth's surface.

(b) If the satellite is stopped suddenly in its orbit and allowed to fall freely on the earth, find the speed with which it hits the surface of earth. Given M = mass of earth and R = Radius of earth

Solution:

(a) Let height above the earth's surface = h then

$$v_{orbital} = \sqrt{\frac{GM}{R+h}} = \frac{1}{2}v_e = \frac{1}{2}\sqrt{\frac{2GM}{R}} = \sqrt{\frac{GM}{2R}} \Rightarrow R + h = 2R \Rightarrow h = R$$

(b) If the satellite is stopped suddenly then it total energy $E_1 = -\frac{GMm}{2R}$

Let its speed be v when it hits the earth's surface then its total energy on earth surface

$$E_2 = \frac{GMm}{R} + \frac{1}{2}mv^2$$

conservation law for mechanical energy yields $E_1 = E_2 \Rightarrow \frac{-GMm}{2R} = -\frac{GMm}{R} + \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{GM}{R}}$

Illustration 34.

Is it possible to place an artificial satellite in an orbit such that it is always visible over Kota ? Write down the reason.

Solution:

No, Kota is not in the equatorial plane.

BEGINNER'S BOX - 4

1. The mean radius of the earth's orbit around the sun is 1.5×10^{11} m. The mean radius of the orbit of mercury around the sun is 6×10^{10} m. Calculate the year of the mercury.

2. If earth describes an orbit round the sun of double its present radius, what will be the year on earth ?

3. If the gravitational force were to vary inversely as m^{th} power of the distance, then the time period of a planet in circular orbit of radius r around the Sun will be proportional to :-

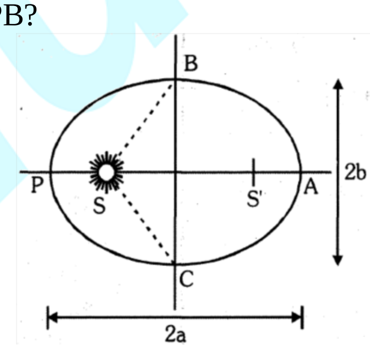
(A) $r^{-3m/2}$ (B) $r^{3m/2}$ (3) $r^{m+1/2}$ (4) $r^{(m+1)/2}$

4. A planet is revolving around the Sun in an elliptical orbit. Its closest distance from the Sun is r_{min}. The farthest distance from the Sun is r_{max}. If the orbital, angular velocity of the planet when it is nearest to the Sun is ω, then the orbital angular velocity at the point when it is at the farthest distance from the Sun is :-

(A) $\left(\sqrt{\frac{r_{min}}{r_{max}}}\right)\omega$ (B) $\left(\sqrt{\frac{r_{max}}{r_{min}}}\right)\omega$ (3) $\left(\sqrt{\frac{r_{max}}{r_{min}}}\right)^2\omega$ (4) $\left(\sqrt{\frac{r_{min}}{r_{max}}}\right)^2\omega$

5. The time period of revolution of moon around the earth is 28 days and radius of its orbit is 4×10^5 km. If $G = 6.67 \times 10^{-11}$ Nrn^2/kg^2 then find the mass of the earth.

6. Let the speed of the planet at the perihelion P in Fig. be v_P and the Sun-planet distance SP be r_P. Relate (r_P, v_P) to the corresponding quantities at the aphelion (r_A, v_A). Will the planet take equal times to traverse BAC and CPB?



7. A comet orbits the sun in a highly elliptical orbit. Does the cornet have a constant

(a) linear speed, (b) angular speed, (c) angular momentum, (d) kinetic energy, (e) potential energy (f) total energy throughout its orbit ? Neglect any mass loss of the cornet when it comes very close to the Sun.

8. A satellite moves in a circular orbit around the earth. The radius of this orbit is one half that of the moon's orbit. Find the time in which the satellite completes one revolution.

9. A small satellite revolves round a planet in an orbit just above planet's surface. Taking the mean density of planet as ρ, calculate the time period of the satellite.

10. Two satellites A and B, having ratio of masses 3 : 1 are in circular orbits of radius rand 4r. Calculate the ratio of total mechanical energies of A to B.

11. A satellite orbits the Earth at a height of 400 km above the surface. How much energy must be expended to rocket the satellite out of the Earth's gravitational influence ? Mass of the satellite = 200 kg; mass of Earth = 6.0×10^{24} kg; radius of the Earth = 6.4×10^{6} m; $G = 6.67 \times 10^{-11}$ N-m^2/kg^2.

12. An artificial satellite orbiting the earth in very thin atmosphere loses its energy gradually due to small but continuous dissipation against atmospheric resistance. Then explain why its speed increases progressively as it comes closer and closer to the earth.

13. Write the answer of the following questions in one word-

(a) What is the orbital speed of Geo-stationary satellite ?

(b) For a satellite moving in an orbit around the earth what is the ratio of kinetic energy to potential energy?

14. An object weighs 10 N at the north pole of the Earth. In a geostationary satellite distant 7R from the centre of the Earth (of radius R), the true weight and the apparent weight are respectively :-

(A) 0, 0 (B) 0.2 N, 0 (C) 0.2 N, 9.8 N (D) 0.2 N, 0.2 N

ANSWERS

BEGINNER'S BOX - 1

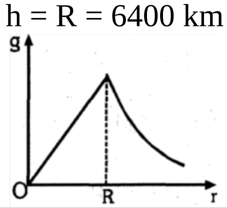
1. Zero **2.** $\sqrt{3}\dfrac{Gm^2}{a^2}$ **3.** (C)

BEGINNER'S BOX - 2

1. A, B **2.** (A) **3.** (A) **4.** (A) **5.** 32 km

6. 0.5% **7.** 4.44 m/s^2 **8.** h = R = 6400 km **9.** 50 km

10. 1920 km **11.** $d = \dfrac{3}{4}R$ **12.**



BEGINNER'S BOX - 3

1. $\dfrac{1}{2}$mgR **2.** $v = v_0 + K \log\left(\dfrac{r}{r_0}\right)$ **3.** $-2200 \times 6.67 \times 10^{-11}$ J/kg

4. (B) **5.** (D) **6.** (D) **7.** (C)

BEGINNER'S BOX - 4

1. $\left(\dfrac{2}{5}\right)^{3/2}$ years **2.** $2\sqrt{2}$ years **3.** (D) **4.** (D)

5. 6.47×10^{24} kg

6. Θ angular momentum $L = m_P r_P v_P = m_P r_A v_A$

$\therefore \quad \frac{v_P}{v_A} = \frac{r_A}{r_P}$ Since $r_A > r_P$, $v_P > v_A$.

The area SBAC bounded by the ellipse and the radius vectors SB and SC is larger than SBPC in Fig. from Kepler's second law, equal areas are swept in equal time intervals. Hence the planet will take a longer time to traverse BAC than CPB.

7. All quantities vary over an orbit except angular momentum and total energy.

8. 9.7 days **9.** $\sqrt{\frac{3\pi}{G\rho}}$ **10.** $\frac{12}{1}$ **11.** 5.89×10^9 J

12. Kinetic energy increases, but potential energy decreases and the sum decreases due to dissipation against friction.

13. (a) 3.1 km/s; (b) $-\frac{1}{2}$ **14.** (B)