INVERSE TRIGONOMETRIC FUNCTIONS

2.1. INTRODUCTION

We have read about functions, one-one onto (bijective) functions and inverse of a function. We have also learnt that inverse of a function f is denoted by f^{-1} and f^{-1} exists if and only if f is a one-one onto function. There are several functions which are not one-one onto and hence their inverse does not exist. We have also read about trigonometric functions are not one-one onto over their natural domains and ranges and hence their, inverse do not exist. But if we restrict their domains and ranges, then they will become one-one onto functions and their inverse will exist. In this chapter we will study inverses of trigomometric functions and their various properties.

2.2. INVERSE OF A FUNCTION

Let $f : A \to B$ If (be a function from A to B) which is one-one onto. Then a function $f^{-1} : B \to A$ (f^{-1} from B to A) is said to be the inverse of the function f if

$$y = f(x) \Leftrightarrow f^{-1}(y) = x$$

i.e., image of x under f is y $\Leftrightarrow$ image of y under f^{-1} is x.

Clearly domain f^{-1} = range f and range f^{-1} = domain f

2.3. PRINCIPAL BRANCHES OF INVERSE TRIGONOMETRIC FUNCTIONS

(i) Definition of $\sin^{-1} x$ (Principal branch of $\sin^{-1} x$) :

$\sin^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ i.e., $\sin^{-1}$ is a function from $[-1, 1]$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Such that $\sin^{-1} x = \theta \Leftrightarrow x = \sin \theta$

where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le \theta < \frac{\pi}{2}$

(ii) Definition of $\cos^{-1} x$ (Principal branch of $\cos^{-1} x$) :

$\cos^{-1} : [-1, 1] \to [0, \pi]$ is such that $\cos^{-1} x = \theta \Leftrightarrow x = \cos \theta$,

where $-1 \le x \le 1$ and $0 \le \theta \le \pi$

(iii) Definition of $\tan^{-1} x$ (Principal branch of $\tan^{-1} x$) :

$\tan^{-1} : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that

$\tan^{-1} x = \theta \Leftrightarrow x = \tan \theta$

where $-\infty < x < \infty$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(iv) Definition of $\cot^{-1} x$: $\cot^{-1}$: $(-\infty, \infty) \to (0, \pi)$ such that

$\cot^{-1} x = \theta \Leftrightarrow x = \cot \theta$, where $-\infty < x < \infty$ and $0 < \theta < \pi$.

(v) **Definition of $\sec^{-1} x$: $\sec^{-1}$:** $(-\infty, -1] \cup [1, \infty) \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$ such that

$\sec^{-1} x = \theta \Leftrightarrow x = \sec\theta$

where, $-\infty < x \le -1$ or $1 \le x < \infty$ and $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$

(vi) **Definition of $\text{cosec}^{-1} x$:**

$\text{cosec}^{-1} : (-\infty, -1] \cup [1, \infty) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

such that $\text{cosec}^{-1} x = \theta \Leftrightarrow x = \text{cosec}\,\theta$

where $-\infty < x \le -1$ or $1 \le x < \infty$ and $-\frac{\pi}{2} \le \theta < 0$ or $0 < \theta \le \frac{\pi}{2}$

Summary

	Function	Domain of the function	Principal value branch
1.	$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
2.	$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
3.	$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
4.	$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$
5.	$y = \sec^{-1} x$	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
6.	$y = \text{cosec}^{-1} x$	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$

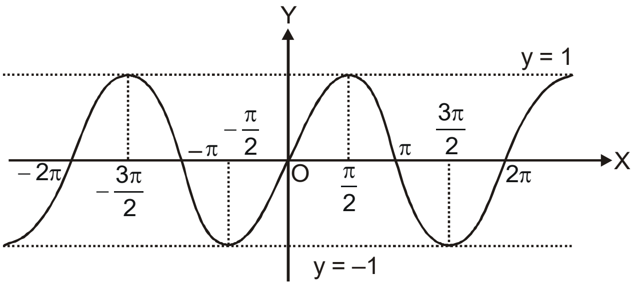
Note : Unless otherwise stated $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1}$, x $\sec^{-1} x$ and $\text{cosec}^{-1} x$ will mean their principal branches.

