ROTATIONAL MOTION

So far we have learnt kinematics and dynamics of translation motion in which all the particles of a body undergo identical motions i.e. at any instant of time all of them have equal velocities and equal accelerations and in any interval of time they all follow identical trajectories. Therefore kinematics of any particle of a body or of its centre of mass in translation motion is representative of kinematics of the whole body. But when a body is in rotational motion, all its particles and the centre of mass do not undergo identical motions. Newton's laws of motion, which are the main guiding laws of mechanics, are applicable to a particle and if applied to a rigid body or system of particles, they predict' the motion of the centre of mass. Therefore, it becomes necessary to investigate how the centre of mass and different particles of a rigid body move when a body rotates.

1. RIGID BODY

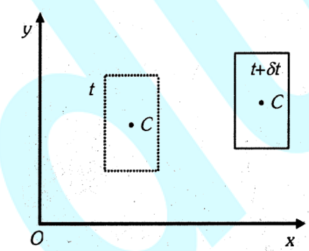
A rigid body is an assemblage of a large number of material particles, which do not change their mutual distances under any circumstance or in other words, the body is not deformed under any circumstance.

Actual material bodies are never perfectly rigid and are deformed under the action of external forces. When these deformations are small enough not to be considered during the course of motion, the body is assumed to be a rigid body. Hence, all solid objects such as stone, ball, vehicles etc are considered as rigid bodies while analyzing their translational as well as rotational motion.

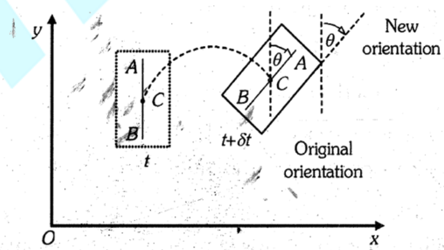
2. ROTATIONAL MOTION OF A RIGID BODY

Any kind of motion is identified by change in position change in orientation or change in both. If a body changes its orientation during its motion it said to be in rotational motion.

In the following figures, a rectangular plate is shown moving in the x-y plane. The point C is its centre of mass. In the first case it does not change its orientation, therefore is in pure translation motion. In the second case it changes its orientation during its motion. It is a combination of translational and rotational motion.



Pure Translation



Combination of translation and rotation

Rotation i.e. change in orientation is identified by the angle through which a linear dimension or a straight line drawn on the body turns. In the figure this angle is shown by θ.

2.1 Types of Motions involving Rotation

Motion of body involving rotation can be classified into following three categories.

I. Rotation about a fixed axis.

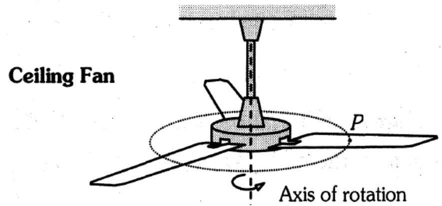
II. Rotation about an axis in translation.

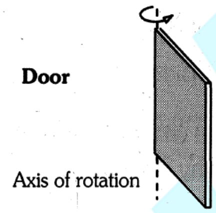
III. Rotation about an axis in rotation

<u>Rotation about a fixed axis</u> :

Rotation of ceiling fan, opening and closing of doors and rotation of needles of a wall' dock etc. come into this category.

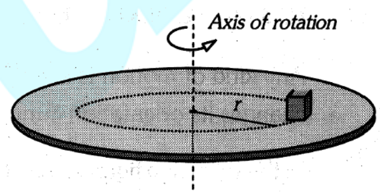
When a ceiling fan rotates, the vertical rod supporting it remains stationary and all the particles on the fan move on circular paths. Circular path of a particle P on one of its blades is shown by dotted circle. Centres of circular paths followed by every particle on the central line through the rod. This central line is known as the axis of rotation and is shown by a dashed line. All the particles on the axis of rotation are at rest, therefore the axis is stationary and the fan is in rotation about this fixed axis.





A door rotates about a vertical line that passes through its hinges. This vertical line is the axis of rotation. In the figure, the axis of rotation is shown by dashed line.

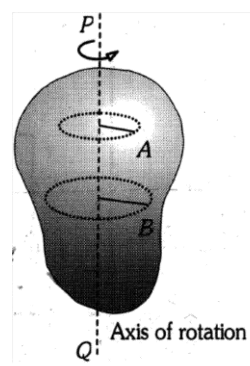
Axis of rotation :



An imaginary line perpendicular to the plane of circular paths of particles of a rigid body in rotation and containing the centres of all these circular paths is known as axis of rotation.

It is not necessary that the axis of rotation should pass through the body. Consider a system shown in the figure, where a block is fixed on a rotating disc. The axis of rotation passes through the center of the disc but not through the block.

Important observations :



Let us consider a rigid body of arbitrary shape rotating about a fixed axis PQ passing through the body. Two of its particles A and B are shown moving on their circular paths.

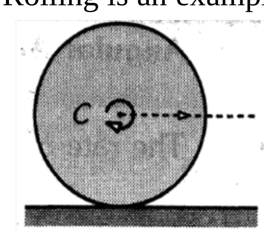
All its particles, not lying on the axis of rotation, move along circular paths with centres on the axis or rotation. All these circular paths are in parallel planes that are perpendicular to the axis of rotation.

All the particles of the body undergo same angular displacement in the same time interval, therefore all of them move with the same angular velocity and angular acceleration.

Particles moving on circular paths of different radii move with different speeds and different magnitudes of linear acceleration. Furthermore, no two particles in the same plane perpendicular to the axis of rotation have same velocity and acceleration vectors.

<u>Rotation about an axis in translation</u>

Rotation about an axis in translation includes a broad category of motions. Rolling is an example of this kind of motion.

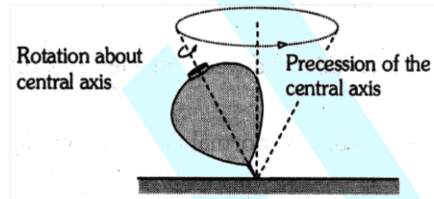


Consider the of a vehicle, moving on straight leveled road. The wheel appears rotating about its stationary axle relative to a reference frame, attached with the vehicle. The rotation of the wheel as observed from this frame is rotation about a fixed axis. Relative to a reference frame fixed with the ground, the wheel appears rotating about the moving axle, therefore, rolling of a wheel is superposition of two simultaneous but distinct motions - rotation about the axle fixed with the vehicle and translation of the axle together with the vehicle.

Rotation about an axis in rotation.

In this kind of motion, the body rotates about an axis which in turn rotates about some other axis. Analysis of rotation about rotating axes is beyond our scope, therefore we shall keep our discussion elementary level only.

As an example consider a rotating top. The top rotates about its central axis of symmetry and this axis sweeps a cone about a vertical axis. The central axis continuously changes its orientation, therefore it is in rotational motion. This type of rotation in which the axis of rotation also rotates and sweeps out cone is known as precession.



Another example of rotation about an axis in rotations a swinging table-fan while running. Table-fan rotates about its shaft along which its axis of rotation passes. When running swings, its shaft rotates about a certain axis.

3. KINEMATICS OF ROTATIONAL MOTION

Angular Displacement (θ)

- When a particle moves in a curved path, the change in the angle traced by its position vector about a fixed point is known as angular displacement.
- Unit : radian
- Dimension : $M^0L^0T^0$ i.e. dimensionless.
- Elementary angular displacement is a vector whereas other angular displacements is a scalar.

Angular Velocity (ω)

- The angular displacement per unit time is defined as angular velocity.

  $\omega = \frac{\Delta\theta}{\Delta t}$, where $\Delta\theta$ is the angular displacement during the time interval Δt.

- Instantaneous angular velocity $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$. Average angular velocity $\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$
- Unit : rad/s
- Dimensions : $[M^0L^0T^{-1}]$, which is same as that of frequency.
- Instantaneous angular velocity is a vector quantity, whose direction is normal to the rotational plane and its direction is given by right hand screw rule.
- If ω be the angular velocity, v the linear velocity and r the radius of path, we have the following relation. $\vec{v} = \vec{\omega} \times \vec{r}$
- If n be the frequency then $\omega = 2\pi n$, If T be the time period then $\omega = 2\pi/T$.
- The angular velocity of a rotating rigid body can be either positive or negative, depending on whether it is rotating in the direction of increasing θ (anticlockwise) or decreasing θ (clockwise).
- The magnitude of angular velocity is called the angular speed which is also represented by ω.

Angular Acceleration (α)

- The rate of change of angular velocity is defined as angular acceleration $\vec{\alpha} = \dfrac{d\vec{\omega}}{dt}$
- Suppose a particle has angular velocity $\vec{\omega}_1$ & $\vec{\omega}_2$ at time t_1 and t_2 respectively

  Then average angular acceleration, $\vec{\alpha} = \dfrac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1}$
- It is a vector quantity, whose direction is along the change in direction of angular velocity.
- Unit : rad/s^2
- Dimensions : $M^0L^0T^{-2}$
- Relation between angular acceleration $\vec{\alpha}$ and tangential acceleration $\vec{\alpha}_t$ is $\vec{\alpha}_t = \vec{\alpha} \times \vec{r}$
- Radial or normal acceleration, $\vec{\alpha}_r = \vec{\omega} \times \vec{v}$. Its direction is along the radius.
- Net acceleration $\vec{a} = \vec{a}_t + \vec{a}_r = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$

Comparison of Linear Motion and Rotational Motion

	Linear Motion		**Rotational Motion**
(i)	If acceleration is 0, v = constant and $s = vt$	(ii)	If acceleration a = constant, then ω = constant and $\theta = \omega t$
(ii)	If acceleration a = constant, then	(ii)	If angular acceleration α = constant, then
	(a) $s = \dfrac{(u+v)}{2}t$		(a) $\theta = \dfrac{(\omega_0+\omega)}{2}t$
	(b) $a = \dfrac{v-u}{t}$		(b) $\alpha = \dfrac{\omega+\omega_0}{t}$
	(c) $v = u + at$		(c) $\omega = \omega_0 + \alpha t$
	(d) $s = ut + \dfrac{1}{2}at^2$		(d) $\theta = \omega_0 t + \dfrac{1}{2}\alpha t^2$
	(e) $v^2 = u^2 + 2as$		(e) $\omega^2 = \omega_0^2 + 2\alpha\theta$
	(f) $S_{n^{th}} = u + \dfrac{a}{2}(2n-1)$		(f) $\theta_{n^{th}} = \omega_0 + \dfrac{\alpha}{2}(2n-1)$
(iii)	If acceleration is not constant, the Above equation will not be applicable. In this case	(iii)	If angular acceleration is not constant, the above equation will not be applicable. In this case
	(a) $v = \dfrac{ds}{dt}$		(a) $\omega = \dfrac{d\theta}{dt}$
	(b) $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = v\dfrac{dv}{ds}$		(b) $\alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2} = v\dfrac{d\omega}{d\theta}$

GOLDEN KEY POINTS

- In a rigid body, angular velocity of any point w.r.t. any other point is constant and is equal to the angular velocity of the rigid body.

Illustrations

Illustration 1.

A wheel is rotating with angular velocity 2 rad/s. It is subjected to a uniform angular acceleration 2.0 rad/s^2.

(a) What angular velocity does the wheel acquire after 10 s?

(b) How many revolutions will it make in this time interval?

Solution:

The wheel is in uniform angular acceleration, Hence –

$\omega = w_o + \alpha t \rightarrow$ Substituting the values of w_o, α and t, we have

$\omega = 2 + 2 \times 10 = 22$ rad/s

$\theta = \theta_o + \frac{1}{2}(\omega_o + \omega)\, t \rightarrow$ Substituting $\theta_o = 0$ for initial position and ω_o from above equation, we have

$$n = \frac{\theta}{2\pi} = \frac{120}{2\pi} \approx 19$$

Illustration 2.

A flywheel rotates with a uniform angular acceleration. Its angular velocity increases from 20π rad/s to 40π rad/s in 10 seconds. How many rotations did it undergo in this period?

[AIPMT (Mains) 2006]

Solution:

Θ $\omega_2 = \omega_1 + \alpha t \therefore 40\pi = 20\pi + 10\alpha \Rightarrow \alpha = 2\pi$ rad/s^2

Angular displacement $\theta = \frac{\omega_2^2 - \omega_1^2}{2\alpha} = \frac{(40\pi)^2 - (20\pi)^2}{2 \times 2\pi} = \frac{1200\pi^2}{4\pi} = 300\theta$ rad

Therefore the number of rotations undergone $= \frac{\theta}{2\pi} = \frac{300\pi}{2\pi} = 150.$

Illustration 3.

If the position vector $(\vec{r})$ of the point is $(\hat{i} + 2\hat{j} + 3\hat{k})$ m and its angular $(\vec{\omega})$ is $(\hat{i} - \hat{j} + \hat{k})$ rad/s, then find the linear velocity of the particle.

Solution:

Linear velocity $\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (-5\hat{i} - 2\hat{j} + 3\hat{k})$ m/s

Illustration 4.

A particle starts rotating from rest according to the formula $\theta = \frac{3t^3}{20} - \frac{t^2}{3}$ radian. Calculate-

(a) the angular velocity at the end of 5 seconds.

(b) angular acceleration at the end of 5 seconds.

Solution:

(a) Angular velocity $\omega = \frac{d\theta}{dt} = \frac{d}{dt}\left[\frac{3t^3}{20} - \frac{t^2}{3}\right] = \frac{3}{20} \times 3t^2 - \frac{1}{3} \times 2t = \frac{9t^2}{20} - \frac{2t}{3}$

Angular velocity at the end of 5 seconds

$= \frac{9}{20} \times 5 \times 5 - \frac{2}{3} \times 5 = \frac{225}{20} - \frac{10}{3} = 11.25 - 3.33 = 7.92$ rad/s.

(b) Angular acceleration : $\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left[\frac{9t^2}{20} - \frac{2t}{3}\right] = \frac{9}{20} \times 2t - \frac{2}{3} = \frac{9t}{10} - \frac{2}{3}$

Angular acceleration at the end of 5 seconds : $\alpha = \frac{9 \times 5}{10} - \frac{2}{3} = 4.5 - 0.67 = 3.83$ rad/s^2.

Illustration 5.

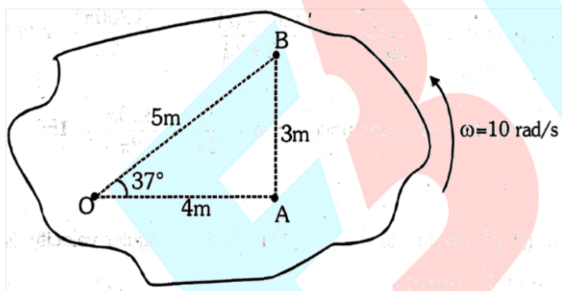
A wheel of perimeter 220 cm rolls on a levelled road at a speed of 9 km/h. How many revolutions does the wheel make per second ?

Solution:

Frequency $n = \frac{\omega}{2\pi} = \frac{v}{2\pi r} = \frac{9 \times \frac{5}{18}}{22} = \frac{25}{22}$ rev/s = 1.136 rev/s.

Illustration 6.

A rigid lamina is rotating about an axis passing perpendicular to its plane through point O as shown in the figure.



