Complex Number

1. The Real Number System

Natural Number (N) : The number which are used for counting are known as Natural Number (also known as set of Positive Integers) i.e.

N = {1, 2, 3, ...}

Whole Number (W) : If ' 0 ' is included in the set of natural numbers then we get the set of Whole Numbers i.e.W = {0, 1, 2, ...}

= {N} + {0}

Integers (Z or I) :If negative natural number is included in the set of whole number then we get set of Integers i.e.

Z or I = {............–3, –2, –1, 0, 1, 2, 3,............}

Rational Numbers (Q) : The numbers which are in the form of p/q (Where p, q ∈ I, q ≠ 0) are called as Rational Number e.g. $\sqrt{2}$, $\frac{2}{3}$, 3, $\frac{1}{3}$, 0.76, 1.2322 etc.

Irrational Numbers : The numbers which are not rational i.e. which can not be expressed in p/q form or whose decimal part is non terminating non repeating but which may represent magnitude of physical quantities. e.g., $5^{1/3}$, π, e,.....etc.

Real Numbers (R) : The set of Rational and Irrational Number is called as set of Real Numbers i.e. $N \subset W \subset Z \subset Q \subset R$

Note :

(i) Number zero is neither positive nor negative but is an even number.

(ii) Square of a real number is always positive.

(iii) Between two real numbers there lie infinite real numbers.

(iv) The real number system is totally ordered, for any two numbers a, b ∈R , we must say, either a < b or b < a or b = a.

(v) All real number can be represented by points on a straight line. This line is called as number line.

(vi) An Integer (Note) is said to be even, if it is divided by 2 other wise it is odd number.

(vii) The magnitude of a physical quantity may be expressed as a real number times, a standard unit.

(viii) Number ' 0 ' is an additive quantity

(ix) Number '1' is multiplicative quantity.

(x) Infinity (∞) is the concept of the number greater than greatest you can imagine. It is not a number, it is just a concept, so we do not associate equality with it.

(xi) Division by zero is meaning less.

(xii) A non zero integer p is called prime if p ≠ ± 1 and its only divisors are ± 1 and ± p.

1.1 Modulus of a Real Number :

The Modulus of a real number x is defined as follows

$|x| = x$ when $x > 0$

0 when $x = 0$

$-x$ when $x < 0$

e.g. $|3| = 3$ $|-6| = -(-6) = 6$

Now $|x-a| = \begin{cases} x-a & \text{when } x \ge a \\ -(x-a) & \text{when } x < a \end{cases}$

1.2 Intervals : Let a, x, b are real number so that

$x \in [a, b] \Rightarrow a \le x \le b$

[a,b] is known as the closed interval a, b

$x \in (a, b) \Rightarrow a < x < b$

(a, b) is known as the open interval a, b

$x \in (a, b] \Rightarrow a < x \le b$

(a, b] is known as semi open, semi closed Interval

$x \in [a, b) \Rightarrow a \le x < b$

[a, b) is known as semi closed, semi open Interval

2. Imaginary Number

Square root of a negative real number is an imaginary number, while solving equation $x^2 + 1 = 0$ we get $x = \pm\sqrt{-1}$ which is imaginary. So the quantity $\sqrt{-1}$ is denoted by 'i' called 'iota' thus $i = \sqrt{-1}$

Further $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$ ....may be expressed as $\pm i\sqrt{2}$, $\pm i\sqrt{3}$, $\pm 2i$ ....

2.1 Integral powers of iota

As we have seen $i = \sqrt{-1}$ so $i^2 = -1$

$i^3 = -i$ and $i^4 = 1$

Hence $n \in N$, $i^n = i, -1, -i, 1$ attains four values according to the value of n, so

$i^{4n+1} = i$, $i^{4n+2} = -1$

$i^{4n+3} = -i$, i^{4n} or $i^{4n+4} = 1$

In other words $i^n = (-1)^{n/2}$ if n is even integer

$i^n = (-1)^{\frac{n-1}{2}}\, i$ if n is odd integer.

Note :–

(i) $i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{1}$

(ii) $\sqrt{a.b} = \sqrt{a}\,.\sqrt{b}$ possible iff both a, b are non-negative. (incorrect). It is also true for one positive and one negative no.

e.g. $\sqrt{(-2)(3)} = \sqrt{-2}\,.\sqrt{3}$

only invalid when both are negative means

$\sqrt{a.b} \neq \sqrt{a}\,.\sqrt{b}$ iff a & b both are negative.