2.4 GRAPHS OF PRINCIPAL BRANCHES OF INVERSE TRIGONOMETRIC FUNCTIONS

(i) **Graph of y = sin x**

Domain = R = $(-\infty, \infty)$

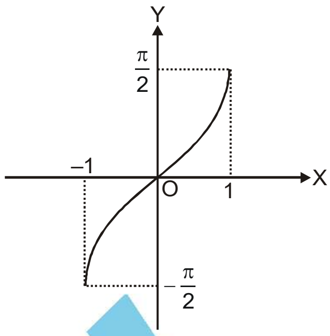
and Range = $[-1, 1]$



Graph of $y = \sin^{-1} x$

Domain = $[-1, 1]$

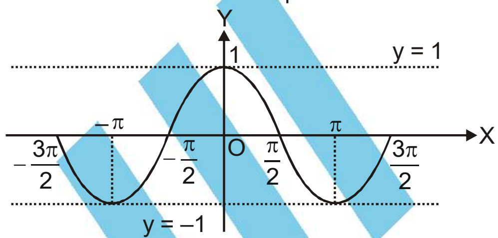
Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



(ii) Graph of $y = \cos x$

Domain = R = $(-\infty, \infty)$

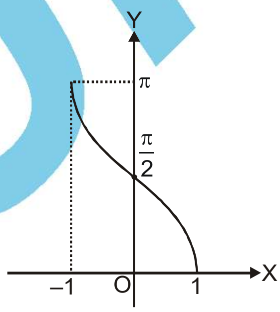
Range = $[-1, 1]$



Graph of $y = \cos^{-1} x$

Domain = $[-1, 1]$

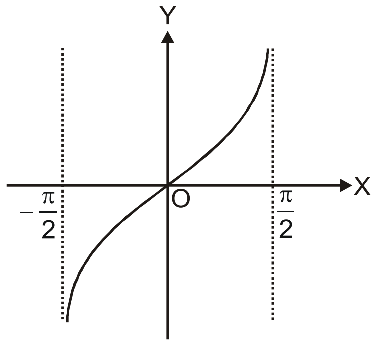
Range = $[0, \pi]$



(iii) Graph of $y = \tan x$

Domain = R – $\left\{x : x = (2n+1)\frac{\pi}{2}, n \in Z\right\}$

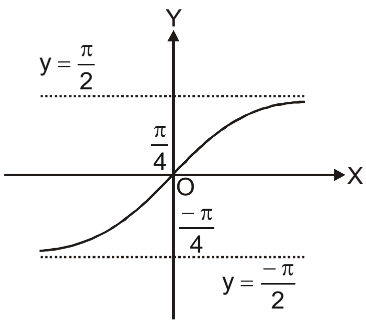
Range = $(-\infty, \infty)$



Graph of $y = \tan^{-1} x$

Domain = R = $(-\infty, \infty)$

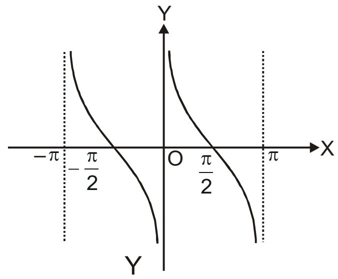
Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



(iv) Graph of y = cot x

Domain = R – {x : x = n π, n $\in$ Z}

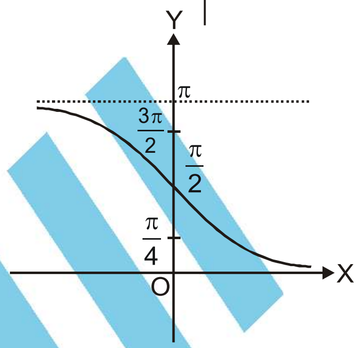
Range = $(-\infty, \infty)$



Graph of y = cot^{-1} x

Domain = $(-\infty, \infty)$

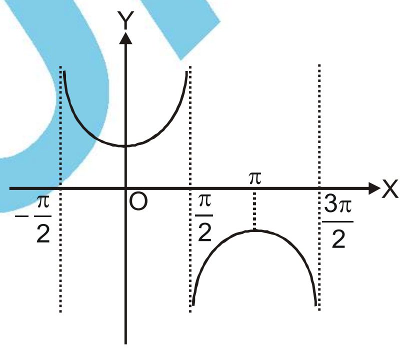
Range = $(0, \pi)$



(v) Graph of y = sec x

Domain = R – $\left\{x : x = (2n+1)\frac{\pi}{2}, n \in Z\right\}$

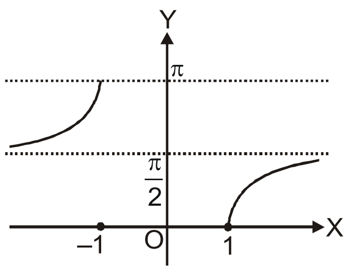
Range = $(-\infty, -1] \cup [1, \infty)$