The angular. velocity of point B w.r.t. A is

(A) 10 rad/s (B) 8 rad/s (C) 6 rad/s (D) 0

Ans. (A)

Solution:

In a rigid body, angular velocity of any point w.r.t. any other point is constant and is equal to the angular velocity of the rigid body.

BEGINNER'S BOX - 1

1. A disc rotates about a fixed axis. Its angular velocity ω varies with time according to the equation $\omega = at + b$.
Initially at t = 0 its angular velocity is 1.0 rad/s and angular position is 2 rad; at the instant t = 2 s, angular velocity is 5.0 rad/s. Determine angular position θ and angular acceleration α when t = 4 s.

2. A wheel of radius 1.5 m is rotating at a constant angular acceleration of 10 rad/s^2. Its initial angular speed is $\left(\frac{60}{\pi}\right)$ rpm. What will be its angular speed and angular displacement at t = 2.0 s?

3. A rigid body rotates about a fixed axis with variable angular speed $\omega = A - Bt$ where A and B are constant. Find the angle through which it rotates before it comes to rest.

4. A car has wheels of radius 0.30 m and is travelling at 36 m/s. Calculate :-

(a) the angular speed of the wheel.

(b) If the wheels describe 40 revolutions before coming to rest with a uniform acceleration.

(i) find its angular acceleration and
(ii) the distance covered.

5. A child's top is spun with angular acceleration $\alpha = 4t^3 - 3t^2 + 2t$ where t is in seconds and a is in radians per second-square. At t =0, the top has angular velocity ω_0 = 2 rad/s and a reference line on it is at an angular position θ_0 = 1 rad.
Statement I : Expression for angular velocity ro = (2 + t;2- t3 + t4) ~ad/s
Statement II : Expression for angular position e = (1 + 2t-3t2 + 4e) rad
(A) Only statement-1 is true
(B) Only statement-11 is true
(C) Both of them are true
(D) None of them are true

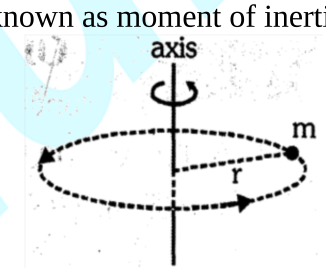
6. On account of the rotation of earth about its axis :-
(A) the linear velocity of objects at equator is greater than that at other places
(B) the angular velocity of objects at equator is more than that of objects at poles
(C) the linear velocity of objects at all places on the earth is equal, but angular velocity is different
(D) the angular velocity and linear velocity are uniform at all places

7. A fly wheel originally at rest is to attain an angular velocity of 36 rad/s in 6 s. The total angle it turns through which it turns in the 6 s is :-
(A) 54 radian (B) 108 radian (C) 6 radian (D) 216 radian

8. A rotating rod starts from rest and acquires a rotational speed n = 600 revolutions/minute in 2 seconds with constant angular acceleration. The angular acceleration of the rod is :-
(A) 10π rad/s^2 (B) 5π rad/s^2 (C) 15π rad/s^2 (D) None of these

4. MOMENT OF INERTIA

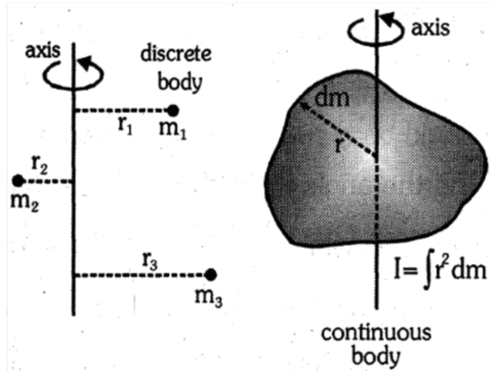
- The measure of the property by virtue of which a body revolving about an axis opposes any change in rotational motion is known as moment of inertia.



- The moment of inertia of a particle with respect to an axis of rotation is equal to the product of its mass and the square of its distance from the rotational axis.
$I = mr^2$, r = perpendicular distance from the axis of rotation

- Moment of inertia of a system of particles

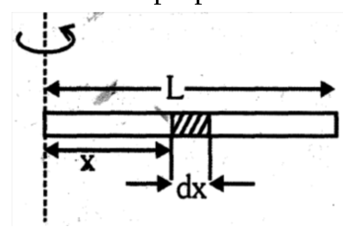
$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ... = \sum mr^2$$



- For a continuous body I = $\int r^2 dm$
- Moment of inertia depends on : (a) mass of the body
  (b) mass distribution of the body
  (c) Position of axis of rotation
- Moment of inertia does not depend on :-
  UNIT : SI : kg-m^2 CGS : g–cm^2
  Dimensions : $[M^1L^2T^0]$
- As the distance of mass increases from the rotational axis, the moment of inertia (M.I.) increase.

Moment of Inertia for symmetrical mass distribution

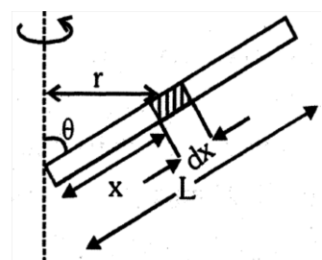
- **Ex. :** Moment of inertia of a rod about an axis passing through its end and perpendicular to length. If the mass of the rod is 'M' & mass of element is dm then



$$Dm = \frac{M}{L}dx$$

$$I = \int r^2\,dm = \int_0^L x^2 \frac{M}{L}dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2}{3}$$

- **Ex. :** Moment of inertia of a rod about an axis inclined at angle 'θ' with the rod & passing through one end.



$$r = x\sin\theta$$

so $$I = \int r^2\,dm = \int_0^L x^2\sin^2\theta\frac{M}{L}dx$$

$$= \frac{M}{L}\sin^2\theta\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2\sin^2\theta}{3}$$

5. RADIUS OF GYRATION (K)

It is the distance from the rotation axis where the mass of the object could to be assumed to be concentrated without altering the moment of inertia of the body about that axis.

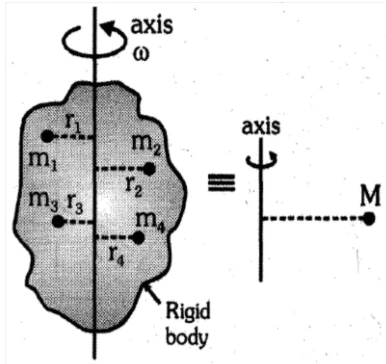
If the mass m of the body were actually concentrated at a distance K from the axis, the moment of inertia about that axis would be mK^2.

$$K = \sqrt{\frac{I}{m}}$$

K has no meaning without axis of rotatin
K is a scalar quantity

The radius of gyration has dimensions of length and is measured in appropriate units of length such as meters

Ex.



for the situation shown $K = \sqrt{\dfrac{m_1r_1^2 + m_2r_2^2 + ...+ m_n r_n^2}{m_1 + m_2 + ...m_n}}$

if $m_1 = m_2 = m_3 \ldots\ldots = m$ then, $M = mn$ $K = \sqrt{\dfrac{r_1^2 + r_2^2 + ...r_n^2}{n}}$ n = total number of particles

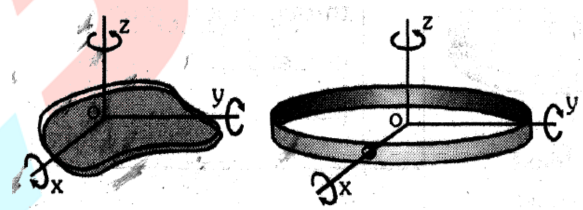
6. THEOREMS OF MOMENT OF INERTIA

- **Theorem of perpendicular axes (applicable only for two dimensional bodies or plane laminae)**

The moment of inertia (MI) of a plane lamina about an axis perpendicular to its plane is equal to the sum of its moments of inertia about any two mutually perpendicular axes in its own plane intersecting each other at the point through which the perpendicular axis passes.

$I_z = I_x + I_y$

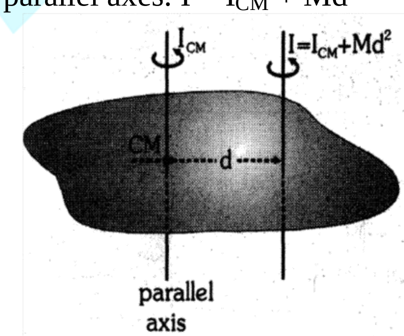
where I_x = MI of the body about X-axis
I_y = MI of the body about Y-axis
I_z = MI of the body about Z-axis



It is applicable only for two dimensional bodies and cannot be employed for three dimensional bodies.

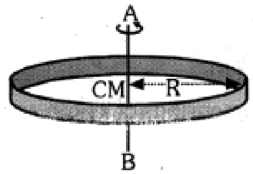
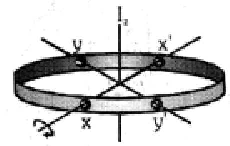
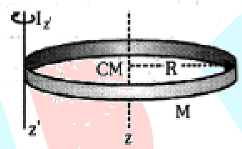
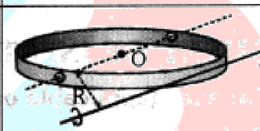
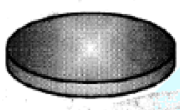
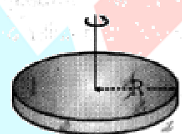
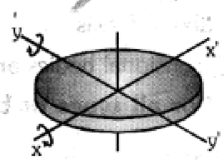
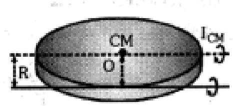
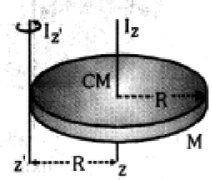
- **Theorem of parallel axes (for all types of bodies)**

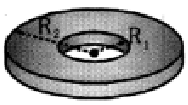
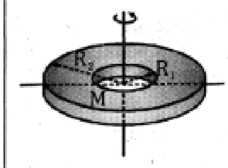
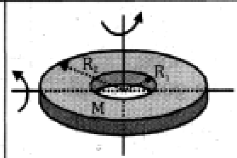
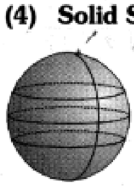
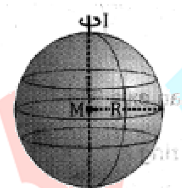
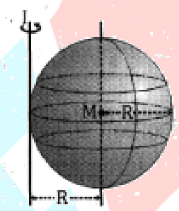
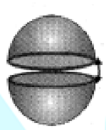
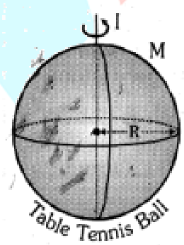
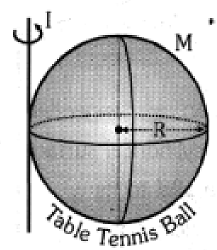
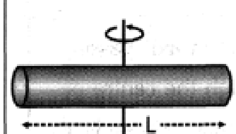
Moment of inertia of a body about any axis is equal to the moment of inertia about a parallel axis passing through the centre of mass plus product of mass of the body and the square of the distance· between these two parallel axes. $I = I_{CM} + Md^2$

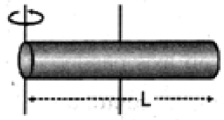
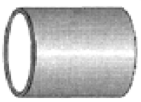
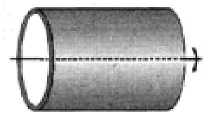
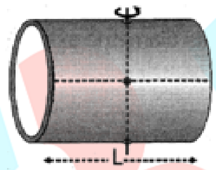
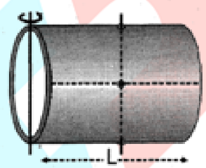
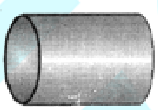
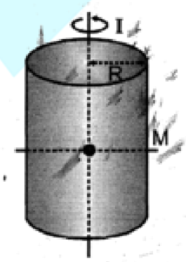
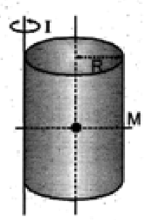
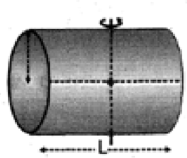


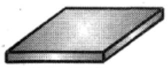
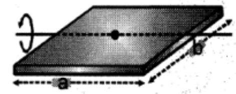
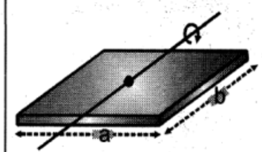
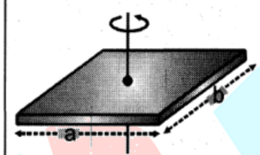
I_{CM} = Moment of inertia about the axis passing through the centre of mass.

7. MOMENT OF INERTIA OF SOME REGULAR BODIES

Type of body		Position of axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
(1) Ring  M = Mass R = Radius	(a)	About an axis perpendicular to its plane and passing through the centre		MR^2	R
	(b)	About the diametric axis		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(c)	About an axis tangential to the ring and perpendicular to its plane		$2MR^2$	$\sqrt{2}R$
	(d)	About an axis tangential to the ring and lying in its plane		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}\,R$
(2) Disc  M = Mass R = Radius	(a)	About an axis passing through the centre and perpendicular to its plane		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(b)	About a diametric axis		$\frac{MR^2}{4}$	$\frac{R}{2}$
	(c)	About an axis tangential to the disc and lying in its plane		$\frac{5}{4}MR^2$	$\frac{\sqrt{5}}{2}R$
	(d)	About an axis tangential to the disc and perpendicular to its plane		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$

Type of body	Position of axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
(3) Annular disc  M = Mass R_1 = Inner Radius R_2 = Outer Radius	(a) About an axis passing through the centre and perpendicular to the plane of disc		$\frac{M}{2}\left[R_1^2+R_2^2\right]$	$\sqrt{\frac{R_1^2+R_2^2}{2}}$
	(b) About a diametric axis		$\frac{M}{4}\left[R_1^2+R_2^2\right]$	$\frac{\sqrt{R_1^2+R_2^2}}{2}$
(4) Solid Sphere  M = Mass R = Radius	(a) About its diametric axis		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
	(b) About a tangent to the Sphere		$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$
(5) Thin Hollow Sphere  (Thin spherical shell) M = Mass R = Radius (Thickness negligible)	(a) About its diametric axis		$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$
	(b) About a tangent to the sphere		$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$
(6) Thin Rod  **Thickness is negligible w.r.t. length**	(a) About an axis passing through the centre of mass and perpendicular to its length		$\frac{ML^2}{12}$	$\frac{L}{\sqrt{12}}$

Type of body		Position of axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
M = Mass L = Length	(b)	About an axis passing through one end and perpendicular to its length		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$
(7) Thin Hollow Cylinder  M = Mass R = Radius L = Length	(a)	About its geometrical axis which is parallel to its length		MR^2	R
	(b)	About an axis which is perpendicular to its length and passes through its centre of mass		$\frac{MR^2}{2}+\frac{ML^2}{12}$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{12}}$
	(c)	About an axis perpendicular to its length and passing through one end of the cylinder		$\frac{MR^2}{2}+\frac{ML^2}{3}$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{3}}$
(8) Solid Cylinder M = Mass R = Radius L = Length 	(a)	About its geometrical axis, which is parallel to its length		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
	(b)	About an axis tangential to the cylindrical surface and parallel to its geometrical axis		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
	(c)	About an axis passing through the centre of mass and perpendicular to its length		$\frac{ML^2}{12}+\frac{MR^2}{4}$	$\sqrt{\frac{L^2}{12}+\frac{R^2}{4}}$

Type of body	Position of axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
(9) Rectangular Plate  M = Mass a = Length b = Breadth	(a) About an axis passing through its centre of mass and perpendicular to side b in its plane		$\frac{Mb^2}{12}$	$\frac{b}{2\sqrt{3}}$
	(b) About an axis passing through its centre of mass and perpendicular to side a in its plane.		$\frac{Ma^2}{12}$	$\frac{a}{2\sqrt{3}}$
	(c) About an axis passing through the centre of mass and perpendicular to its plane		$\frac{M(a^2+b^2)}{12}$	$\sqrt{\frac{a^2+b^2}{12}}$

GOLDEN KEY POINTS

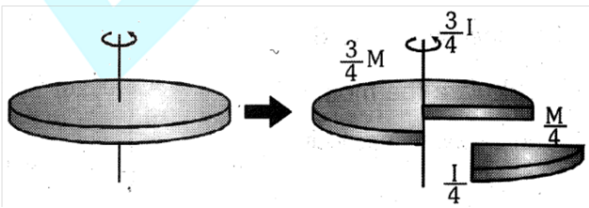
- Moment of inertia is not a vector as direction CW or ACW is not to be specified and also not a scalar as it has different values in different directions (i.e. about different axes), It is a tensor quantity.