(iii) ' i ' is neither positive, zero nor negative, Due to this reason order relations are not defined for imaginary numbers.

3. Complex Number

A number of the form $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$ is called a complex number where x is called as real part and y is called imaginary part of complex number and they are expressed as

$Re(z) = x$, $Im(z) = y$

Here if $x = 0$ the complex number is purely Imaginary and if $y = 0$ the complex number is purely Real.

A complex number may also be defined as an ordered pair of real numbers any may be denoted by the symbol (a, b). If we write z = (a, b) then a is called the real part and b the imaginary part of the complex number z.

Note :

(i) Inequalities in complex number are not defined because 'i' is neither positive, zero nor negative so $4 + 3i < 1 + 2i$ or $i < 0$

or $i > 0$ is meaning less.

(ii) If two complex numbers are equal, then their real and imaginary parts are separately equal. Thus if $a + ib = c + id$

$\Rightarrow a = c$ and $b = d$

so if $z = 0 \Rightarrow x + iy = 0 \Rightarrow x = 0$ and $y = 0$

The student must note that

$x, y \in R$ and $x, y \neq 0$. Then if

$x + y = 0 \Rightarrow x = y$ is correct

but $x + iy = 0 \Rightarrow x = -iy$ is incorrect

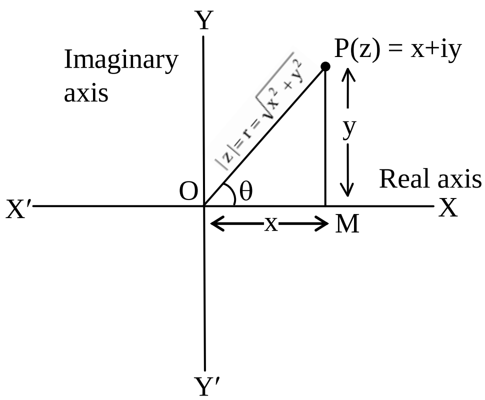
Hence a real number cannot be equal to the imaginary number, unless both are zero.

(iii) The complex number 0 is purely real and purely imaginary both.

3.1 Representation of a Complex Number :

(a) Cartesian Representation :

The complex number $z = x + iy = (x, y)$ is represented by a point P whose coordinates are refered to rectangular axis xox´ and yoy´, which are called real and imaginary axes respectively. Thus a complex number z is represented by a point in a plane, and corresponding to every point in this plane there exists a complex number such a plane is called Argand plane or Argand diagram or complex plane or gussian plane.



Note :

(i) Distance of any complex number from the origin is called the modulus of complex number and is denoted by |z|. Thus, $|z| = \sqrt{x^2 + y^2}$.

(ii) Angle of any complex number with positive direction of x-axis is called amplitude or argument of z. Thus, amp (z) = arg (z) = θ $= \tan^{-1}\frac{y}{x}$.

(b) Polar Representation : If z = x + iy is a complex number then z = r (cos θ + i sin θ) is a polar form of complex number z where x = r cos θ, y = r sin θ and $r = \sqrt{x^2 + y^2} = |z|$.

(c) Exponential Form : If z = x + iy is a complex number then its exponential form is $z = r e^{i\theta}$ where r is modulas and θ is amplitude of complex number.

(d) Vector Representation : If z = x + iy is a complex number such that it represent point P(x, y) then its vector representation is $z = \overrightarrow{OP}$

3.2 Algebraic operations with Complex Number:

Addition (a + ib) + (c + id) = (a + c) + i (b + d)

Subtraction (a + ib)–(c + id) = (a – c) + i (b – d)

Multiplication $(a + ib)(c + id) = ac + iad + ibc + i^2bd$
$= (ac – bd) + i(ad + bc)$

Division $\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$

(when at least one of c and d is non zero)

$$= \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$$

3.2.1 Properties of Algebraic operations with Complex Number

Let z, z_1, z_2 and z_3 are any complex number then their algebraic operation satisfy following properties-

Commutativity : $z_1 + z_2 = z_2 + z_1$ & $z_1 z_2 = z_2 z_1$

Associativity : $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

and $(z_1 z_2) z_3 = z_1(z_2 z_3)$

Identity element : If O = (0, 0) and 1 = (1, 0) then z + 0 = 0 + z = z and z.1 = 1. z = z. Thus 0 and 1 are the identity elements for addition and multiplication respectively.