Graph of y = sec^{-1} x

Domain = $(-\infty, -1] \cup [1, \infty)$

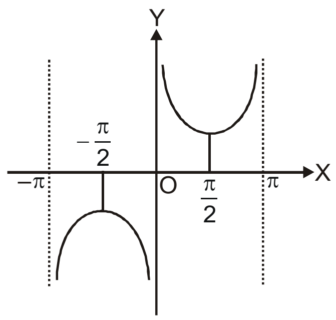
Range = $\left[\mathbf{0}, \frac{\pi}{\mathbf{2}}\right) \cup \left(\frac{\pi}{\mathbf{2}}, \pi\right] = [0, \pi] - \left\{\frac{\pi}{2}\right\}$



(vi) Graph of y = cosec x

Domain = R – {x : x = n π, n $\in$ Z}

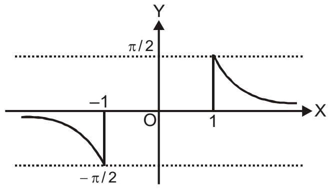
Range = $(-\infty, -1] \cup [1, \infty)$



Graph of $y = \text{cosec}^{-1} x$

Domain = $(-\infty, -1] \cup [1, \infty)$

Range = $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$



2.5. SOME IMPORTANT PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

Property I :

(i) $\sin(\sin^{-1} x) = x$ for all $x \in [-1, 1]$

(ii) $\cos(\cos^{-1} x) = x$, for r all $x \in [-1, 1]$

(iii) $\tan(\tan^{-1} x) = x$, for all $x \in R$

(iv) $\cot(\cot^{-1} x) = x$, for all $x \in R$.

(v) $\sec(\sec^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$ i.e., for all $x \le -1$ or $x \ge 1$

(vi) $\text{cosec}(\text{cosec}^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$ i.e., for all $x \le -1$ or $x \ge 1$

Property II :

(i) $\sin^{-1}(\sin x) = x,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos x) = x,\ x \in [0, \pi]$

(iii) $\tan^{-1}(\tan x) = x,\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iv) $\cot^{-1}(\cot x) = x,\ x \in (0, \pi)$

(v) $\sec^{-1}(\sec x) = x,\ x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ i.e., $x \in (0, \pi) - \left\{\frac{\pi}{2}\right\}$

(vi) $\text{cosec}^{-1}(\text{cosec}\ x) = x,\ x \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ i.e., $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$$\sin^{-1}(\sin x) = \begin{cases} -\pi - x & \text{if } -\frac{3\pi}{2} \le x \le -\frac{\pi}{2} \\ x, & \text{if } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ x - 2\pi & \text{if } \frac{3\pi}{2} \le x \le \frac{5\pi}{2} \\ 3\pi - x, & \text{if } \frac{5\pi}{2} \le x \le \frac{7\pi}{2} \text{ and so on} \end{cases}$$

$$\cos^{-1}(\cos x) = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \\ x - 2\pi, & 2\pi \le x \le 3\pi \\ 4\pi - x, & 3\pi \le x \le 4\pi \text{ and so on} \end{cases}$$

$$\tan^{-1}(\tan x) = \begin{cases} x+\pi, & -\dfrac{3\pi}{2} < x < -\dfrac{\pi}{2} \\ x, & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ x-\pi, & \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \\ x-2\pi, & \dfrac{3\pi}{2} < x < \dfrac{5\pi}{2} \text{ and so on} \end{cases}$$

Property III :

(i) $\sin^{-1} x = \text{cosec}^{-1}\dfrac{1}{x}$, $-1 \le x \le 1$ and $x \ne 0$

$\text{cosec}^{-1} x = \sin^{-1}\dfrac{1}{x}$, $x \le -1$ or $x \ge 1$

(ii) $\cos^{-1}x = \sec^{-1}\dfrac{1}{x}$, $-1 \le x \le 1$

$\sec^{-1} x = \cos^{-1}\dfrac{1}{x}$, $x \le -1$ or $x \ge 1$

(iii) $\tan^{-1}x = \cot^{-1}\left(\dfrac{1}{x}\right)$, $x > 0 = -\pi + \cot^{-1}\left(\dfrac{1}{x}\right)$, $x < 0$

(iv) $\cot^{-1} x = \tan^{-1}\left(\dfrac{1}{x}\right)$, $x > 0 = \pi + \tan^{-1}\left(\dfrac{1}{x}\right)$, $x < 0$

Property IV :

(i) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$
(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, for all $x \in [-1, 1]$
(iii) $\tan^{-1}(-x) = -\tan^{-1} x$, for all $x \in R$
(iv) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, for all $x \in R$
(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$ i.e., for all $|x| \ge 1$
(vi) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty]$ i.e., for all $|x| \ge 1$

Property V :

(i) $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$, for all $x \in [-1, 1]$