- If a wheel is to be made by using two different materials then for larger moment of inertia, larger density material should lie in the periphery .

- Radius of gyration depends on
  (i) Axis of rotation
  (ii) Distribution of mass of body

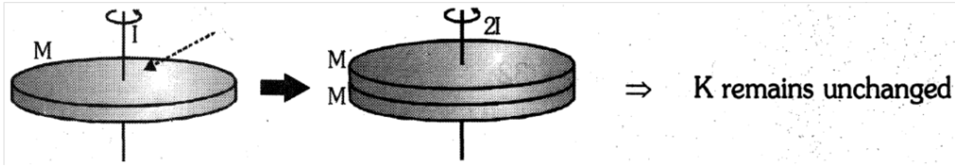
- Radius of gyration does not depend on
  (i) Mass of the body
  (ii) Angular quantities (angular displacement, angular velocity etc.)

- For Symmetrical separation radius of gyration remains unchanged .

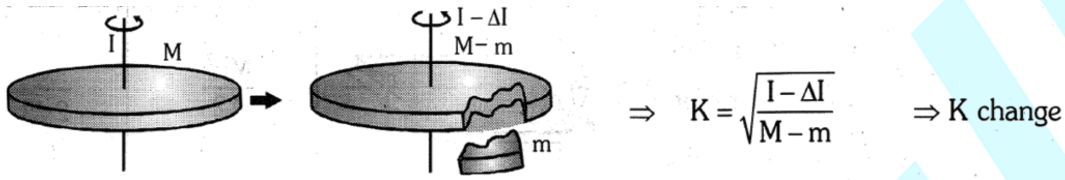


$\Rightarrow$ K remains unchanged

- For Symmetrical attachment radius of gyration remains unchanged.

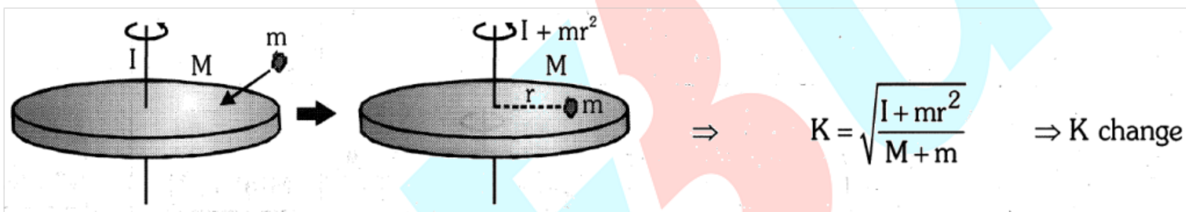


- For asymmetrical separation radius of gyration will change.



Here ΔI = M. I. of detached mass w.r.t. same axis

- For asymmetrical attachment radius of gyration will change.

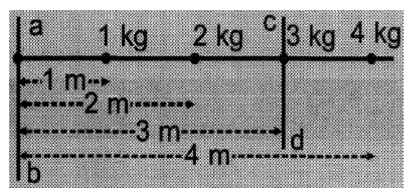


Illustrations

Illustration 7.

Calculate the M.I. of the given particles system about axis a–b and c–d

Solution:



$I_{a-b} = 1 \times (1)^2 + 2 \times (2)^2 + 3 \times (3)^2 + 4 \times (4)^2 = 100$ kg-m^2

$I_{c-d} = 1 \times (2)^2 + 2 \times (1)^2 + 3 \times (0)^2 + 4 \times (1)^2 = 100$ kg-m^2

Illustration 8.

Calculate the moment of inertia about the rotational axis XX' in following figures.

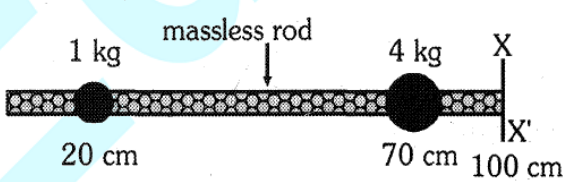


fig (a)

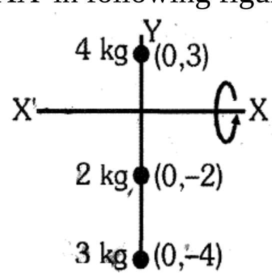


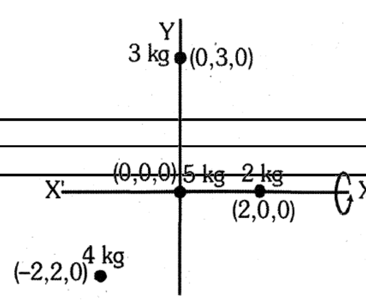
fig (b) (not to scale)

Solution:

(a) $I_{xx'} = 4 \times (0.3)^2 + 1 \times (0.8)^2 = 1$ kg-m^2

(b) $I_{xx'} = 4 \times (3)^2 + 2 \times (2)^2 + 3 \times (4)^2 = 92$ kg-m^2

Illustration 9.



Four bodies of masses 5 kg, 2 kg, 3 kg, and 4 kg are respectively placed at positions (0,0,0), (2,0,0), (0, 3, 0) and (–2, –2, 0). Calculate the moment of inertia of the system of bodies about x- axis, y-axis and z-axis respectively.

Solution:

$I_x = 3 \times (3)^2 + 4 \times (2)^2 = 43$ units

$I_y = 2 \times (2)^2 + 4 \times (2)^2 = 24$ units

$I_z = 2 \times (2)^2 + 3 \times (3)^2 + 4\ (f\sqrt{2}) = 8 + 27 + 32$

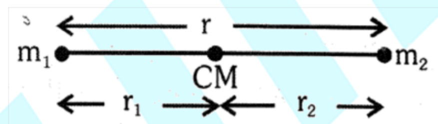
$= 67$ units ($I_z = I_x + I_y = 43 + 24 = 67$)

Illustration 10.

Two masses m_1 and m_2 are placed at a separation r. Find out the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the line joining the masses.

Solution:

$m_1r_1 = m_2r_2$ and $r_1 + r_2 = r \Rightarrow r_1 = \dfrac{m_2 r}{m_1 + m_2}$, $r_2 = \dfrac{m_1 r}{m_1 + m_2}$

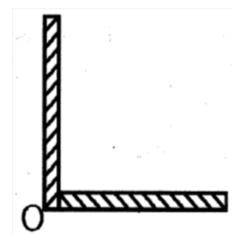


Moment of inertia $I = m_1r_1^2 + m_2r_2^2 = m_1\left(\dfrac{m_2 r}{m_1 + m_2}\right)^2 + m_2\left(\dfrac{m_1 r}{m_1 + m_2}\right)^2 = \left(\dfrac{m_1 m_2}{m_1 + m_2}\right)r^2$

Note : Here $I = \mu r^2$ where μ (reduced mass) $= \dfrac{m_1 m_2}{m_1 + m_2}$.

Illustration 11.

Two identical rods each of mass M and length L are kept according to figure. Find the moment of inertia of rods about an axis passing through O and perpendicular to the plane of rods. **[AIPMT (Mains) 2004]**



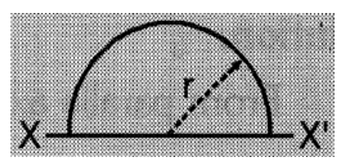
Solution:

Θ Moment of inertia of each rod about an axis passing through an end $= \dfrac{ML^2}{3}$

$$\therefore I_{\text{given system}} = \frac{ML^2}{3} + \frac{ML^2}{3} = \frac{2ML^2}{3}.$$

Illustration 12.

A uniform wire of length f and mass M is bent in the shape of a semicircle of radius r as shown in figure. Calculate moment of inertia about the axis XX'



Solution.

Length of the wire $\lambda = \pi r \Rightarrow r = \dfrac{l}{\pi} \Rightarrow$ $I_{xx'} = \dfrac{Mr^2}{2} = \dfrac{Ml^2}{2\pi^2}$

Illustration 13.

Calculate the moment of inertia of an annular disc about an axis which lies in the plane of the disc and tangential to the (i) inner circle and (ii) outer circle. Mass of the disc is M and its inner radius is R_1 and outer radius is R_2.

Solution:

(i) M.I. about an axis tangential to the inner circle is $I_{AB} = \dfrac{M}{4}(R_1^2 + R_2^2) + MR_1^2$

(ii) M.I. about an axis tangential to the outer circle is $I_{CD} = \frac{M}{4}(R_1^2 + R_2^2) + MR_2^2$

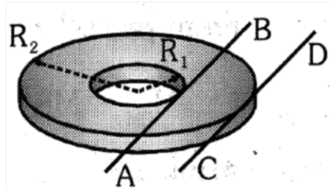
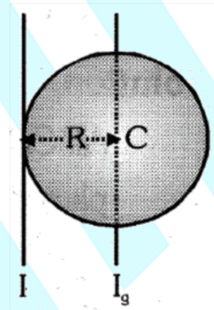


Illustration 14.

The moment of inertia of a sphere is 40 kg-m^2 about its diametric axis. Determine the moment of inertia about any tangent.



Solution:

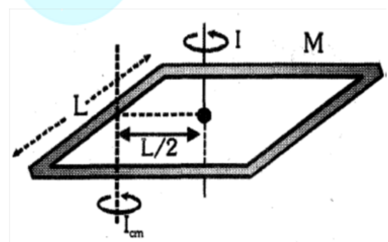
Given that $\frac{2}{5}MR^2 = 40 \Rightarrow MR^2 = 100$ kg-m^2.

By theorem of parallel axes,

$I = I_{CM} + MR^2 = \frac{2}{5}Mr^2 + MR^2 = \frac{7}{5}Mr^2 = \frac{7}{5} \times 100 = 140$ kg-m^2.

Illustration 15.

Four rods are arranged in the form of a square. Calculate the moment of inertia of the system of rods about an axis passing through the centre and perpendicular to the plane of the square (Assume that each rod has a mass M and length L)



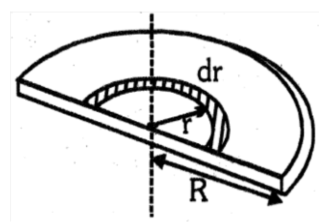
Solution:

From parallel axes theorem,

$$I = 4I_{CM} + 4M\left(\frac{L}{2}\right)^2 = 4\left(\frac{ML^2}{12}\right) + 4\frac{ML^2}{4} = \frac{3}{4}ML^2.$$

Illustration 16.

Calculate the moment of Inertia of a semicircular disc of mass M and radius R about an axis passing through its centre and perpendicular to its plane.



Solution:

Let us assume a ring of radius 'r' & thickness 'dr'

$$dm = \frac{M}{\frac{\pi R^2}{2}}(\pi r dr) = \frac{2Mrdr}{R^2}$$

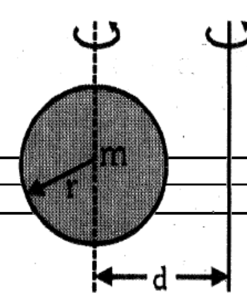
$$I = \int r^2 dm = \int_0^R r^2 \frac{2Mr}{R^2} dr = \frac{2M}{R^2}\int_0^R r^3 dr = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R \Rightarrow I = \frac{MR^2}{2}$$

Illustration 17.

The radius of gyration of a solid sphere of radius r about a certain axis is r. Calculate the distance of that axis from the centre of the sphere.

Solution:

From parallel axes theorem,

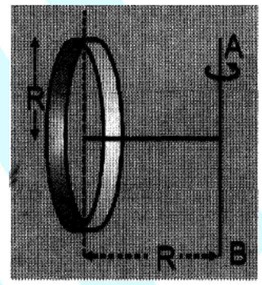


$\because \quad I = I_{CM} + md^2 \quad \therefore\ mr^2 = \frac{2}{5}mr^2 + md^2$

$\Rightarrow \quad d = \sqrt{\frac{3}{5}}r = \sqrt{0.6}\ r.$

Illustration 18.

Find the moment of inertia of the ring shown in figure about the axis AB.



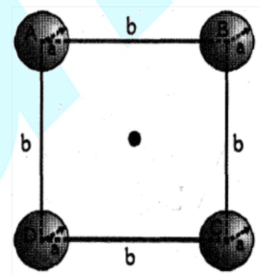
Solution:

From parallel axes theorem,

$$I_{AB} = I_{CM} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2.$$

Illustration 19.

Four spheres each of diameter 2a and mass M are placed with their centres lying on the four corners of a square of side b. Calculate the moment of inertia of the system about one side of the square taken as the axis.



Solution:

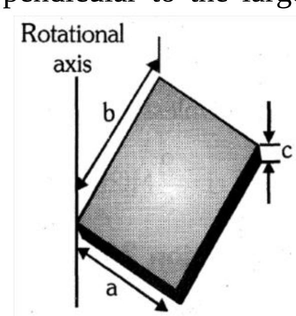
ABCD is square of side b and four spheres each of mass M and radius a are placed at the four corners. Suppose we have to calculate the moment of inertia of the system about the side BC as the axis. Therefore the moment of inertia of the system (M.I.).

= M.l. of sphere at A about BC+Ml of sphere at B about BC+Ml of sphere at C about BC+Ml of sphere at D about BC

$$= \left(\frac{2}{5}Ma^2 + Mb^2\right) + \frac{2}{5}Ma^2 + \frac{2}{5}Ma^2 + \left(\frac{2}{5}Ma^2 + Mb^2\right) = \frac{8}{5}Ma^2 + 2Mb^2 = \frac{2}{5}M(4a^2 + 5b^2).$$

Illustration 20.

The uniform solid block shown in figure has mass M and edge dimensions a, b, and c. Calculate its rotational inertia about an axis passing through one corner and perpendicular to the large faces.