Inverse element : Additive inverse of z is – z and multiplicative inverse of z is $\frac{1}{z}$.

Cancellation Law :

$$\left.\begin{aligned} z_1 + z_2 &= z_1 + z_3 \\ z_2 + z_1 &= z_3 + z_1 \end{aligned}\right\} \Rightarrow z_2 = z_3$$

and $z_1 \neq 0$ $$\left.\begin{aligned} z_1 z_2 &= z_1 z_3 \\ z_2 z_1 &= z_3 z_1 \end{aligned}\right\} \Rightarrow z_2 = z_3$$

Distributivity : $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

and $(z_2 + z_3) z_1 = z_2 z_1 + z_3 z_1$

3.3 Conjugate Complex Number :

The complex numbers z = (a, b) = a + ib and $\bar{z}$ = (a, – b) = a – ib where b ≠ 0 are said to be complex conjugate of each other (Here the complex conjugate is obtained by just changing the sign of i) e.g.conjugate of z = – 3 + 4i is $\bar{z} = -3 - 4i$.

Note : Image of any complex number in x-axis is called its conjugate.

3.3.1 Properties of Conjugate Complex Number

Let $z = a + ib$ and $\bar{z} = a - ib$ then

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2a = 2\ \text{Re}(z) =$ purely real

(iii) $z - \bar{z} = 2ib = 2i\ \text{Im}(z) =$ purely imaginary

(iv) $z\bar{z} = a^2 + b^2 = |z|^2$

(v) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(vi) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(vii) $\overline{re^{i\theta}} = re^{-i\theta}$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(ix) $\overline{z^n} = (\bar{z})^n$

(x) $\overline{z_1 z_2} = \bar{z}_1\ \bar{z}_2$

(xi) $z + \bar{z} = 0$ or $z = -\bar{z}$

$\Rightarrow$ z = 0 or z is purely imaginary

(xii) $z = \bar{z} \Rightarrow$ z is purely real

4. Modulus of a Complex Number

If $z = x + iy$ then modulus of z is equal to $\sqrt{x^2 + y^2}$ and it is denoted by $|z|$. Thus

$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}$

Note :

Modulus of every complex number is a non negative real number.

4.1 Properties of modulus of a Complex Number

(i) $|z| \geq 0$

(ii) $-|z| \leq \text{Re}(z) \leq |z|$

(iii) $-|z| \leq \text{Im}(z) \leq |z|$

(iv) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(v) $z\bar{z} = |z|^2$

(vi) $|z_1 z_2| = |z_1|\ |z_2|$

(vii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} (z_2 \neq 0)$

(viii) $|z|^n = |z^n|, n \in N$

(ix) $|z| = 1 \Leftrightarrow \bar{z} = \frac{1}{z}$

(x) $z^{-1} = \frac{\bar{z}}{|z|^2}$

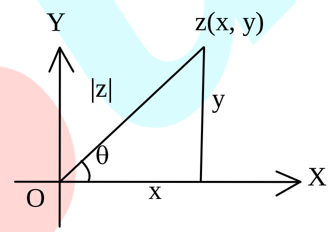
(xi) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1\bar{z}_2)$

(xii) $|z_1+z_2|^2 + |z_1-z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

(xiii) $|re^{i\theta}| = r$

5. Amplitude or Argument of a Complex Number

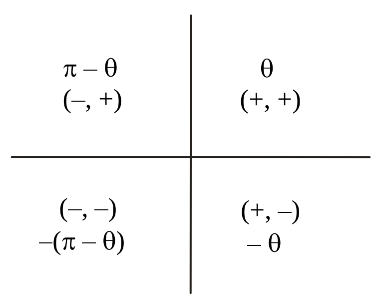
The amplitude or argument of a complex number z is the inclination of the directed line segment representing z, with real axis.



If $z = x + iy$ then

$\text{amp}(z) = \tan^{-1}\left(\frac{y}{x}\right)$

For finding the argument of any complex number first check that the complex number is in which quadrant and then find the angle θ and amplitude using the adjacent figure.