(ii) $\tan^{-1}x + \cot^{-1} x = \dfrac{\pi}{2}$, for all $x \in R$

(iii) $\sec^{-1} x + \text{cosec}^{-1} x = \dfrac{\pi}{2}$ for all $x \in (-\infty, -1] \cup [1, \infty)$ i.e., for all $|x| \ge 1$

Property VI :

(i) $$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

Property VII :

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}, \text{ if } xy > -1$$

This result can be established by putting –y in place of y in the results of property V using the fact that $\tan^{-1}(-y) = -\tan^{-1} y$.

$$\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } x > 0,\ y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } x < 0,\ y > 0 \text{ and } xy < -1 \end{cases}$$

Property VIII :

$$\sin^{-1} x + \sin^{-1} y = \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \\ & \text{or if } xy < 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } 0 < x, y \le 1 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } -1 \le x, y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

Property IX :

$$\sin^{-1} x - \sin^{-1} y = \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \\ & \text{or if } xy < 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } 0 < x \le 1,\ -1 \le y \le 0 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } -1 \le x < 0,\ 0 < y \le 1 \text{ and } x^2 + y^2 > 1 \end{cases}$$

Property X :

$$\cos^{-1} x + \cos^{-1} y = \begin{cases} \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right), & \text{if } -1 \le x, y \le 1 \text{ and } x + y \ge 0 \\ 2\pi - \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right), & \text{if } -1 \le x, y \le 1 \text{ and } x + y \le 0 \end{cases}$$

Property XI :

$$\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1 \le x, y \le 1 \text{ and } x \le y \\ -\cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\}, & \text{if } -1 \le y \le 0,\ 0 < x \le 1 \text{ and } x \ge y \end{cases}$$

This result can be established in the same way as in property X.

Property XII :

(i) $$2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } x > 1 \\ -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right), & \text{if } x < -1 \end{cases}$$

Property XIII :

(i) $$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } x < -1 \end{cases}$$

(ii) $$2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{if } -\infty < x \le 0 \end{cases}$$

Property XIV : (i) $$\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$= \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$$

Where $x \ge 0$

(ii) $$\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$= \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\frac{1}{x} = \operatorname{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

Where $x > 0$

(iii) $$\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$$

$$= \cot^{-1}\left(\frac{1}{x}\right) = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$$

Property XV :

(i) $$2\sin^{-1}x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -\dfrac{1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}), & \text{if } \dfrac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin(2x\sqrt{1-x^2}), & \text{if } -1 \le x \le -\dfrac{1}{\sqrt{2}} \end{cases}$$

Property XVI :

(i) $$2\cos^{-1}x = \begin{cases} \cos^{-1}(2x^2-1), & \text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1), & \text{if } -1 \le x \le 0 \end{cases}$$

Property XVII :

$$3\sin^{-1}x = \begin{cases} \sin^{-1}(3x-4x^3), & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - \sin^{-1}(3x-4x^3), & \text{if } \dfrac{1}{2} \le x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3), & \text{if } -1 \le x \le -\dfrac{1}{2} \end{cases}$$

Property XVIII :

$$3\cos^{-1}x = \begin{cases} \cos^{-1}(4x^3-3x), & \text{if } \dfrac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x), & \text{if } -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x), & \text{if } -1 \le x \le -\dfrac{1}{2} \end{cases}$$

Property XIX :

$$3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right), & \text{if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$$

SOLVED PROBLEMS

Ex.1 ***Write each of the folloiwng functions in the simplest form :***

(i) $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)$, $|x| < a$ *(ii)* $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$, $a > 0;\ \frac{-a}{\sqrt{3}} \le x \le \frac{a}{\sqrt{3}}$

Sol. (i) Let x = a sin θ. Then,

$$\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right) = \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

Thus, $\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right) = \sin^{-1}\left(\frac{x}{a}\right)$

(ii) Let x = a tan θ. Then

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = \tan^{-1}\left(\frac{3a^3\tan\theta-a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right) = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1}\left(\frac{x}{a}\right)$$

Thus, $\tan^{-1}\left(\frac{3a^3x-x^3}{a^3-3ax^2}\right) = 3\tan^{-1}\left(\frac{x}{a}\right)$

Ex.2 ***Simplify :***

(i) $\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right)$, *if* $\frac{a}{b}\tan x > -1$

(ii) $\tan\frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right)$, $|x| < 1,\ y > 0$ *and* $xy < 1$

Sol. (i) We have

$$\tan^{-1}\left(\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right) = \tan^{-1}\left(\frac{\frac{a\cos x-b\sin x}{b\cos x}}{\frac{b\cos x+a\sin x}{b\cos x}}\right) = \tan^{-1}\left(\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right)$$

$$= \tan^{-1}\left(\frac{p-q}{1+pq}\right), \text{ where } p = \frac{a}{b} \text{ and } q = \tan x$$

$$= \tan^{-1}p - \tan^{-1}q = \tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}(\tan x) = \tan^{-1}\left(\frac{a}{b}\right) - x$$