Solution :

Use the parallel - axes theorem. The rotational inertia of a rectangular slab about an axis through the centre and perpendicular to the large face is given by

$$I_{cm} = \frac{M}{12}(a^2 + b^2)$$

A parallel axis through a corner is at distance $h = \sqrt{(a/2)^2 + (b/2)^2}$

Form the centre, so $\quad I = I_{cm} + Mh^2 = \frac{M}{12}(a^2 + b^2) + \frac{M}{4}(a^2 + b^2) = \frac{M}{3}(a^2 + b^2).$

Illustration 21.

Four spheres each of mass M and diameter 2a are placed at the corners of square of side bas shown below. Ca1culate the moment of inertia about

(a) axis BB' (b) axis AA'

Solution:

(a) Axis BB' passes through the centre and perpendicular to the plane

$$I_{BB'} = 4 \times I_{CM} + 4M\left[\frac{b}{\sqrt{2}}\right]^2 = 4\times\frac{2}{5}Ma^2 + 4M\times\frac{b^2}{2} = 4M\left[\frac{2}{5}a^2+\frac{b^2}{2}\right]$$

(b) Axis AA' passes through the centre of the spheres

$$I_{AA'} = 4I_{CM} + 2M\left[\frac{b}{\sqrt{2}}\right]^2 = 4\times\frac{2}{5}Ma^2 + 2M\times\frac{b^2}{2} = \frac{8}{5}Ma^2 + Mb^2$$

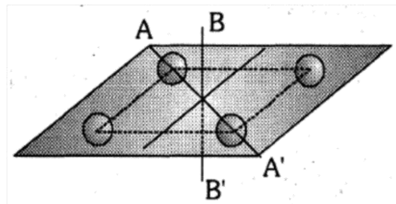
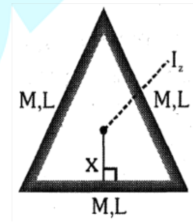


Illustration 22.

Three rods are arranged in the form of an equilateral triangle. Calculate the M.I. about an axis passing through the geometrical centre and perpendicular to the plane of the triangle (Assume that mass and L respectively).

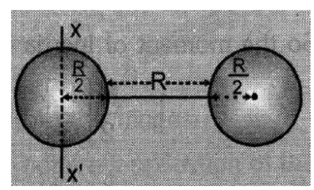


Solution:

$$I = 3I_{CM} + 3M\,x^2 = \frac{3ML^2}{12} + 3M\left(\frac{L}{2\sqrt{3}}\right)^2 = \frac{ML^2}{4}+\frac{ML^2}{4} = \frac{ML^2}{2}.$$

Illustration 23.

Diameter of each spherical shell is R and mass M they are joined by a light and massless rod. Calculate the moment of inertia of the system about xx' axis.

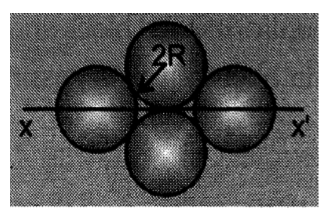


Solution:

$$I_{system} = \frac{2}{3}M\left(\frac{R}{2}\right)^2 + \left[\frac{2}{3}M\left(\frac{R}{2}\right)^2 + M(2R)^2\right] = \frac{1}{3}MR^2 + 4MR^2 = \frac{13}{3}MR^2$$

Illustration 24.

The moment of inertia of a sphere about its diameter is I. Four such spheres are arranged as shown in figure. Find the moment of inertia of the system about the axis XX'. (radius of each sphere is 2R).



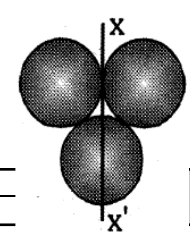
Solution:

Unless explicitly mentioned any sphere should be assumed to be solid.

$$I = \frac{2}{5}M(2R)^2 \quad \text{or,} \quad M(2R)^2 = \frac{5}{2}I \Rightarrow I_{system} = I + 2[I + M(2R)^2] + I = 2I + 2\times\frac{5}{2}I + I = 9I.$$

Illustration 25.

Adjoining diagram shows three rings, each of which has a mass M and radius R. Find the moment of inertia of this system about the axis XX'

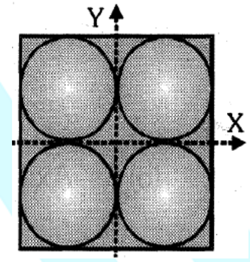


Solution:

$$I_{system} = 2 \times I_{upper} + I_{lower} = 2 \times \frac{3}{2}MR^2 + \frac{1}{2}MR^2 = \frac{7}{2}MR^2$$

Illustration 26.

Four holes of radius R are cut from a thin square plate of side 4R and mass M. Determine the moment of inertia of the remaining portion about Z-axis.



Solution:

M = Mass of the square plate before the holes were cut.

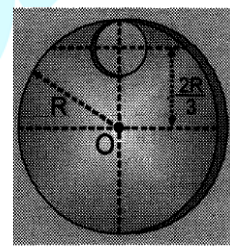
Mass of each hole $m = \left[\frac{M}{16R^2}\right]\pi R^2 = \frac{\pi M}{16}$.

$\therefore$ Moment of inertia of the remaining portion,

$$I = I_{square} - 4I_{hole} = \frac{M}{12}(16R^2 + 16R^2) - 4\left[\frac{mR^2}{2} + m(\sqrt{2}R)^2\right] = \frac{8}{3}MR^2 - 10mR^2 = \left[\frac{8}{3} - \frac{10\pi}{16}\right]MR^2.$$

Illustration 27.

A thin uniform disc has a mass 9M and. radius R. A disc of radius $\frac{R}{3}$ is cutoff as shown in figure. Find the moment of inertia of the remaining disc about an axis passing through O and perpendicular to the plane of disc.



Solution:

As the mass is uniformly distributed on the disc,

so mass density (per unit area) $= \frac{9M}{\pi R^2}$. Mass of removed portion $= \frac{9M}{\pi R^2} \times \left[\frac{R}{3}\right]^2 = M$

so the moment of inertia of the removed portion about the stated axis by theorem of parallel axes is:

$$I_1 = \frac{M}{2}\left[\frac{R}{3}\right]^2 + M\left[\frac{2R}{3}\right]^2 = \frac{MR^2}{2} \quad \text{...(i)}$$

The moment of inertia of the original complete disc about the stated axis is I_2 the $I_2 = 9M\frac{R^2}{2}$ ...(ii)

So the moment of inertia of the left over disc shown in fig. is $I_2 - I_1$.

i.e. $I_2 - I_1 = 4MR^2$.

BEGINNER'S BOX - 2

1. Point masses of 1, 2, 3 and 10 kg are lying at the points (0, 0), (2m, 0) (0, 3m) and(–2m, –2m) respectively in x-y plane. Find the moment of inertia of this system about y-axis. (in kg-m^2).

2. The moment of inertia of a ring and a disc of same mass about their diameters are same. What will be the ratio of their radii ?

3. Two rings have their moments of inertia in the ratio 4 : 1· and their diameters are in the ratio 4 : 1. Find the ratio of their masses.

4. Two bodies of masses 1 kg and 2 kg are attached to the ends of a 2 m long weightless rod. This system is rotating about an axis passing through the middle point of rod and perpendicular to length. Calculate the M.I. of system.

5. If the moment of inertia of a disc about an axis tangential and parallel to its surface be I, then what will be the moment of inertia about an axis tangential but perpendicular to the surface ?

6. A disc is recast into a thin hollow cylinder of same radius. Which will have larger moment of inertia? Explain.

7. Why spokes are fitted in a cycle wheel ?

8. A circular disc A of radius r is made from an iron plate of thickness t and another circular disc B of radius 4r is made from an iron plate of thickness t/4. Find the relation between the moment of inertia l_A and 1_B .

9. A square plate of side λ has mass per unit area μ. Find its moment of inertia about an axis passing through the centre and perpendicular to its plane.

10. Three masses m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of side a. What is the moment of inertia of the system about an axis along the median passing through m_1 ?

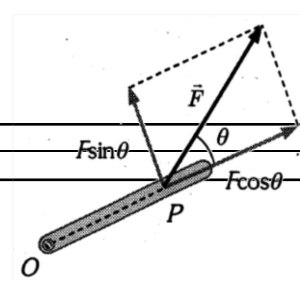
11. Two rings having the same radius and mass are placed such that their centres are at a common point and their planes are perpendicular to each other. Find the moment of inertia of the system about an axis passing through the centre and perpendicular to the plane of one of the rings.
(M = mass of each ring and R = radius).

12. Moment of inertia of a sphere about its diameter is $2/5MR^2$. What is its moment of inertia about an axis perpendicular to its two diameters and passing through their point of intersection ?

8. TORQUE

Torque is the rotational analogue of force and expresses the tendency of a force applied to an object to cause rotation in it about a given point.

Consider a rod pivoted at the point O. A force $\vec{F}$ is applied on it at the point P. The component F cosθ of the force along the rod is

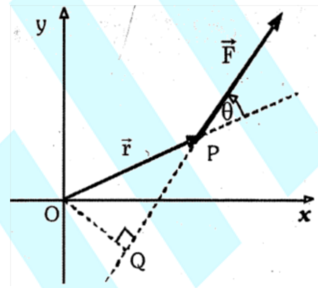


counterbalanced by the reaction force of the pivot and cannot contribute in rotating the rod. It is the component F $\sin\theta$ of the force perpendicular to the rod, which is responsible for rotation of the rod. Moreover, farther is the point P from O, where the force is applied easier it is to rotate the rod.

This explains why handle on a door is attached as far as possible away from the hinges.

Magnitude of torque of a force is proportional to the product of distance of point of application of the force from the pivot and the magnitude of the perpendicular component F $\sin\theta$ of the force. Denoting torque by symbol τ, the distance of point of application of force from the pivot by r, we can write

$$\tau_0 \propto rF\sin\theta$$



Since rotation has sense of direction, torque should also be a vector. Its direction is given by the right hand rule. Now we can express torque by the cross product of $\vec{r}$ and $\vec{F}$.

$$\tau_o = \vec{r} \times \vec{F} \qquad [1]$$

The magnitude of torque

Torque = Force x perpendicular distance of line of action of force from the axis of rotation.

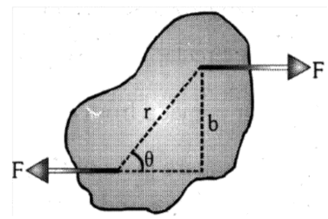
$$\tau = r\,F\sin\theta$$

so Moment of force is known as torque

- **Unit of torque :** N–m (same as that of work or energy)
- **Dimensions of torque :** $[ML^2T^{-2}]$
- **Unit of torque (N–m) cannot be written as joule, because joule is used specifically for work or energy**
- **Couple of forces·**

  When two forces of equal magnitude act on different points and in opposite directions with distinct lines of action these force form a couple. This couple tries to rotate body. Moment of couple

  $$\tau = Fr\sin\theta = Fb$$



- Rotation of a door about a hinge, rotation of grinding wheel about a pivot or unbolting a nut by a pipe–wrench can be cited as examples involving torque. These examples represent rotational effects.

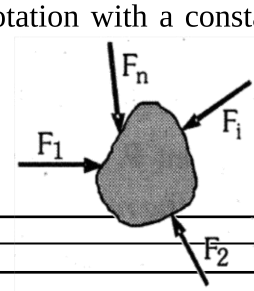
$$\tau = \text{constant} \Rightarrow Fr\sin\theta = \text{constant} \Rightarrow F = \frac{\text{constant}}{r\sin\theta}$$

Longer the arm and greater the value of $\sin\theta$, lesser will be the force required for producing desired rotational effect. This fact explains is why, it is much easier to rotate a body about a given axis when the force is applied at maximum distance from the axis of rotation and normal to the arm.

9. ROTATIONAL EQUILIBRIUM

A rigid body is said to be in a state of rotational equilibrium if its angular acceleration is zero. Therefore a body in rotational equilibrium must either be at rest or in rotation with a constant angular velocity.

If a rigid body is in rotational equilibrium under the action of several coplanar forces, the resultant torque of all the forces about any axis perpendicular to the plane containing the forces must be zero.



In the figure a body is shown under the action of several external coplanar forces F_1, F_2,…..F_i, and F_n.

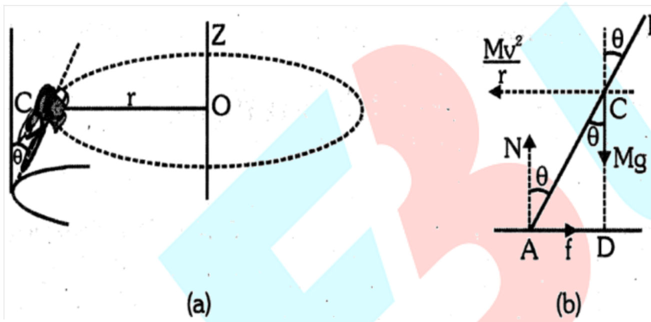
$$\sum \vec{\tau}_p = 0$$

Here P is a point in the plane of the forces about which we calculate torque' of all the external forces acting on the body.

Note: (i) A body cannot be in rotational equilibrium under the action of a single force unless the line of action passes through the axis of rotation.
(ii) If a body is in rotati6'nal equilibrium under the action of three forces, the lines of action of the three forces must be either concurrent or parallel.

10. BENDING OF A CYCUST ON A HORIZONTAL TURN

Suppose a cyclist is moving at a speed υ on a circular horizontal road of radius r. Consider the cycle and the rider together as the system. The centre of mass C (figure a) of the system is going in a circle with the centre at O and radius r.



As seen from a frame rotating with the same angular velocity as the system is in equilibrium

So $F_{net} = 0$ and $\tau_{net} = 0$

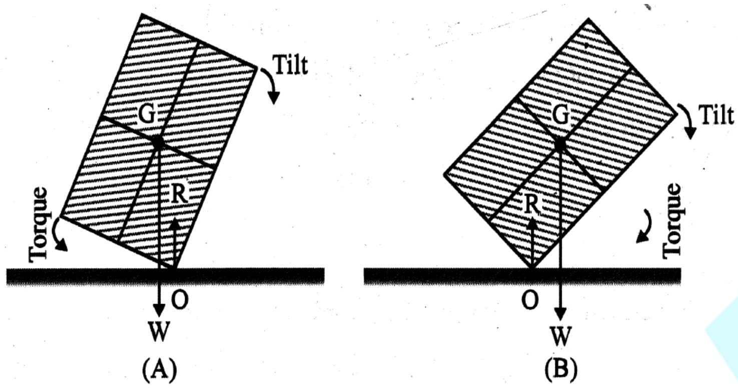
for rotational equilibrium

$$\tau_A = 0 \Rightarrow \quad Mg\,(AD) = \frac{M\upsilon^2}{r}(CD) \Rightarrow \frac{AD}{CD} = \frac{\upsilon^2}{rg} \Rightarrow \tan\theta = \frac{\upsilon^2}{rg}$$

Thus the cyclist bends at an angle $\tan^{-1}\left(\frac{\upsilon^2}{rg}\right)$ with the vertical.

