Ellipse

1. Definition

An ellipse is the locus of a point which moves in such a way that its distance form a fixed point is in constant ratio to its distance from a fixed line. The fixed point is called the **focus** and fixed line is called the **directrix** and the constant ratio is called the **eccentricity of a ellipse** denoted by (e).

In other word, we can say an ellipse is the locus of a point which moves in a plane so that the sum of it distances from fixed points is constant.

2. Equation of an Ellipse

2.1 Standard Form of the equation of ellipse

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} \quad (a > b)$$

Let the distance between two fixed points S and S' be 2ae and let C be the mid point of SS′.

Taking CS as x- axis, C as origin.

Let P(h, k) be the moving point Let SP+ SP′ = 2a (fixed distance) then

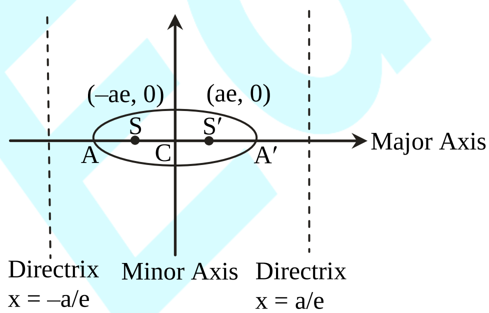
$SP+S'P = \sqrt{\{(h-ae)^2+k^2\}}+\sqrt{\{(h+ae)^2+k^2\}} = 2a$

$h^2(1-e^2) + k^2 = a^2(1-e^2)$

Hence Locus of P(h, k) is given by.

$x^2(1-e^2) + y^2 = a^2(1-e^2)$

$\Rightarrow \dfrac{x^2}{a^2}+\dfrac{y^2}{a^2(1-e^2)} = 1$



Let us assume that $a^2(1-e^2) = b^2$

∴ The standard equation will be given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

2.1.1 Various parameter related with standard ellipse :

Let the equation of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (a > b)

(i) **Vertices of an ellipse :**

The points of the ellipse where it meets with the line joining its two foci are called its vertices.

For above standard ellipse A. A′ are vertices

$A \equiv (a, 0), A' \equiv (-a, 0)$

(ii) **Major axis :**

The chord AA′ joining two vertices of the ellipse is called its major axis.

Equation of major axis : y = 0

Length of major axis = 2a

(iii) **Minor axis :**

The chord BB′ which bisects major axis AA′ perpendicularly is called minor axis of the ellipse.

Equation of minor axis x = 0

Length of minor axis = 2b

(iv) **Centre :**

The point of intersection of major axis and minor axis of an ellipse is called its centre.

For above standard ellipse

centre = C(0, 0)

(v) **Directrix :**

Equation of directrices are x= a/e and x = – a/e.

(vi) **Focus :** S (ae, 0) and S′ (– ae, 0) are two foci of an ellipse.

(vii) **Latus Rectum :** Such chord which passes through either focus and perpendicular to the major axis is called its latus rectum.

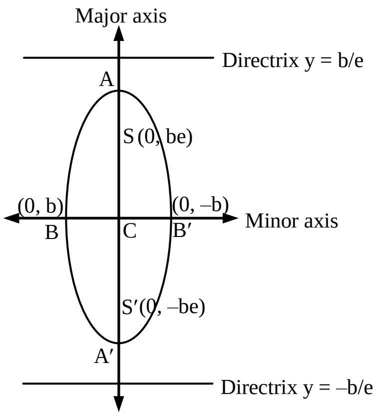
(viii) **Length of Latus Rectum :**

Length of Latus rectum is given by $\dfrac{2b^2}{a}$.

(ix) **Relation between constant a, b, and e**

$b^2 = a^2(1-e^2) \Rightarrow e = \sqrt{1-\dfrac{b^2}{a^2}}$

3. Second form of Ellipse



$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$ when $a < b$.

For this ellipse

(i) centre : (0, 0)

(ii) vertices : (0, b) ; (0, – b)

(iii) foci : (0, be) ; (0, – be)

(iv) major axis : equation x = 0, length = 2b

(v) minor axis : equation y = 0, length = 2a

(vi) directrices : y = b/e, y = – b/e

(vii) length of latus ractum = $2a^2/b$

(viii) eccentricity : $e = \sqrt{1-\frac{a^2}{b^2}}$

4. General equation of the ellipse

The general equation of an ellipse whose focus is (h,k) and the directrix is the line ax + by + c = 0 and the eccentricity will be e. Then let $P(x_1,y_1)$ be any point on the ellipse which moves such that SP = ePM

$$\Rightarrow (x_1-h)^2+(y_1-k)^2=\frac{e^2(ax_1+by_1+c)^2}{a^2+b^2}$$

Hence the locus of (x_1,y_1) will be given by

$(a^2 + b^2) [(x – h)^2 + (y – k)^2] = e^2(ax + by + c)^2$

Which is the equation of second degree from which we can say that any equation of second degree represent equation of an ellipse.

Note : Condition for second degree in X & Y to represent an ellipse is that if $h^2 = ab < 0$ & $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

5. Parametric forms of the Ellipse

Let the equation of ellipse in standard form will be given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Then the equation of ellipse in the parametric form will be given by $x = a\cos\phi$, $y = b\sin\phi$ where ϕ is the eccentric angle whose value vary from $0 \le \phi < 2\pi$. Therefore coordinate of any point P on the ellipse will be given by $(a\cos\phi, b\sin\phi)$.

6. Point and Ellipse

Let $P(x_1, y_1)$ be any point and let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the equation of an ellipse.

The point lies outside, on or inside the ellipse as if

$$S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0,\ =0,\ <0$$

7. Ellipse and a Line

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the given line be y = mx + c.

Solving the line and ellipse we get

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

i.e. $(a^2m^2 + b^2)x^2 + 2mca^2x + a^2(c^2 - b^2) = 0$

above equation being a quadratic in x.

$\therefore$ discriminant $= 4m^2c^2a^4 - 4a^2(a^2m^2 + b^2)(c^2 - b^2)$

$= b^2\{(a^2m^2 + b^2) - c^2\}$

Hence the line intersects the ellipse in

(i) two distinct points if $a^2m^2 + b^2 > c^2$

(ii) in one point if $c^2 = a^2m^2 + b^2$

(iii) does not intersect if $a^2m^2 + b^2 < c^2$

$\therefore\ y = mx \pm\sqrt{(a^2m^2+b^2)}$ touches the ellipse and condition for tangency $c^2 = a^2m^2 + b^2$.

Hence $y = mx \pm\sqrt{(a^2m^2+b^2)}$, touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $\left(\frac{\pm a^2m}{\sqrt{a^2m^2+b^2}},\frac{\pm b^2}{\sqrt{a^2m^2+b^2}}\right)$.

8. Equation of the Tangent

(i) The equation of the tangent at any point (x_1, y_1) on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

(ii) The equation of tangent at any point 'ϕ' is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1.$$