(ii) Let x = tan θ and y = tan φ. Then

$$\tan \frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right)$$

$$= \tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right) = \tan\left[\frac{1}{2}(2\tan^{-1}x)+\frac{1}{2}[2\tan^{-1}(y)]\right]$$

$$= \tan(\tan^{-1}x + \tan^{-1}y) = \tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} = \frac{x+y}{1-xy}$$

Ex.3 ***Prove that***

(i) $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$ ***(ii)*** $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Sol. (i) We have

$$\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\left(\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}\right) = \tan^{-1}\left(\frac{125}{250}\right) = \tan^{-1}\frac{1}{2}$$

(ii) We have

$$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{7}}{1-\frac{1}{2}\times\frac{1}{7}}\right) = \tan^{-1}\left(\frac{9}{13}\right) \quad \text{..............(1)}$$

Now, $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{1}{2} + \left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}\right)$

$$= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{9}{13} = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{9}{13}}{1-\frac{1}{2}\times\frac{9}{13}}\right) = \tan^{-1}\left(\frac{31}{17}\right)$$

Ex.4 ***Find the value of :***

(i) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ ***(ii)*** $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ ***(iii)*** $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

Sol. (i) We know that $\sin^{-1}(\sin x) = x$

$$\therefore \quad \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{2\pi}{3}$$

But, $\frac{2\pi}{3} \notin \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, which is the principal branch of $\sin^{-1}x$

However, $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\pi-\frac{2\pi}{3}\right) = \sin\frac{\pi}{3}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Hence, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$

(ii) We know that $\tan^{-1}(\tan x) = x$

$$\Rightarrow \quad \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \frac{3\pi}{4}$$

But, $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal branch of $\tan^{-1} x$

However, $\tan\frac{3\pi}{4} = \tan\left(\pi + \left(-\frac{\pi}{4}\right)\right) = \tan\left(-\frac{\pi}{4}\right)$ and $\left(-\frac{\pi}{4}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence, $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = -\frac{\pi}{4}$

(iii) We know that $\cos^{-1}(\cos x) = x$

$$\Rightarrow \quad \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{7\pi}{6}$$

But, $\frac{7\pi}{6} \notin [0, \pi]$ which is the principal branch of $\cos^{-1} x$

However, $\cos\frac{7\pi}{6} = \cos\left(2\pi - \frac{5\pi}{6}\right) = \cos\frac{5\pi}{6}$ and $\frac{5\pi}{6} \in (0, \pi)$

Hence, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \frac{5\pi}{6}$

Ex.5 ***Prove that***

$$\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{2} - \frac{1}{2}\cos^{-1} x, \text{ for } -\frac{1}{\sqrt{2}} \le x \le 1$$

Sol. Let $x = \cos 2\theta$. Then, $\sqrt{1+x} = \sqrt{1+\cos 2\theta} = \sqrt{2\cos^2\theta} = \sqrt{2}\cos\theta$

and $\sqrt{1-x} = \sqrt{1-\cos 2\theta} = \sqrt{2\sin^2\theta} = \sqrt{2}\sin\theta$

$$\therefore \quad \text{LHS} = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right) = \tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right) = \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) = \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right]$$

$$= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}(x) = \text{RHS}$$

Ex.6 ***Prove that***

$$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in (0, 1)$$

Sol. Let $x = \tan^2\theta$. Then,

$\sqrt{x} = \tan\theta$; and so $\tan^{-1}(\sqrt{x}) = \tan^{-1}(\tan\theta) = \theta$

Further, $\text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta) = \frac{1}{2} \times 2\theta = \theta$

Hence, LHS = RHS

Ex.7 ***Solve each of the following for x :***

(i) $\sin^{-1}\left(\frac{8}{x}\right) + \sin^{-1}\left(\frac{15}{x}\right) = \frac{\pi}{2}$ *(ii)* $\sin^{-1}(x) - \cos^{-1}(x) = \sin^{-1}(3x-2)$, *if* $x > 0$

(iii) $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$ *(iv)* $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

Sol. (i) We have $\sin^{-1}\left(\frac{8}{x}\right) + \sin^{-1}\left(\frac{15}{x}\right) = \frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}\left(\frac{15}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{8}{x}\right)$

$\Rightarrow \quad \frac{15}{x} = \sin\left(\frac{\pi}{2} - \sin^{-1}\frac{8}{x}\right) \Rightarrow = \cos\left(\sin^{-1}\frac{8}{x}\right)$

$\Rightarrow \quad \frac{15}{x} = \sqrt{1-\left(\frac{8}{x}\right)^2} \quad \Rightarrow \quad \frac{225}{x^2} = 1 - \frac{64}{x^2}$

$\Rightarrow \quad \frac{289}{x^2} = 1 \quad \Rightarrow \quad x = \pm 17$

But, $x = -17$ cannot satisfy the given equation.

