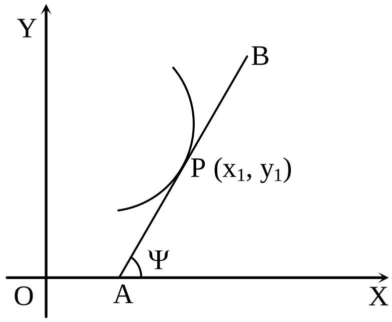
Application of Derivative (Tangent & Normal)

1. Geometrical Interpretation of the Derivative

If $y = f(x)$ be a given function, then the differential coefficient $f'(x)$ or at the point $P(x_1, y_1)$ is the trigonometrical tangent of the angle ψ (say) which the positive direction of the tangent to the curve at P makes with the positive direction of x- axis $\left(\frac{dy}{dx}\right)$, therefore represents the slope of the tangent.

Thus

$$f'(x) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \tan\psi$$



Thus

(i) The inclination of tangent with x- axis.

$$= \tan^{-1}\left(\frac{dy}{dx}\right)$$

(ii) Slope of tangent $= \frac{dy}{dx}$

(iii) Slope of the normal $= -\frac{dx}{dy}$

2. Equation of Tangent

(a) Equation of tangent to the curve $y = f(x)$ at $A(x_1,y_1)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1)$$

(i) If the tangent at $P(x_1,y_1)$ of the curve $y = f(x)$ is parallel to the x- axis (or perpendicular to y- axis) then $\psi = 0$ i.e. its slope will be zero.

$$m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$$

The converse is also true. Hence the tangent at (x_1,y_1) is parallel to x- axis.

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$$

(ii) If the tangent at $P(x_1, y_1)$ of the curve $y = f(x)$ is parallel to y-axis (or perpendicular to x-axis) then $\psi = \pi/2$, and its slope will be infinity i.e.

$$m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \infty$$

The converse is also true. Hence the tangent at (x_1, y_1) is parallel to y- axis

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \infty$$

(iii) If at any point $P(x_1, y_1)$ of the curve $y = f(x)$, the tangent makes equal angles with the axes, then at the point P, $\psi = \pi/4$ or $3\pi/4$, Hence at P, $\tan\psi = dy/dx = \pm 1$. The converse of the result is also true. thus at (x_1,y_1) the tangent line makes equal angles with the axes.

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \pm 1$$

3. Length of Intercepts Made on Axes by the Tangent

Equation of tangent at any point (x_1, y_1) to the curve $y = f(x)$ is

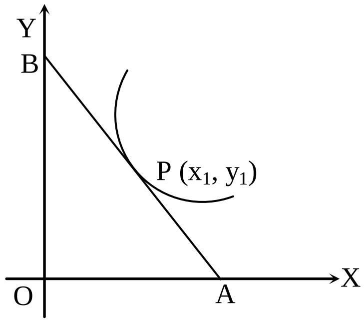
$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1) \quad ....(1)$$

Equation of x- axis, $y = 0$ ....(2)

Equation of y – axis, $x = 0$ ....(3)

Solving (1) and (2), we get.

$$x = x_1 - \left\{\frac{y_1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}\right\}$$



$$\therefore\ x - \text{intercept} = OA = x_1 - \left\{ \frac{y_1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} \right\}$$

Similarly solving (1) and (3), we get

y– intercept

$$OB = y_1 - x_1 \left(\frac{dy}{dx}\right)_{(x_1,y_1)}$$

4. Length of Perpendicular from origin to the Tangent

The length of perpendicular from origin (0,0) to the tangent drawn at the point (x_1, y_1) of the curve y = f(x).

$$p = \left| \frac{y_1 - x_1\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \right|$$

Explanation : The equation of tangent at point $P(x_1, y_1)$ of the given curve

$$y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$$

p = perpendicular from origin to tangent

$$= \left| \frac{y_1 - x_1\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \right|$$

5. Equation of Normal

The equation of normal at (x_1, y_1) to the curve y = f(x) is

$$(y - y_1) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} (x - x_1)$$

or $$(y - y_1) \cdot \left(\frac{dy}{dx}\right)_{(x_1,y_1)} + (x - x_1) = 0$$

5.1 Some facts about the normal

(i) The slope of the normal drawn at point $P(x_1,y_1)$ to the curve $y = f(x) = -\left(\frac{dx}{dy}\right)_{(x_1,y_1)}$

(ii) If normal makes an angle of θ with positive direction of x– axis then

$$-\frac{dx}{dy} = \tan\theta \text{ or } \frac{dx}{dy} = -\cot\theta$$

(iii) If normal is parallel to x– axis then

$$-\frac{dx}{dy} = 0 \text{ or } \frac{dy}{dx} = \infty$$

(iv) If normal is parallel to y – axis then

$$-\left(\frac{dx}{dy}\right) = \infty \text{ or } \frac{dy}{dx} = 0$$

(v) If normal is equally inclined from both the axes or cuts equal intercept then

$$-\left(\frac{dx}{dy}\right) = \pm 1 \text{ or } \left(\frac{dy}{dx}\right) = \pm 1$$

(vi) The length of perpendicular from origin to normal is

$$P' = \left| \frac{x_1 + y_1\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \right|$$

(vii) The length of intercept made by normal on x- axis is $= x_1 + y_1\left(\frac{dy}{dx}\right)$

and length of intercept on y - axis is

$$= y_1 + x_1\left(\frac{dx}{dy}\right)$$

6. Angle of Intersection of two Curves

If two curves $y = f_1(x)$ and $y = f_2(x)$ intersect at a point P, then the angle between their tangents at P is defined as the angle between these two curves at P.