GOLDEN KEY POINTS

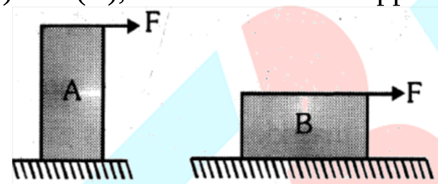
- Torque is an axial vector, i.e. Its direction is always perpendicular to the plane containing vector $\vec{r}$ and $\vec{F}$ and its direction is determined by the right hand screw rule.
- Generally positive sign is given to all torques acting on turn a body anticlockwise and a minus to all torques tending to turn it clock wise.
- Wrench with longer handle is more useful than one with shorter handle.
- A rigid body is said to be in equilibrium, if it is in translational as well as rotational equilibrium. To analyze such problems conditions for both the equilibriums must be applied.



On tilting, a body Will restore its initial position due to torque of weight about the point O till the line of action of weight passes through O. Upon on tilting it further a the body topples due to torque of weight about O; the line of action of action of weight does not pass through the base.

Toppling :

For the shown situations (A) and (B), more chances of toppling is for (A).

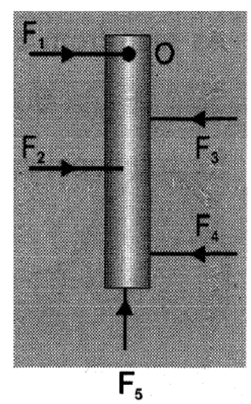


In case of toppling, normal reaction must pass through the end point about which the body topples.

Illustrations

Illustration 28.

Different forces are applied on a pivoted scale as shown. Which of the following forces will produce torque ?



Solution:

$F_1 \neq 0$, $\tau_1 = 0$

(As line of action of force passes from the pivoted point O)

$F_2 \neq 0$ and produces τ_2 (Anti clock wise)

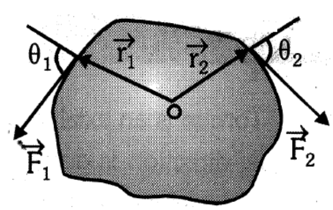
$F_3 \neq 0$ produces τ_3 (Clockwise)

$F_4 \neq 0$ produces τ_4 (Clockwise)

$F_5 \neq 0$, $\tau_5 = 0$; as line of action of force passes from the pivoted point O.

Illustration 29.

The body in shown figure is pivoted at O and two forces act on it as shown.



(a) Find an expression for the net torque on the body about the pivot.

(b) If $r_1 = 1.30$ m, $r_2 = 2.15$ m, $F_1 = 4.20$ N, $F_2 = 4.90$ N, $\theta_1 = 75°$ and $\theta_2 = 60°$, what is the net torque about the pivot?

Solution:

(a) Take a torque that tends to cause a counter clockwise rotation to be positive and a torque that tends to cause a clockwise rotation to be negative. Thus positive torque of magnitude

$r_1F_1 \sin\theta_1$ is associated with $\vec{F}_1$ and a negative torque of magnitude $r_2F_2 \sin\theta_2$ is associated with $\vec{F}_2$. Both of these are about O. The net torque about O is $\tau = r_1F_1\sin\theta_1 - r_2F_2\sin\theta_2$

(b) Substitute the given values to obtain τ = (1.30 m) (4.20 N) sin 75°– (2.15 m) (4.90 N) sin 60°
= –3.85 N-m.

Illustration 30.

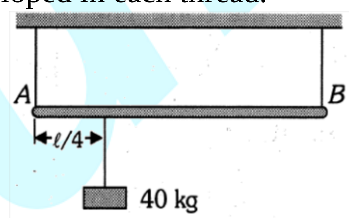
A force 10 (–k) N acts on the origin of the coordinate system. Find the torque about the point (1m, –1m, 0).

Solution:

Torque $\vec{\tau} = \vec{r} \times \vec{F} = \left[(0-1)\hat{i} + (0+1)\hat{j} + (0-0)\hat{k}\right] \times (-10)\hat{k} = 10(-\hat{i} - \hat{j}) = -10(\hat{i} + \hat{j})$ N-m

Illustration 31.

A uniform rod of 20 kg is hanging in a horizontal position with the help of two threads. It also supports a 40 kg mass as shown in the figure. Find the tensions developed in each thread.

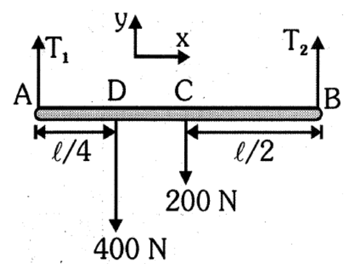


Solution:

Free body diagram of the rod is shown in the figure.

Translational equilibrium requires

$\sum F_y = 0 \Rightarrow$ $T_1 + T_2 = 400 + 200 = 600$ N ...(i)



Rotational equilibrium : Applying the condition about A, we get T_2.

$\sum \vec{\tau}_A = \vec{0} \Rightarrow$ $-400(\lambda/4) - 200(\lambda/2) + T_2\lambda = 0$

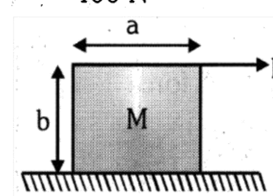
$T_2 = 200$ N

From equation (i)

$T_1 = 400$ N.

Illustration 32.

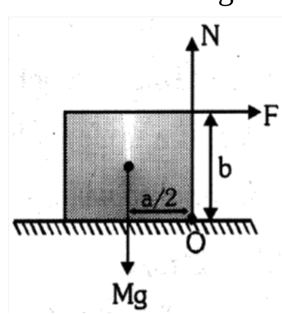
Find the minimum value of F for the block to topple about an edge.



Solution:

When the block is about to topple the normal reaction N shifts to the edge through O.

FBD during toppling

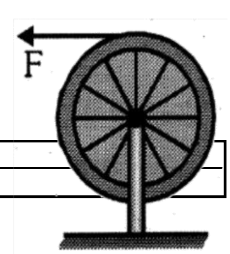


Taking torque about O

$F(b) = Mg\left(\frac{a}{2}\right)$ $\Rightarrow$ $F_{min} = \frac{Mga}{2b}$.

Illustration 33.

A string is wrapped around the rim of a wheel of moment of inertia 0.20 kg-m^2 and radius 20 cm. The wheel is free to rotate about its axis. Initially the



wheel is at rest. The string is now pulled by a force of F = 20 N. Find the angular velocity of the wheel after 5 s.

Solution:

Angular impulse = change in angular momentum

$$(\tau t = I\omega - 0) \Rightarrow \omega = \frac{\tau t}{I} = \frac{FRt}{I} = \frac{20 \times 0.2 \times 5}{0.2} = 100 \text{ rad/s}$$

Illustration 34.

A fixed pulley. of radius 20 cm and moment of inertia 0.32 kg.m^2 about its axle has a massless cord wrapped around its rim. A 2 kg mass M is attached to the end of the cord. The pulley can rotate about its axis without any friction. Find the acceleration of the mass M. (Assume g = 10 m/s^2)

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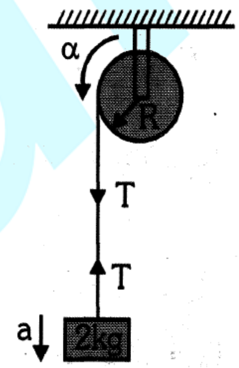
Solution:



For the motion of the block $2g - T = 2a$

For the motion of the pulley $\tau = TR = I\alpha$

$$\because a = \alpha R \qquad \therefore T = \frac{Ia}{R^2} \Rightarrow 2g - \frac{Ia}{R^2} = 2 \Rightarrow a = \frac{g}{1 + \frac{I}{2R^2}}$$

$$\Rightarrow a = \frac{10}{1 + \frac{0.32}{2 \times 0.2 \times 0.2}} = \frac{10}{1+4} \; \frac{10}{5} = 2 \text{ m/s}^2$$

BEGINNER'S BOX - 3

1. A force $\vec{F} = (\hat{i} - 2\hat{j} + 5\hat{k})$ N is acting at a point $\vec{r} = (-2\hat{i} + \hat{j} + 3\hat{k})$ m. Find torque about the origin.

2. The density of a rod AB increases continuously from A to B. Is it easier to set it in rotation by clamping it at A and applying a perpendicular force at B or by clamping it at B and applying the force at A ? Explain your answer.

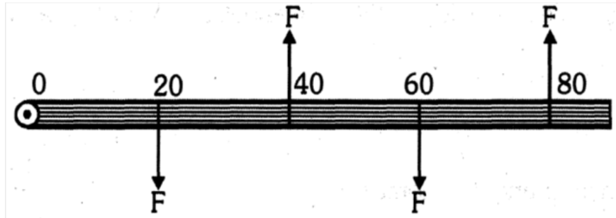
3. A uniform disc of radius 20 cm and mass 2 kg can rotate about a fixed axis through the centre and perpendicular to its plane. A massless cord is round along the rim or the disc. If a uniform force of 2 newtons is applied on the cord, tangential acceleration of a point on the rim of disc will be :-

(A) 1 m/s^2 (B) 2 m/s^2 (3) 3.2 m/s^2 (4) 1.6 m/s^2

4. A door 1.6 m wide requires a force of 1 N to be applied at the free end to open or close it. The force that is required to be applied at a point 0.4 m distant from the hinges for opening or closing the door is :-

(A) 1.2 N (B) 3.6 N (C) 2.4 N (D) 4 N

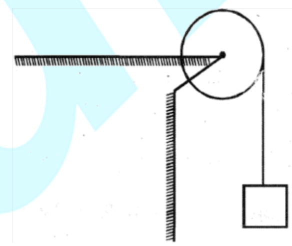
5. Four equal and parallel forces are acting on a rod (as shown in figure at distances of 20 cm, 40 cm, 60 cm and 80 cm respectively from one end of the rod. Under the influence of these forces the rod



(1) is at rest
(2) experiences a torque
(3) experiences a linear motion
(4) experiences a torque and also a linear motion

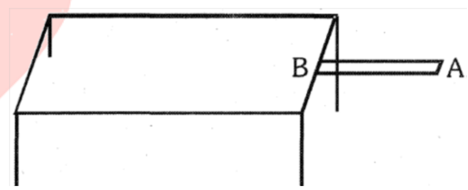
6. A weightless rod is acted upon by upward parallel forces of 2 N and 4 N at ends A and B respectively. The total length of the rod AB = 3m. To keep the rod in equilibrium a force of 6 N should act in the following manner:
(A) downward at any point between A and B
(B) downward at the midpoint of AB
(C) downward at a point C such that AC = 1 m
(D) downward at a point D such that BD = 1 m

7. Figure shows a uniform disc, with mass M = 2.4 kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block of mass m = 1.2 kg hangs from a massless cord which is wrapped around the rim of the disc. The tension in the cord is :



(A) 12 N
(B) 20 N
(C) 24 N
(D) 6 N

8. A thin uniform stick of length λ and mass m is held horizontally with its end B hinged on the edge of a table. Point A is suddenly released. The acceleration of the centre of mass of the stick at the time of release is :- **[AIPMT (Mains) 2007]**



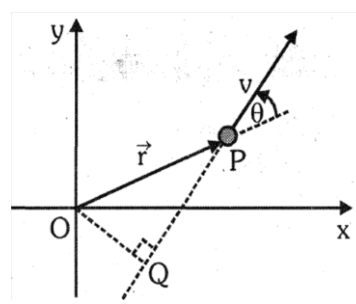
(A) $\frac{3}{4}g$
(B) $\frac{3}{7}g$
(C) $\frac{2}{7}g$
(D) $\frac{1}{7}g$

11. **ANGULAR MOMENTUM**

Angular momentum of a particle

Angular momentum $\vec{L}_o$ about the origin O of a particle of mass m moving with velocity $\vec{v}$ is defined as the moment of its linear momentum $\vec{p} = m\vec{v}$ about the point O.

$$\vec{L}_o = \vec{r} \times (m\vec{v})$$

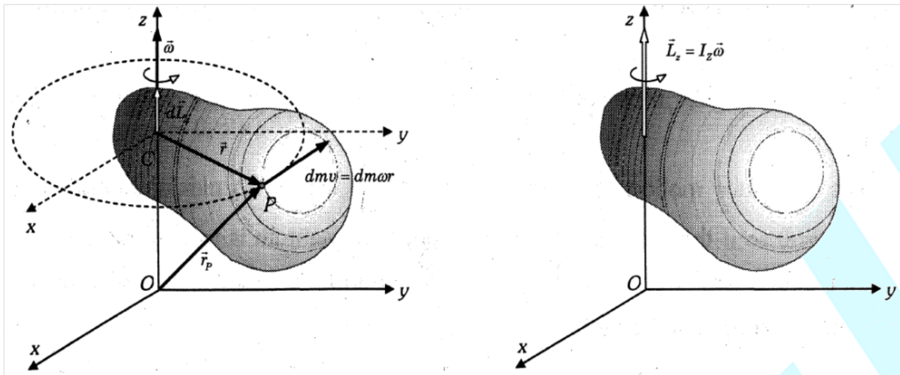


Angular Momentum of a Rigid Body

Angular momentum is a measure of the quantity of rotational motion in a body. The angular momentum of a system of particles is the sum of the angular moment all the particles forming the system. A rigid body is an assemblage of large number of particles maintaining their mutual distances intact under all circumstances.

Angular Momentum about an axis

The following figure shows a particle P of a rigid body rotating about the z-axis angular momentum $d\vec{L}_z = \vec{r} \times (dm\vec{v}) = r^2\, dm\vec{\omega}$. It is along the z-axis i.e. axis of rotation. In the next figure total angular momentum $\vec{L}_z = \int d\vec{L}_z = I_z\vec{\omega}$ about the axis of rotation is shown. It is also along the axis of rotation.



Relation between torque & angular momentum

For a rotating body

$$\vec{L} = I\vec{\omega} \Rightarrow \frac{d\vec{L}}{dt} - I\frac{d\vec{\omega}}{dt} \qquad [\because I \text{ is constant.}]$$

$$\frac{d\vec{L}}{dt} = I\vec{\alpha} = \vec{\tau} \qquad \left[\because \frac{d\vec{\omega}}{dt} = \vec{\alpha} \text{ and } \vec{\tau} = I\vec{\alpha}\right] -$$

The rate of change of angular momentum is equal to the net torque acting on the body. This expression is the rotational analogue of $\frac{d\vec{p}}{dt} = \vec{F}$ and hence is referred to as newtons II law for rotational motion.

Angular Impulse & Angular Impulse-Momentum principle

If a large torque acts on a body for a small time then angular impulse = $\vec{\tau}dt$

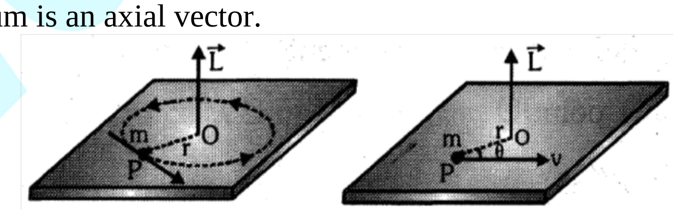
Angular impulse = $\vec{\tau}_{av}\Delta t = \Delta\vec{L}$ = change in angular momentum $\left[\because \vec{\tau} = \frac{\Delta\vec{L}}{\Delta t}\right]$

Action of angular impulse is to change the angular momentum. It has same units, dimensions and direction as the angular momentum.