$\therefore \quad x = 17$

(ii) Let $\sin^{-1} x = \alpha$ and $\cos^{-1} x = \beta$. Then

$x = \sin\alpha$ and $x = \cos\beta$

$\Rightarrow \quad \cos\alpha = \sqrt{1-x^2} = \sin\beta$

From the given condition, we have

$\alpha - \beta = \sin^{-1}(3x-2)$

$\Rightarrow \quad 3x - 2 = \sin(\alpha - \beta)$

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\Rightarrow \quad 3x - 2 = x^2 - (1 - x^2)$

$\Rightarrow \quad 3x - 2 = 2x^2 - 1$

$\Rightarrow \quad 2x^2 - 3x + 1 = 0$

$\Rightarrow \quad (2x-1)(x-1) = 0$

$\Rightarrow \quad x = 1, \frac{1}{2}$

(iii) We have, $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$

$\tan^{-1}\left(\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right) = \tan^{-1}\frac{8}{31}$

$\Rightarrow \quad \frac{2x}{1-(x^2-1)} = \frac{8}{31}$, assuming that $(x+1)(x-1) < 1$

i.e., $x^2 - 1 < 1$

$\Rightarrow \quad \frac{2x}{2-x^2} = \frac{8}{31}$ i.e., $x^2 < 2$

$\Rightarrow \quad 62x = 16 - 8x^2$

$\Rightarrow \quad 8x^2 + 62x - 16 = 0$

$\Rightarrow \quad 4x^2 + 31x - 8 = 0$

$\Rightarrow \quad x = \dfrac{-31 \pm \sqrt{961+128}}{8} = \dfrac{-31 \pm 33}{8} = \dfrac{1}{4}$ or -8

When $x = -8$, $x^2 = 64 \nless 2$. Hence, we reject this value of x. Hence, the required value is $x = \dfrac{1}{4}$.

(iv) We have, $\tan^{-1} 2x + \tan^{-1} 3x = \dfrac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{2x+3x}{1-(2x)(3x)}\right) = \dfrac{\pi}{4}$

$\Rightarrow \quad \dfrac{5x}{1-6x^2} = 1 \Rightarrow 6x^2 + 5x - 1 = 0$

$\Rightarrow \quad x = \dfrac{-5 \pm \sqrt{25+24}}{12} = \dfrac{-5 \pm 7}{12} = -1$ or $\dfrac{1}{6}$

Now the formula,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy} \text{ holds only when } xy < 1$$

Thus, when $x = -1$, $(2x)(3x) = (-2)(-3) = 6 > 1$

So, we reject $x = -1$ and accept $x = \dfrac{1}{6}$

Ex.8 ***Solve for x :*** $2\tan^{-1} x = \sin^{-1}\dfrac{2a}{1+a^2} - \cos^{-1}\dfrac{1-b^2}{1+b^2}$

Sol. Let $a = \tan\theta$ and $b = \tan\phi$.

Then, $\sin^{-1}\left(\dfrac{2a}{1+a^2}\right) = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right)$

$= \sin^{-1}(\sin 2\theta) = 2\theta$

and $\cos^{-1}\left(\dfrac{1-b^2}{1+b^2}\right) = \cos^{-1}\left(\dfrac{1-\tan^2\phi}{1+\tan^2\phi}\right)$

$= \cos^{-1}(\cos 2\phi) = 2\phi$

Hence, RHS $= 2\theta - 2\phi = 2(\theta - \phi)$

$= 2(\tan^{-1} a - \tan^{-1} b)$

So, the given equation implies

$2\tan^{-1} x = 2(\tan^{-1} a - \tan^{-1} b), \quad -1 < a < 1, \quad -1 < b < 1$

$\Rightarrow \quad \tan^{-1} x = \tan^{-1} a - \tan^{-1} b$

$= \tan^{-1}\left(\dfrac{a-b}{1+ab}\right); \quad ab > -1$

$\Rightarrow \quad x = \dfrac{a-b}{1+ab}; \quad ab > -1$

Ex.9 ***Solve :***

(i) $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\mathrm{cosec}\, x)$

(ii) $\tan^{-1}\dfrac{1-x}{1+x} = \dfrac{1}{2}\tan^{-1} x \ (x > 0)$

(iii) $\sin[2\cos^{-1}\{\cot(2\tan^{-1} x)\}] = 0$

Sol. (i) We have $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\, x)$

$\Rightarrow \tan^{-1}\left(\dfrac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}\left(\dfrac{2}{\sin x}\right)$