Note :

(i) Principle value of any complex number lies between $-\pi < \theta \leq \pi$.

(ii) Amplitude of a complex number is a many valued function. If θ is the argument of a complex number then $(2n\pi+\theta)$ is also argument of complex number.

(iii) Argument of zero is not defined.

(iv) If a complex number is multiplied by iota (i) its amplitude will be increased by $\pi/2$ and will be decreased by $\pi/2$, if is multiplied by –i.

(v) Amplitude of complex number in I and II quadrant is always positive and in III and IV is always negative.

5.1 Properties of argument of a Complex Number

(i) amp (any real positive number) = 0

(ii) amp (any real negative number) = π

(iii) amp $(z - \bar{z}) = \pm \pi/2$

(iv) amp $(z_1 . z_2)$ = amp (z_1) + amp (z_2)

(v) amp $\left(\frac{z_1}{z_2}\right)$ = amp (z_1) – amp (z_2)

(vi) amp $(\bar{z})$ = – amp (z) = amp (1/z)

(vii) amp (– z) = amp (z) $\pm \pi$

(viii) amp (z^n) = n amp (z)

(ix) amp (iy) = $\pi/2$ if $y > 0$

= $-\pi/2$, if $y < 0$

(x) amp (z) + amp $(\bar{z})$ = 0

6. Square root of a Complex Number

The square root of z = a + ib is -

$$\sqrt{a+ib} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right] \text{ for } b > 0$$

$$\text{and } \pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right] \text{ for } b < 0$$

Note :

(i) The square root of i is $\pm\left(\frac{1+i}{\sqrt{2}}\right)$ (Here b = 1)

(ii) The square root of – i is $\pm\left(\frac{1-i}{\sqrt{2}}\right)$ (Here b = –1)

(iii) The square root of ω is $\pm\,\omega^2$

(iv) The square root of ω^2 is $\pm\,\omega$

7. Triangle Inequalities

(i) $|z_1 \pm z_2| \le |z_1| + |z_2|$

(ii) $|z_1 \pm z_2| \ge |z_1| - |z_2|$

8. Miscellaneous Results

(i) If ABC is an equilateral triangle having vertices z_1, z_2, z_3 then $z_1^2 + z_2^2 + z_3^2$

$= z_1z_2 + z_2z_3 + z_3z_1$

or $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0.$

(ii) If z_1, z_2, z_3, z_4 are vertices of parallelogram then $z_1 + z_3 = z_2 + z_4$.

(iii) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers represented by points P and Q respectively in Argand Plane then -

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= |(x_2 - x_1) + i(y_2 - y_1)| = |z_2 - z_1|$$

(iv) If a point P divides AB in the ratio of m : n, then $z = \frac{mz_2 + nz_1}{m+n}$ where z_1, z_2 and z represents the point A, B and P respectively.

(v) $|z - z_1| = |z - z_2|$ represents a perpendicular bisector of the line segment joining the points z_1 and z_2.

(vi) Let P be any point on a circle whose centre C and radius r, let the affixes of P and C be z and z_0 then $|z - z_0| = r$.

(a) Again if $|z - z_0| < r$ represent interior of the circle of radius r.

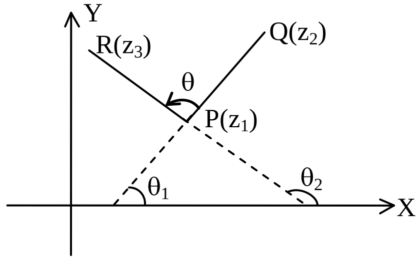
(b) $|z - z_0| > r$ represent exterior of the circle of radius r.

(vii) Let z_1, z_2, z_3 be the affixes of P, Q, R respectively in the Argand Plane. Then from the figure the angle between PQ and PR is.

$\theta = \theta_2 - \theta_1$

$= \arg.\ \overrightarrow{PR} - \arg \overrightarrow{PQ}$

$= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$



(a) If z_1, z_2, z_3 are collinear, thus $\theta = 0$ therefore

$\frac{z_3 - z_1}{z_2 - z_1}$ is purely real.

(b) If z_1, z_2, z_3 are such that $PR \perp PQ$,

$\theta = \pi / 2$ So $\frac{z_3 - z_1}{z_2 - z_1}$ is purely imaginary.