So angular impulse given to a rigid body is equal to the change in its angular momentum. This statement is also known as angular impulse-momentum principle.

GOLDEN KEY POINT

- Angular momentum is an axial vector.



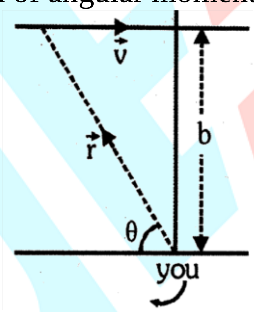
- As torque $(\vec{r} \times \vec{F})$ is defined as the 'moment of force', Angular momentum $(\vec{r} \times \vec{p})$ is also define-as the moment of linear momentum.
- In cartesean coordinates, angular momentum is expressed as :

$$\vec{L} = (\vec{r} \times \vec{p}) = m(\vec{r} \times \vec{v}) \; (\because \vec{p} = m\vec{v}) \; = m\left[(x\hat{i} + y\hat{j} + z\hat{k}) \times (v_x\hat{i} + v_y\hat{j} + v_z\hat{k})\right] \qquad \vec{L} = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix}$$

i.e. $\vec{L} = m[\hat{i}(yv_z - zv_y) - \hat{j}(xv_z - zv_x) + \hat{k}(xv_y - yv_x)]$

- As the magnitude of the angular momentum is L = mvr sinθ
  - (a) For θ = 0° or 180° i.e. $\vec{r}$ and $\vec{v}$ are parallel or anti-parallel ∴ sinθ = 0 ; L will be minimum. If the axis of rotation is on the line of motion the of moving particle then angular momentum is minimum and zero.
  - (b) For θ = 90°, i.e., angular momentum is maximum when $\vec{r}$ and $\vec{v}$ are orthogonal ∴ sinθ = 1 ; L will be maximum:
    i.e. in case of circular motion of a particle, angular momentum is maximum and is mvr.

- As $|\vec{L}|$ = mvr sin-θ so if the point about which angular momentum is to be determined is not on the line of motion i.e., θ ≠ 0 or 180°then $|\vec{L}| \neq 0$ i.e., a particle in translatory motion always has an angular momentum unless the point is on the line of motion
- Direction of angular momentum due to linear motion : Imagine ypur position at the point about which you want to determine the angular momentum spread the palm of your right hand along the position vector and curl the fingers in the direction of velocity of particle then the thumb of your right hand shows the direction of angular momentum (Right hand thumb rule).



$\vec{L} = mvb(-\hat{k})$ here $b = r \sin\theta$

$\vec{L} = mvr\sin\theta(-\hat{k}) \quad \Rightarrow \quad \vec{L} = m(\vec{r} \times \vec{v})$

a particle is moving parallel to X-axis or Y -axis or along any straight line with constant velocity then its angular momentum about the origin will remain constant.

Illustration 35

A particle of mass 0.01 kg having position vector $\vec{r} = (10\hat{i} + 6\hat{j})$ metre is moving 'with a velocity $5\hat{i}$ m/s. Calculate its angular momentum about the origin.

Solution

$\vec{p} = m\vec{v} = 0.01 \times 5\hat{i} = 0.05\hat{i} \quad \Rightarrow \quad \vec{L} = \vec{r} \times \vec{p} = (10\hat{i} + 6\hat{j}) \times 0.05\hat{i} = 0.3\,(-\hat{k}) = -0.3\,\hat{k}$ J/s.

Illustration 36.

A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad/s; the radius of the cylinder is 0.25 m. What is the magnitude of the angular momentum of the cylinder about its axis?

Solution

M = 20 kg, ω = 100 rad/s R = 0.25 m

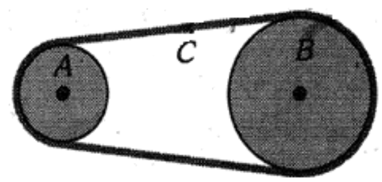
Moment of inertia of the cylinder about its axis

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 20 \times (0.25)^2 = 0.625n \text{ kg-m}^2$$

Angular momentum $L = I\omega = 0.625 \times 100 = 62.5$ J-s '

Illustration 37.

A belt moves two pulleys A and Bas shown in the figure. The pulleys are mounted on two fixed horizontal axles. Radii of the pulleys A and Bare 50 cm B are 50 cm and 80 cm respectively. Pulley A is driven at constant angular acceleration of 0.8 rad/s^2 until pulley B acquires an angular velocity of the pulleys.



(a) Find the acceleration of a point C on the belt and angular acceleration of the pulley B.
(b) How long does it take for the pulley B to acquire an angular velocity of 10 rad/s?

Solution:

Since the belt does not slide on the pulleys, magnitudes of velocity and acceleration of any point on the belt are same as that of any point on the periphery of either of the two pulleys.

$\vec{a}_T = \vec{\alpha} \times \vec{r}$ $a_C = \alpha_A r_A = \alpha_B r_B$

Substituting $r_A = 0.5$ m, $r_B = 0.8$ m and $\alpha_A = 0.8$ rad/s^2,

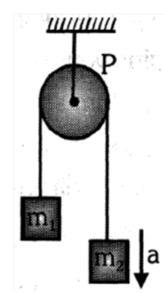
we have $a_C = 0.4$ m/s^2 and $\alpha_B = \frac{a_C}{r_B} = \frac{\alpha_A r_A}{r_B} = 0.5$ rad/s^2

Θ $\omega = \omega_0 + \alpha t \Rightarrow t = \frac{\omega_B - \omega_{Bo}}{\alpha_B}$

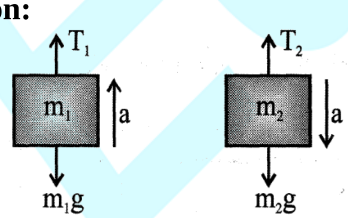
Substituting $\omega_{Bo} = 0$, $\omega_B = 10$ rad/s and $\alpha_B = 0.5$ rad/s^2, we have t = 20 s

Illustration 38.

In the figure, the blocks have unequal masses m_1 and m_2 ($m_1 < m_2$). m_2 has a downward acceleration a. The pulley P has a radius r, and some mass. The string does not slip on the pulley. Find the acceleration of block and angular acceleration of pulley. (I = moment of inertia of pulley)



Solution:



$T_1 - m_1g = m_1a$ ……(1)

$m_2g - T_2 = m_2a$ ……(2)

Total torque = $I\alpha$

$(T_2 - T_1)R = I\alpha$ ……(3)

$a = \alpha R$ ……(4)

By Eq. (1), (2) (3) and (4) $\Rightarrow$ a = $\dfrac{(m_2 - m_1)}{m_1 + m_2 + \dfrac{I}{r^2}} g$ $\quad \Rightarrow \quad$ $a = \dfrac{a}{R}$

Illustration 39.

The diameter of a disc is 0.5 m and its mass is 16 kg. What torque will increase its angular velocity from zero to 120 rotations/minute in 8 seconds?

Solution:

Moment of inertia of the disc $I = \dfrac{1}{2}MR^2 = \dfrac{1}{2} \times 16 \times \left[\dfrac{0.5}{2}\right]^2 = \dfrac{1}{2}$ kg-m^2

Angular velocity $\omega = \dfrac{120 \times 2\pi}{60} = 4\pi$ rad/s and change in angular momentum $\Delta L = I\omega - 0 = I\omega$

$\therefore$ Angular impulse is $\tau \times t = \Delta L \Rightarrow \tau = \dfrac{\Delta L}{t} = \dfrac{1}{8} \times \dfrac{1}{2} \times 4\pi = \dfrac{\pi}{4}$ N-m

Illustration 40.

During the launch from a board, a diver's angular speed about her centre of mass changes from zero to 6.20 rad/s in 220 ms. Her rotational inertia about the centre of mass is 12.0 kg-m^2. During the launch, what is the magnitude of :

(a) her average angular acceleration

(b) the average external torque acting on her from the board?

Solution:

(a) From the kinematic equation $\omega = \omega_0 + \alpha t$, we get $\alpha = \dfrac{\omega - \omega_0}{t} = \dfrac{6.20\,\text{rad/s}}{220 \times 10^{-3}\text{s}}$ = 28.2 rad/s^2

(b) If is the rotational inertia of the diver, then the magnitude of the torque acting on her is $\tau = I\alpha = (12.0\ \text{kg.m}^2)(28.2\ \text{rad/s}^2) = 3.38 \times 10^2$ N-m.

Illustration 41.

A spherical shell has a radius of 1.90 m. An applied torque of 960 N-m gives the shell an angular acceleration of 6.20 rad/s^2 about an axis passing through the centre of the shell. What are

(a) the rotational inertia of the shell about that axis and

(b) the mass of the shell?

Solution:

(a) Use $\tau = I\alpha$ where τ is the net torque acting on the shell, I is the rotational inertia of the shell, and α as its angular acceleration. This gives $I = \dfrac{\tau}{\alpha} = \dfrac{960\ \text{N}\cdot\text{m}}{6.20\ \text{rad/s}^2}$ = 155. Kg-m^2

(b) The rotational inertia of the shell is given by $I = \left(\dfrac{2}{3}\right)MR^2$ $\therefore$ $M = \dfrac{3I}{2R^2} = \dfrac{3(155\text{kg}\cdot\text{m}^2)}{2(1.90\text{m})^2}$ = 64.4 kg

BEGINNER'S BOX - 4

1. If the earth is a point mass of 6×10^{24} kg revolving around the sun at a distance of 1.5×10^8 km and in time of T= 3.14×10^7 seconds, then the angular momentum of the earth around the sun is:

(A) 1.2×10^{18} kg-m^2/s
(B) 1.8×10^{20} kg-m^2/s
(C) 1.5×10^{37} kg-m^2/s
(D) 2.7×10^{40} kg-m^2/s

2. If a particle moves along a straight line whose equation is y = mx then what will be its angular momentum about the origin ? Explain your answer.

3. Two gear wheels which are meshed together have radii of 0.50 cm and 0.15 cm. The number of revolutions undergone by the smaller one when the larger goes through 3 revolutions is :-
(A) 5 revolutions (B) 20 revolutions (C) 1 revolution (D) 10 revolutions

4. A uniform circular disc of mass 200 g and radius 4.0 cm is rotated about one of its diameter at an angular speed of 10 rad/s. Find the kinetic energy of the disc and its angular momentum about the axis of rotation.

12. CONSERVATION OF ANGULAR MOMENTUM (COAM)

If the angular impulse of all the external force about an axis vanishes over a certain time interval, then the angular momentum of the system about the same axis remains unchanged in that time interval.

If $\sum \int_{t_1}^{t_2} \vec{\tau}_o \, dt = 0$, we have $\vec{L}_i = \vec{L}_f$

$$\vec{\tau} = \frac{\Delta \vec{L}}{\Delta t} \Rightarrow \tau = \frac{L_f - L_i}{\Delta t} = \frac{I\omega_f - I\omega_i}{\Delta t} = \frac{I(\omega_f - \omega_i)}{\Delta t}$$

If the resultant external torque acting on a system is zero then the total angular momentum of the system remains constant.

If $\tau = 0$ then $\frac{\Delta L}{\Delta t} = 0 \Rightarrow$ L = constant $\Rightarrow$ $L_f = L_i$ or $I_1\omega_1 = I_2\omega_2$

If a system is isolated from its surrounding any internal interaction between its different parts cannot alter its total angular momentum.

The condition of zero net angular impulse required for conservation of angular momentum can be realized in the following cases :

- If no external force acts, the angular impulse about all axes will be zero and hence angular momentum remains conserved about all axes.
- If net torque of all the external forces or torques of each individual force about an axis vanishes the angular momentum about that axis will be conserved.
- If all the external forces are finite in magnitude and the concerned time interval is infinitely small, the angular momentum remains conserved.

The principle of conservation of angular momentum governs a wide range of physical processes from subatomic to celestial world. The following examples explicate some of these applications.

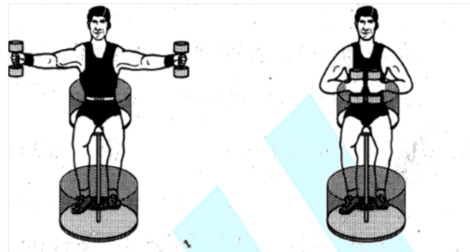
Examples Based On COAM

Spinning Ice Skater.

A spinning ice skater or a ballet dances can control her moment of inertia by extending or folding her arms and make use of conservation of angular momentum to perform their spins. In doing so no external force is needed and if we ignore effects of friction from the ground and the air, the angular momentum can be assumed to be conserved. When she spreads her hands or legs away from the spin axis, her moment of inertia increases therefore her angular velocity decreases and when she brings her hands or legs closer her moment of inertia decreases and consequently her angular velocity increases.

Student on rotating turntable

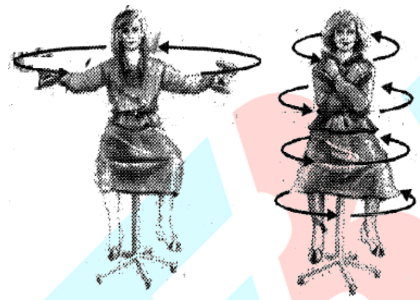
The student with the dumbbells and the turntable make an isolated system on which no external torque acts (if we ignore fraction in the bearings of the turntable and air resistance). Initially, the student has his arms stretched. When he draws the dumbbells close to his body, his angular velocity increases due to conservation of angular momentum.



Larger moment of inertia and smaller angular velocity

Smaller moment of inertia and larger angular velocity

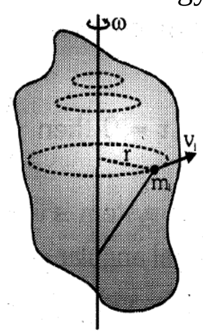
- If a lady skating on ice while spinning folds her arms then her M.l. decreases and consequently 'ω' increases.



13. ROTATIONAL KINETIC ENERGY (KINETIC ENERGY OF ROTATION)

- The energy possessed due to rotational motion of a body is known as rotational kinetic energy .

- A rigid body is rotating about an axis with a uniform angular velocity ω. The body is assumed to be composed of particles of masses m_l m_2 ....
The linear velocity, of the particles are $v_1 = \omega r_1$, $v_2 = \omega r_2$, ...
Therefore the total kinetic energy of the rotating body is

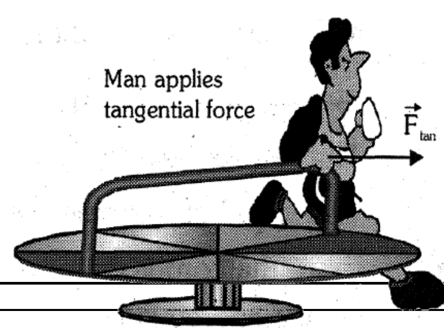


$$KE_r = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + ... = \frac{1}{2}(m_1r_1^2 + m_2r_2^2 + ...)\omega^2 = \frac{1}{2}I\omega^2$$

- It is a scalar quantity.
- Rotational kinetic energy KE_r = $\frac{1}{2}I\omega^2 = \frac{1}{2}I\times\frac{4\pi^2}{T^2} = \frac{1}{2}MK^2\omega^2$ = $\frac{1}{2}I\times\frac{v^2}{R^2} = \frac{L.\omega}{2} = \frac{L^2}{2I}$
- If external torque acting on a body is equal to zero ($\tau = 0$), L = constant $\Rightarrow E \propto \frac{1}{1} \propto \frac{1}{MK^2}$

- **Work and Power in Rotational Motion**

Suppose a tangential force $\vec{F}_{tan}$ acts on the rim of a pivoted disc [for example a man rotating a merry-go-round on a play ground, while simultaneously running along with it]. The disc rotates through an infinitesimal angle de about a fixed axis during an infinitesimal time interval dt.