$\Rightarrow \dfrac{2\cos x}{1-\cos^2 x} = \dfrac{2}{\sin x}$ $\Rightarrow$ $\sin x \cos x = 1 - \cos^2 x = \sin^2 x$

$\Rightarrow \sin x(\cos x - \sin x) = 0$ $\Rightarrow$ $\sin x = 0$ or $\cos x - \sin x = 0$

when $\sin x = 0$, $x = n\pi$, $n \in Z$

when $\cos x - \sin x = 0$ or $\tan x = 1$, $x = n\pi + \dfrac{\pi}{4}$

(ii) We have, $\tan^{-1}\dfrac{1-x}{1+x} = \dfrac{1}{2}\tan^{-1} x$ $(x > 0)$

$\Rightarrow \tan^{-1} 1 - \tan^{-1} x = \dfrac{1}{2}\tan^{-1} x$ $\left(\because \tan^{-1}\dfrac{x-y}{1+xy} = \tan^{-1} x - \tan^{-1} y\right)$

$\Rightarrow \dfrac{\pi}{4} = \tan^{-1} x + \dfrac{1}{2}\tan^{-1} x = \dfrac{3}{2}\tan^{-1} x$

$\Rightarrow \tan^{-1} x = \dfrac{2}{3} \times \dfrac{\pi}{4} = \dfrac{\pi}{6}$ $\Rightarrow x = \tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$

(iii) We have, $\sin[2\cos^{-1}\{\cot(2\tan^{-1} x)\}] = 0$

$\Rightarrow \sin\left[2\cos^{-1}\left\{\cot\left[\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)\right]\right\}\right] = 0$ $\left(\because 2\tan^{-1} x = \tan^{-1}\dfrac{2x}{1-x^2}\right)$

$\Rightarrow \sin\left[2\cos^{-1}\left\{\cot\left(\cot^{-1}\dfrac{1-x^2}{2x}\right)\right\}\right] = 0$ $\left(\because \cot^{-1} x = \tan^{-1}\dfrac{1}{x}\right)$

$\Rightarrow \sin\left[2\cos^{-1}\left(\dfrac{1-x^2}{2x}\right)\right] = 0$

$\Rightarrow \sin\left[\sin^{-1}\left\{2\left(\dfrac{1-x^2}{2x}\right)\sqrt{1-\left(\dfrac{1-x^2}{2x}\right)^2}\right\}\right] = 0$ $(\because 2\cos^{-1} x = \sin^{-1}(2x\sqrt{1-x^2}))$

$\Rightarrow \left(\dfrac{1-x^2}{x}\right)\sqrt{1-\left(\dfrac{1-x^2}{2x}\right)^2} = 0$ $\Rightarrow$ $\dfrac{1-x^2}{x} = 0$ or $\sqrt{1-\left(\dfrac{1-x^2}{2x}\right)^2} = 0$

$\Rightarrow 1 - x^2 = 0$ or $\left(\dfrac{1-x^2}{2x}\right)^2 = 1$ $\Rightarrow$ $x = \pm 1$ or $(1-x^2)^2 = 4x^2$

Now, $(1-x^2)^2 = 4x^2$ implies

$(1-x^2)^2 - (2x)^2 = 0$

$\Rightarrow (1 - x^2 - 2x)(1 - x^2 + 2x) = 0$

$\Rightarrow 1 - x^2 - 2x = 0$ or $1 - x^2 + 2x = 0$

$\Rightarrow x^2 + 2x - 1 = 0$ or $x^2 - 2x - 1 = 0$

$\Rightarrow x = -1 \pm \sqrt{2}$ or $x = 1 \pm \sqrt{2}$

Hence, $x = \pm 1, -1 \pm \sqrt{2}, 1 \pm \sqrt{2}$

EXERCISE – I — UNSOLVED PROBLEMS

Q.1 Evaluate the following :–

(i) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$ (ii) $\cos^{-1}(\cos 10)$ (iii) $\tan\left[\frac{1}{2}\left(\cos^{-1}\frac{\sqrt{5}}{3}\right)\right]$

Q.2 Prove that

$$\tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \tan^{-1}\left(\frac{c-a}{1+ca}\right)$$

$$= \tan^{-1}\left(\frac{a^3-b^3}{1+a^3b^3}\right) + \tan^{-1}\left(\frac{b^3-c^3}{1+b^3c^3}\right) + \tan^{-1}\left(\frac{c^3-a^3}{1+c^3a^3}\right)$$

Q.3 Prove that

$$\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$$

Q.4 Prove that

$$2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right) = \cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right) \text{ for } 0 < b \le a \text{ and } x \ge 0$$

Q.5 If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$, prove that

$xy + yz + zx = 1$

Q.6 If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, prove that
$x^2 + y^2 + z^2 + 2xyz = 1$

Q.7 If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, prove that

$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

Q.8 Solve : $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

Q.9 Solve : $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$

$\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$

Q.10 If $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$, find x.