But slopes of tangents at P are $\left(\frac{dy}{dx}\right)_1$ and $\left(\frac{dy}{dx}\right)_2$,

so at P their angle of intersection ϕ is given by

$$\tan\phi = \pm\frac{(dy/dx)_1 - (dy/dx)_2}{1 + (dy/dx)_1 (dy/dx)_2}$$

The other angle of intersection will be $(180^\circ - \phi)$

Note : If two curves intersect orthogonally i.e. at right angle then $\phi = \frac{\pi}{2}$ so the condition will be

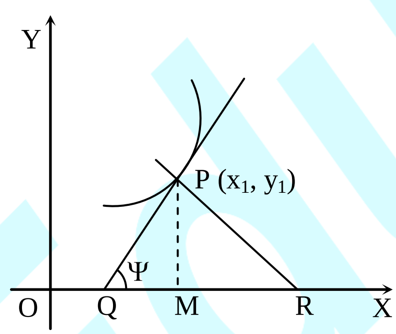
$$\left(\frac{dy}{dx}\right)_1 . \left(\frac{dy}{dx}\right)_2 = -1$$

7. Length of Tangent, Normal, Sub Tangent & Sub Normal

Let tangent and normal to the curve y = f (x) at a point P (x,y) meets the x– axis at points Q and R respectively. Then PQ and PR are called length of tangent and normal respectively, at point P. Also if PM be the perpendicular from P on x- axis, then QM and MR are called length of sub tangent and subnormal respectively at P. So from the diagram at P (x,y)

(i) length of tangent = PQ

$= y \operatorname{cosec}\psi$



$$= y\frac{\sqrt{1 + (dy/dx)^2}}{(dy/dx)}$$

(ii) length of the normal $= PR = y \sec\psi$

$= y\sqrt{1 + (dy/dx)^2}$

(iii) length of sub tangent $= QM = y \cot\psi$

$= y / \left(\frac{dy}{dx}\right)$

(iv) length of sub normal $= MR$

$= y \tan\psi$

$= y\left(\frac{dy}{dx}\right)$

8. Point of Inflexion

If at any point P, the curve is concave on one side and convex on other side with respect to x- axis, then the point P is called the point of inflexion. Thus P is a point of inflexion if at P,

$$\frac{d^2y}{dx^2} = 0, \text{ but } \frac{d^3y}{dx^3} \neq 0$$

Also point P is a point of inflexion if f'' (x) = f'''(x) = ... = $f^{n-1}(x) = 0$ and f^n (x) 0 for odd n.

9. Rolle's Theorem

If a function f defined on the closed interval [a, b], is

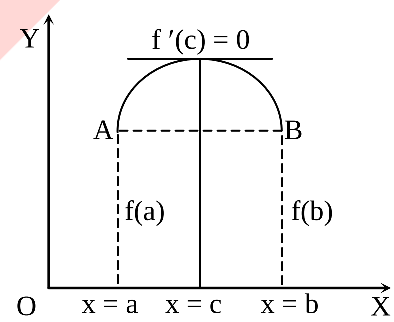
(i) Continuous on [a, b],

(ii) Derivable on (a, b) and

(iii) f(a) = f(b), then there exists atleast one real number c between a and b (a < c < b) such that f′(c) = 0

Geometrical interpretation

Let the curve y = f(x), which is continuous on [a, b] and derivable on (a, b), be drawn.



The theorem states that between two points with equal ordinates on the graph of f, there exists atleast one point where the tangent is parallel to x-axis.

Algebraic interpretation

Between two zeros a and b of f (x) (i.e., between two roots a and b of f(x) = 0) there exists atleast one zero of f′(x).

10. Lagrange's Mean Value Theorem

If a function f defined on the closed interval [a, b], is

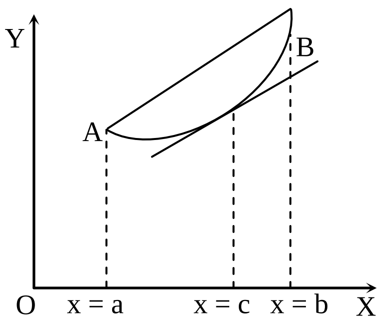
(i) Continuous on [a, b] and

(ii) Derivable on (a, b), then there exists atleast one real number c between a and b (a < c < b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical interpretation

The theorem states that between two points A and B on the graph of f there exists atleast one point where the tangent is parallel to the chord AB.



11. Derivative as a Rate Measure

Let y = f(x) be a function of x. Let Δy be the change in y corresponding to a small change Δx in x. Then $\frac{\Delta y}{\Delta x}$ represents the change in y due to a unit change in x. In other words, $\frac{\Delta y}{\Delta x}$ represents the average rate of change of y w.r.t. x as x changes from x to x + Δx.

As Δx → 0, the limiting value of this average rate of change of y with respect to x in the interval [x, x + Δx] becomes the intantaneous rate of change of y w.r.t. x.

Thus, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ = instantaneous rate of change of y w.r.t. x

$$\Rightarrow \frac{dy}{dx} = \text{rate of change of y w.r.t. x} \left[\because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}\right]$$

The word " instantaneous" is often dropped.

Hence, $\frac{dy}{dx}$ represents the value of change of y w.r.t. x for a definite value of x.

Related rates :

Generally we come across with the problems in which the rate of change of one of the quantities involved is required corresponding to the given rate of change of another quantity. For example, suppose the rate of change of volume of a spherical balloon is required when the rate of change of its radius is given. In such type of problems, we must find a relation connecting such quantities and differentiate this relation w.r.t. to time.