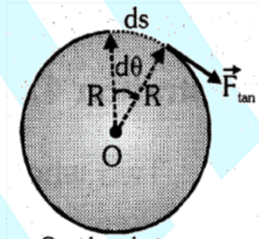
The work done by the force $\vec{F}_{tan}$ while any point on the rim moves a distance ds is $dW = F_{tan}\, ds$.
If dθ the corresponding is angular displacement, then $ds = Rd\theta$ and $\therefore dW = F_{tan}\, Rd\theta$
Torque due to the force $\vec{F}_{tan}$ is $\tau = F_{tan}R$ $\therefore dW = \tau d\theta$ ……(i)

$$W = \int_{\theta_1}^{\theta_2} \tau d\theta \text{ (work done by a torque)} \quad ....(ii)$$

If the torque remains constant while the angle changes by a finite amount $\Delta\theta = \theta_2 - \theta_1$ then

$$W = t(\theta_2 - \theta_1) = \tau\Delta\theta \quad \text{(work done by a constant torque)} \quad ....(iii)$$

Work done by constant torque is the product of torque and the angular displacement.



Overhead view
of merry-go-round

Let τ represent the net torque on the body so from equation $\tau = I\alpha$.
assuming the body to be rigid so that its moment of inertia I remains constant,

$$\tau d\theta = (I\alpha)d\theta = I\frac{d\omega}{dt}d\theta = I\frac{d\theta}{dt}d\omega = I\omega d\omega$$

Since t is the net torque, the integral in equation (ii) yields the total work done on the rotating body.

$$W_{total} = \int_{\omega_1}^{\omega_2} I\omega\, d\omega = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

When a torque executes work on a rotating rigid body, the kinetic energy of the body changes by an amount equal to the work done.

- **Work-energy theorem in rotational motion**

  Work done by the torque = Change in kinetic energy of rotation $W = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$

  The change in the rotational kinetic energy of a rigid body equals the work done by external torques exerted on the body. This equation. is analogous to equation corresponding to the work energy theorem applicable for a particle.

- **Rotational power**

  The power associated with the work done by torque acting on a rotating body :
  Divide both sides of equation $dW = \tau d\theta$ by the time interval dt during which the angular displacement de occurs.

  $\therefore \frac{dW}{dt} = \tau\frac{d\theta}{dt}$. Instantaneous rotational power $P_r = \tau\omega$. In general, $P_r = \vec{\tau}.\vec{\omega}$.

GOLDEN KEY POINTS

- If a system is isolated from its surrounding
  i.e. any internal interaction between different parts of a system cannot alter its total angular momentum.
  If $\tau = 0$ then $I_1\omega_1 = I_2\omega_2$
  $MK_1^2\, 2\pi n_1 = MK_2^2\, 2\pi n_2$ $(\because I = MK^2, \omega = 2\pi n)$
  $\Rightarrow K_1^2 n_1 = K_2^2 n_2$.
- The angular velocity of a planet about the sun increases due to decrease in I when it comes closer to the sun.
- The speed of the inner layers of the whirlwind in a tornado is alarmingly high.

- If external torque on a system is zero, then its angular momentum remains conserved. However the rotational kinetic energy is not conserved.

  $I_1\omega_1 = I_2\omega_2 \Rightarrow \frac{1}{2}I_1^2\omega_1^2 = \frac{1}{2}I_2^2\omega_2^2 \Rightarrow I_1 \times \frac{1}{2}I_1\omega_1^2 = I_2 \times \frac{1}{2}I_2\omega_2^2 \Rightarrow I_1K_{r_1} = I_2K_{r_2}$

  If $I_1 > I_2$ then $K_{r_1} < K_{r_2}$ and vice-versa.

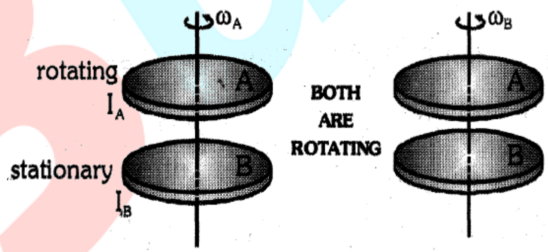
  So if the moment of inertia decreases, the rotational kinetic energy increases in absence of external torque and vice versa.
- A body rotating about a fixed axis essentially possesses rotating kinetic energy.
- Moment of inertia of a rigid body about a given axis is. numerically equal to twice its rotational kinetic energy

  when it is rotating with unit angular velocity. $\left(I = \frac{2K_r}{\omega^2}\right)$
- Rotational kinetic energy depends upon the axis of rotation.

Illustrations

Illustration 42.

Two wheels A and Bare placed coaxially. Wheel A has moment of inertia I_A and angular velocity ω_A while B is stationary. On clubbing them with a clutch they move jointly by an angular velocity 'ω' then find the M.I. of 'B'.



Solution:

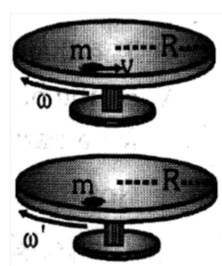
The phenomenon occurs in the absense of external torque, so applying law of COAM, we have

$\Theta\ I_A\ \omega_A = (I_A + I_B)\ \omega \quad \Rightarrow I_B = \frac{I_A\omega_A - I_A\omega}{\omega}$

$= I_A\left(\frac{\omega_A}{\omega} - 1\right)$

Illustration 43.

A cockroach of mass 'm' is moving with velocity v in anticlockwise sense on the rim of a disc of radius R. The M.I. of the disc about the axis is 'I' and it is rotating in clockwise direction with an angular velocity 'ω'. If the cockroach stops then calculate the angular velocity of disc.



Solution:

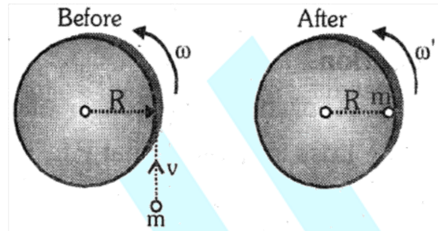
The phenomenon occurs in the absense of external torque, so applying law of COAM, we have

$I_{disc}\ \omega - mvR = (I_{disc} + mR^2)\ \omega' = \frac{I\omega - mvR}{I + mR^2}$.

Illustration 44.

A solid cylinder of mass 'M' and radius 'R' is rotating along its axis with angular velocity ω without friction. A particle of mass 'm' moving with velocity v collides against the cylinder and sticks to its rim. After the impact calculate angular velocity of cylinder.

Solution:



Initial angular momentum of cylinder = $I\omega$

Initial angular momentum of particle = mvR

Before collision the total angular momentum $L_1 = l\omega + mvR$

After collision the total angular momentum $L_2 = (I + mR^2)\omega'$

$L_1 = L_2 \Rightarrow (l + mR^2)\omega' = I\omega + mvR$.

New angular velocity $\omega' = \dfrac{I\omega - mvR}{I + mR^2}$

Note: Initial kinetic energy of the system = $\dfrac{1}{2}I\omega^2 + \dfrac{1}{2}mv^2$.

Final kinetic energy of the system = $\dfrac{1}{2}\,(I + mR^2)\omega'^2$

Illustration 45.

Keeping the mass of earth constant if its radius is halved then calculate the duration of the day.

$$I_1\omega_1 = I_2\omega_2 \quad \Rightarrow \frac{2}{5}MR^2 \times \frac{2\pi}{T_1} = \frac{2}{5}M\left(\frac{R}{2}\right)^2 \times \frac{2\pi}{T_2}$$

$$\Rightarrow T_2 = \frac{T_1}{4} \qquad \Theta\ T_1 = 24\text{ h} \qquad \therefore\ T_2 = 6\text{ h}.$$

Illustration 46.

Explain with reason that if ice melts at polar region then moment of inertia of earth increases, angular velocity ω decreases and the duration of the day become longer.

Solution:

If ice in the polar region melts, the resulting water will flow towards the equator and consequently moment of inertia of earth will increase because more mass has accumulated at equator which is at more distant from the rotational axis as compared to the poles.

So from law of COAM if I increases then ω decreases. Time period (T) = $\dfrac{2\pi}{\omega} \Rightarrow$ T increases.

Due to increase in time period, the duration of day and night will increase.

Illustration 47.

A rotating table has angular velocity 'ω' and moment of inertia I_1. A person of mass 'm' stands on the centre of rotating table. If the person moves a distance r along its radius then what will be the final angular velocity of the rotating table ?

Solution:

The process takes place in the absense of external torque, so law of COAM can be applied,

Initial angular momentum = Final angular momentum $I_1\omega_1 = (I_1 + mr^2)\,\omega_2 \Rightarrow \omega_2 = \dfrac{I_1\omega}{I_1 + mr^2}$.

Illustration 48.

A light rod carries three equal masses A, B and C as shown in the figure. What will be the velocity of B in the vertical position of the rod, if it is released from horizontal position as shown in the figure ?

(1) $\sqrt{\frac{8gl}{7}}$ (2) $\sqrt{\frac{4gl}{7}}$

(3) $\sqrt{\frac{2gl}{7}}$ (4) $\sqrt{\frac{10gl}{7}}$

Solution:

Applying law of conservation of mechanical energy

Loss in gravitational P.E. = Gain in rotational K.E. i.e.,

$$mg\frac{l}{3}+mg\left(\frac{2l}{3}\right)+mgl=\frac{1}{2}\left(m\left(\frac{l}{3}\right)^2+m\left(\frac{2l}{3}\right)+ml^2\right)\omega^2$$

$$\Rightarrow \omega = \sqrt{\frac{36g}{14l}} \Rightarrow v_B = \omega\lambda_B = \frac{2l}{3}\sqrt{\frac{36g}{14l}} = \sqrt{\frac{8gl}{7}}$$

Illustration 49.

A fly wheel (in shape of ring) of mass 0.2 kg and radius 10 cm is rotating with $\frac{5}{\pi}$ rev/s about an axis perpendicular to its plane passing through its centre. Calculate the angular momentum and kinetic energy of the fly wheel. **[AIPMT 2006]**

Solution:

Angular velocity $\omega = 2\pi \times \frac{5}{\pi} = 10$ rad/s

Moment of inertia $I = mr^2 = (0.2)(0.1)^2 = 2 \times 10^{-3}$ kg-m^2

Angular momentum= $I\omega = 2 \times 10^{-3} \times 10 = 2 \times 10^{-2}$ J-s or 2×10^{-2} kg-m^2/s

Kinetic energy = $\frac{1}{2}I\omega^2 = \frac{1}{2} \times 2 \times 10^{-3} \times (10)^2 = 0.1$ J.

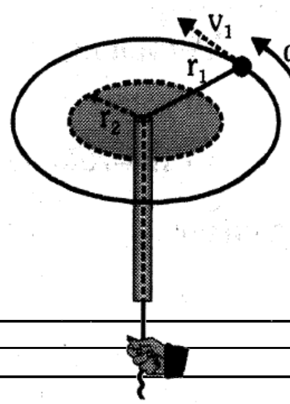
Illustration 50.

A point mass is tied to one end of a cord whose other end passes through a vertical hollow tube, caught in one hand. The point mass is being rotated in a horizontal circle of radius 2 m with a speed of 4 m/s. The cord is then pulled down so that the radius of the circle reduces to 1 m. Compute the new linear and angular velocities of the point mass and also the ratio of kinetic energies in the initial and final states.

Solution:

The force on the point mass due to cord is radial. and hence the torque about torque about of rotation is zero. Therefore, the angular momentum must remain constant as the cord is shortened. Let mass of the particle be m let it rotate initially in circle of radius r_1 with linear velocity v_1 and angular velocity ω_1. Further let the corresponding quantities in the final state be radius r_2, linear velocity v_2 and angular velocity ω_2.

Θ Initial angular momentum = Final angular momentum



$\therefore \quad I_1\omega_1 \; I_2\omega_2 \Rightarrow \quad mr_1^2\frac{v_1}{r_1} = mr_2^2\frac{v_2}{r_2} \Rightarrow r_1v_1 = r_2v_2$

$\therefore \quad v_2 = \frac{r_1}{r_2}v_1 = \frac{2}{1} \times 4 = 8$ m/s and $\omega_2 = \frac{v_2}{r_2} = \frac{8}{1} = 8$ rad/s

$$\frac{\text{Final K.E.}}{\text{Initial K.E.}} = \frac{\frac{1}{2}I_2\omega_2^2}{\frac{1}{2}I_1\omega_1^2} = \frac{mr_2^2 \times \left[\frac{v_2}{r_2}\right]^2}{mr_1^2 \times \left[\frac{v_1}{r_1}\right]^2} = \frac{v_2^2}{v_1^2} = \frac{(8)^2}{(4)^2} = 4.$$

Illustration 51.

A thin meter scale is kept vertical by placing its lower end hinged on floor. It is allowed to fall. Calculate the velocity of its upper end when it hits the floor .

Solution:

Loss in PE $\left(\frac{mgl}{2}\right)$ = gain in rotational KE $\left(\frac{1}{2}I\omega^2 = \frac{1}{2}\frac{ml^2}{3}\times\frac{v^2}{l^2}\right) \Rightarrow v = \sqrt{3gl}$.

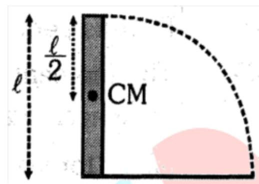


Illustration 52.

If the rotational kinetic energy of a body is increased by 300% then determine the percentage increase in its angular momentum.

Solution:

Percentage increase in angular momentum = $\frac{L_2 - L_1}{L_1} \times 100$

Now, $L = \sqrt{2IE}$ $\quad \therefore L \propto \sqrt{E} \Rightarrow E_1 = E$ and $E_2 = E + \frac{300}{100}E \Rightarrow E_2 = 4E$

Increase in angular momentum = $\frac{\sqrt{E_2} - \sqrt{E_1}}{\sqrt{E_1}} \times 100 = \frac{\sqrt{4E} - \sqrt{E}}{\sqrt{E}} \times 100 = 100\%$

Illustration 53.

The power output of an automobile engine is advertised to be 200 H.P. at 6000 rpm. What is the corresponding torque generated.

Solution:

$P = 200$ H.P. $= 200 \times 746 = 1.49 \times 10^5$ W

& $\quad \omega = 6000$ rev/min $= 6000 \times \frac{2\pi}{60} = 628$ rad/s

$\tau = \frac{P}{\omega} = \frac{1.49\times10^5}{628} = 237.5$ N–m

BEGINNER'S BOX - 5

1. The angular velocity of a body changes from ω_1 to ω_2 without applying external torque. Then find the ratio of initial radius of gyration to final radius of gyration.