Q.11 If $-1 \le x, y, z \le 1$, such that $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$,
find the value of

$$x^{2000} + y^{2001} + z^{2002} - \frac{9}{x^{2000} + y^{2001} + z^{2002}}$$

Q.12 Prove that

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \text{..........} + \tan^{-1}\frac{1}{n^2+n+1} = \tan^{-1}\left(\frac{n}{n+2}\right)$$

Q.13 Sum to n terms the series

$$\tan^{-1}\left(\frac{x}{1+1\cdot 2x^2}\right) + \tan^{-1}\left(\frac{x}{1+2\cdot 3x^2}\right) + \tan^{-1}\left(\frac{x}{1+3\cdot 4x^2}\right) + \text{..........}$$

Q.14 Establish the algebraic relation between x, y, z if $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ are in A.P. and if further x, y, z are also in A.P., their prove that x = y = z

Q.15 If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$ and $x + y + z = \sqrt{3}$, then prove that x = y = z

Q.16 Solve that equation for x ;

$$3\sin^{-1}\frac{2x}{1+x^2} - 4\cos^{-1}\frac{1-x^2}{1+x^2} + 2\tan^{-1}\frac{2x}{1-x^2} = \frac{\pi}{3}$$

Q.17 If $\sin(\pi\cos\theta) = \cos(\pi\sin\theta)$, prove that

$$\theta = \pm\frac{1}{2}\sin^{-1}\frac{3}{4}$$

Q.18 Prove that $\tan^{-1}\frac{yz}{xr} + \tan^{-1}\frac{zx}{yr} + \tan^{-1}\frac{xy}{zr} = \frac{\pi}{2}$,

where $x^2 + y^2 + z^2 = r^2$

Q.19 Prove that

$$\tan^{-1}\left(\frac{a_1x-y}{a_1y+x}\right) + \tan^{-1}\left(\frac{a_2-a_1}{a,\, a_2+1}\right) + \tan^{-1}\left(\frac{a_3-a_2}{a_2a_3+1}\right) + \text{.......} + \tan^{-1}\left(\frac{a_n-a_{n-1}}{a_na_{n-1}+1}\right) + \tan^{-1}\left(\frac{1}{a_n}\right) = \tan^{-1}\left(\frac{x}{y}\right)$$

Q.20 If a_1, a_2, a_3, ... form an A.P. with common difference d (a > 0, d > 0) prove that

$$\tan^{-1}\frac{d}{1+a_1a_2} + \tan^{-1}\frac{d}{1+a_2a_3} + \text{.........} + \tan^{-1}\frac{d}{1+a_na_{n+1}} = \tan^{-1}\frac{a_{n+1}-a_n}{1+a_1a_{n+1}}$$

EXERCISE – II BOARD PROBLEMS

Q1. Prove that following : $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$.

Q2. Using principal value, evaluate the following : $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

Q3. Prove the following : $\cos^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2},\ x \in \left(0, \frac{\pi}{4}\right)$

OR

Solve for x : $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\, x)$

Q4. Write the principal value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

Q5. Prove the following : $\tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

OR

Prove the following : $\cos[\tan^{-1}(\sin\{\cot^{-1} x\})] = \sqrt{\frac{1+x^2}{2+x^2}}$

Q6. Find the value of $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$.

Q7. Prove that : $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x,\ -\frac{1}{\sqrt{2}} \le x \le 1$

Q8. Prove the following : $\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$

Q9. Show that :

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$$

OR

Solve the following equation :

$$\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$$

Answers

EXERICISE – 1 (UNSOLVED PROBLEMS)

1. (i) $\frac{2\pi}{3}$ **(ii)** $(4\pi - 10)$ **(iii)** $\left(\frac{3-\sqrt{5}}{2}\right)$ **8.** $x = \frac{1}{6}$ **9.** $x = \frac{1}{2}, y = 1$ **10.** $x = \pm 1, \pm(1 \pm \sqrt{2})$

11. 0 **13.** $S = \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$ **16.** $x = \frac{1}{\sqrt{3}}$

EXERICISE – 2 (BOARD PROBLEMS)

Q2. π **Q3.** $\frac{\pi}{4}$ **Q4.** $\frac{5\pi}{6}$ **Q6.** $\frac{\pi}{5}$