2. A stone is attached to a string and is revolved in a circular path with constant angular velocity ω. In this state its angular momentum is L. If the length of the string is reduced to half and again it is revolved with the same angular velocity ω then find its angular momentum.

3. A thin uniform thin rod of mass m and length λ is suspended from one of its ends and is rotated at the rate of f rotations per second. Find the rotational kinetic energy of the rod.

4. A wheel of moment of inertia 10 kg-m^2 rotates at the rate of 10 revolutions per minute. Find the work done in increasing its. speed to 5 times of its initial value.

5. A solid ball of mass 1 kg and radius 3 cm is rotating about its own axis with an angular velocity of 50 radians per second. Find its kinetic energy of rotation.

6. When sand is poured gradually on the edge of a rotating disc, what will be the change in its angular velocity?

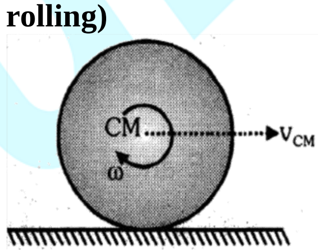
7. The moment of inertia of a wheel about its axis of rotation is 3.0 MKS units. Its kinetic energy is 600 J. Find its period of rotation.

8. The moment of inertia of a wheel is 1000 kg-m^2. At a given instant, its angular velocity is 10 rad/s. After the wheel rotates through an angle of 100 radians, its angular velocity increases to 100 rad/s. Calculate the,
(a) torque applied on the wheel. (b) increase in the rotational kinetic energy.

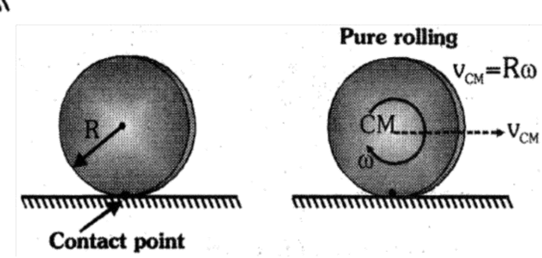
14. ROWNG MOTION

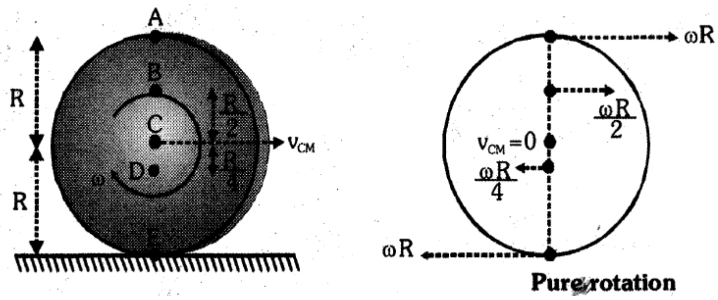
When a body performs translatory motion as well as rotatory motion combinedly then it is said to undergo rolling motion. The velocity of the centre of mass represents linear motion while angular velocity about the centroidal axis represents rotatory motion.
Total energy in rolling = Translatory kinetic energy + Rotatory kinetic energy.

- **Rolling without slipping (Pure rolling)**

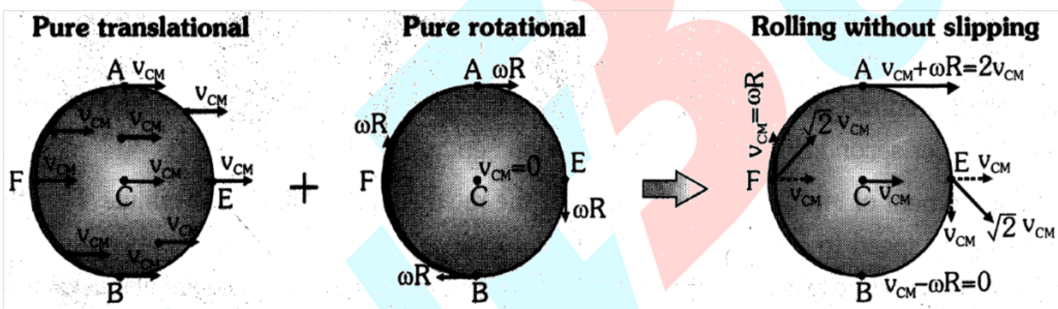


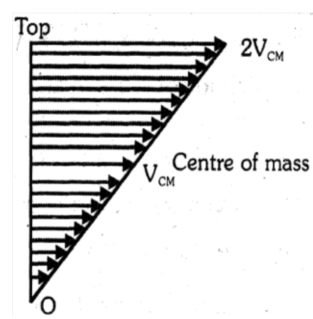
- If the relative velocity of the point of contact of the rolling body with the surface is zero then it is known as pure rolling.
- If a body is performing rolling then the velocity of any point of the body with respect to the surface is given by
$\vec{v} = \vec{v}_{CM} + \vec{\omega} \times \vec{R}$





$$
\begin{array}{lllllll}
v_A & = & v_{CM} + \omega R & = & v_{CM} + v_{CM} & = & 2v_{CM} \\
v_B & = & v_{CM} + \dfrac{\omega R}{2} & = & v_{CM} + \dfrac{v_{CM}}{2} & = & \dfrac{3}{2}v_{CM} \\
v_C & = & v_{CM} + 0 & = & v_{CM} & & \\
v_D & = & v_{CM} - \dfrac{\omega R}{4} & = & v_{CM} - \dfrac{v_{CM}}{4} & = & \dfrac{3}{4}v_{CM} \\
v_E & = & v_{CM} - \omega R & = & v_{CM} - v_{CM} & = & 0
\end{array}
$$





Pure Translatory Motion + Pure Rotatory Motion = Rolling motion
For pure rolling motion of the above mentioned body, we have

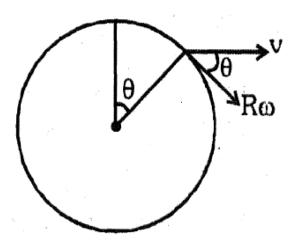
$v_A = 2v_{CM}$

$v_E = \sqrt{2}\ v_{CM}$

$v_F = \sqrt{2}\ v_{CM}$

$v_B = 0$

- **Velocity at a point on the rim of a rolling sphere**



$v_{net} = \sqrt{v^2 + R^2\omega^2 + 2vR\omega\cos\theta}$

For pure rolling v = Rω

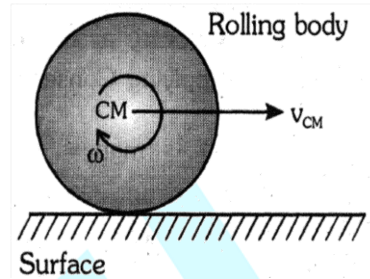
$= \sqrt{v^2 + v^2 + 2v^2\cos\theta} = \sqrt{2v^2 + 2v^2\cos\theta}$

$= \sqrt{2v^2(1+\cos\theta)} = 2v\cos\dfrac{\theta}{2}$ (since $1 + \cos\theta = 2\cos^2\dfrac{\theta}{2}$),

$v_{net} = 2v\cos\dfrac{\theta}{2}$

- **Rolling Kinetic Energy**

Rolling Kinetic Energy $E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \Rightarrow \frac{1}{2}mv^2 + \frac{1}{2}mK^2\left(\frac{v^2}{R^2}\right)$



Rolling Kinetic Energy $E = \frac{1}{2}mv^2\left(1+\frac{K^2}{R^2}\right)$

$$E_{translation} : E_{rotation} : E_{Total} = 1 : \frac{K^2}{R^2} : 1+\frac{K^2}{R^2}$$

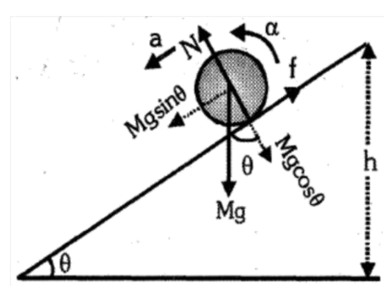
Body	$\frac{K^2}{R^2}$	$\frac{E_{trans}}{E_{rotation}} = \frac{1}{\left(\frac{K^2}{R^2}\right)}$	$\frac{E_{trans}}{E_{total}} = \frac{1}{\left(1+\frac{K^2}{R^2}\right)}$	$\frac{E_{rotation}}{E_{total}} = \frac{K^2/R^2}{\left(1+\frac{K^2}{R^2}\right)}$
Ring	1	1	$\frac{1}{2}$	$\frac{1}{2}$
Disc	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Solid sphere	$\frac{2}{5}$	$\frac{5}{2}$	$\frac{5}{7}$	$\frac{2}{7}$
Spherical shell	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{3}{5}$	$\frac{2}{5}$
Solid cylinder	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Hollow cylinder	1	1	$\frac{1}{2}$	$\frac{1}{2}$

Angular momentum of a body in combined translational and rotational motion

Suppose a body is rotating about an axis passing through its centre of mass with an angular velocity ω_{cm} and moving translationally with a linear velocity v_{CM}. Then, the angular momentum of the body about a point P outside the body in the lab frame is given by, $\vec{L}_P = \vec{L}_{cm} + \vec{r}\times\vec{p}_{cm}$, where $\vec{r}$ is the position vector of the centre of mass with respect to point P. Hence, $\vec{L}_P = \vec{L}_{cm} + \vec{r}\times m\vec{v}_{cm}$

15. ROLLING MOTION ON AN INCLINED PLANE

A body of mass M and radius R is rolling down a plane inclined at an angle θ with the horizontal. The body rolls without slipping. The centre of mass of the body r moves in a straight line. External forces acting on the body are :



- Weight Mg of the body vertically downwards through its center of mass.
- The normal reaction N of the inclined plane.
- The frictional force f acting upwards and parallel to the inclined plane.

For linear motion $\quad Mg\sin\theta - f = Ma_{CM}$; For angular motion $\quad \tau = fR = I\alpha$

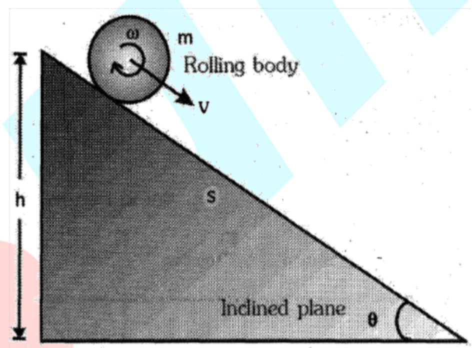
From the condition for pure rolling $a_{CM} = \alpha R \Rightarrow Mg\sin\theta - f = M\left(\frac{fR^2}{I}\right) = M\left(\frac{fR^2}{MK^2}\right) \Rightarrow f = \frac{Mg\sin\theta}{\left(1+\frac{R^2}{K^2}\right)}$

But $f \le \mu Mg\cos\theta \Rightarrow \frac{Mg\sin\theta}{\left(1+\frac{R^2}{K^2}\right)} \le \mu Mg\cos\theta \Rightarrow \mu \ge \frac{\tan\theta}{\left(1+\frac{R^2}{K^2}\right)} \Rightarrow$ for pure rolling $\mu_{min} = \frac{\tan\theta}{\left(1+\frac{R^2}{K^2}\right)}$

Rolling Motion on an inclined plane

(i) Velocity at the bottom of the inclined plane

Applying Conservation of mechanical energy principle



$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}mK^2\left(\frac{v^2}{R^2}\right)$$

$$mgh = \frac{1}{2}mv^2\left(1+\frac{K^2}{R^2}\right) \quad \text{.....(1)}$$

$$h = s\sin\theta \quad \text{.....(2)}$$

from (1) & (2)

$$v_{Rolling} = \sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}} = \sqrt{\frac{2gs\sin\theta}{1+\frac{K^2}{R^2}}}$$

When the body slides $\frac{K^2}{R^2} = 0$

$\therefore$ Velocity when the body slides $\quad v_{sliding} = \sqrt{2gh} = \sqrt{2gs\sin\theta} \Rightarrow v_{sliding} > v_{rolling}$

(ii) Acceleration of the body

$v^2 = u^2 + 2as \qquad (\because u = 0)$

$\therefore \quad v^2 = 2as$

$$\frac{2gs\sin\theta}{\left(1+\frac{K^2}{R^2}\right)} = 2as \Rightarrow a_{rolling} = \frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}$$

When body slides $\frac{K^2}{R^2} = 0$

Acceleration when the body slides $\quad a_{sliding} = g\sin\theta \quad \Rightarrow \quad a_{sliding} > a_{rolling}$

(iii) Time taken by the rolling body to reach the bottom

$$s = \frac{1}{2}at^2 \quad \Rightarrow \quad t = \sqrt{\frac{2s}{a}}, \quad s = \frac{h}{\sin\theta}$$

$$t_{rolling} = \sqrt{\frac{2s}{g\sin\theta}\left(1+\frac{K^2}{R^2}\right)} = \sqrt{\frac{2h}{g\sin\theta}\left(1+\frac{K^2}{R^2}\right)} = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{K^2}{R^2}\right)}$$

When the body slides, $\frac{K^2}{R^2} = 0$

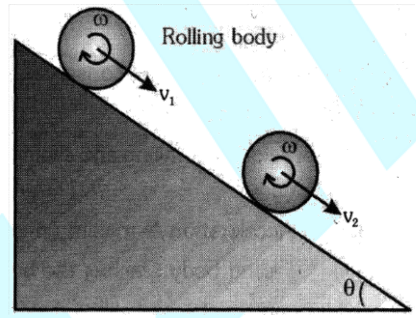
Time of descent when the body slides $t_{sliding} = \sqrt{\frac{2s}{g\sin\theta}} = \sqrt{\frac{2h}{g\sin^2\theta}} = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}$

SO $t_{rolling} > t_{sliding}$

Note: If different bodies are allowed to roll down on an inclined plane then the body with

(i) least $\frac{K^2}{R^2}$ will reach first (ii) maximum $\frac{K^2}{R^2}$ will reach last

(iii) equal $\frac{K^2}{R^2}$ will reach together



(iv) Change in kinetic energy due to rolling $v_2 > v_1$

$$= \frac{1}{2}mv_2^2\left(1+\frac{K^2}{R^2}\right) - \frac{1}{2}mv_1^2\left(1+\frac{K^2}{R^2}\right)$$

$$= \frac{1}{2}m\left(1+\frac{K^2}{R^2}\right)\left(v_2^2 - v_1^2\right)$$

(v) $v_{rolling} = \frac{v_{sliding}}{\sqrt{1+\frac{K^2}{R^2}}}$, $a_{rolling} = \frac{a_{sliding}}{1+\frac{K^2}{R^2}}$, $t_{rolling} = t_{sliding}\sqrt{1+\frac{K^2}{R^2}}$

$V_{rolling} < v_{sliding}$ $a_{rolling} < a_{sliding}$ $t_{rolling} > t_{sliding}$

Comparision between formula of translatory motion and rotatory motion

Translatory Motion	**Rotatory Motion**
• $\vec{F} = \frac{d\vec{p}}{dt}$	• $\vec{\tau} = \frac{d\vec{L}}{dt}$
• $\vec{F} = m\vec{a}$	• $\vec{\tau} = I\vec{\alpha}$
• Linear momentum $(\vec{p})$, $\vec{p} = m\vec{v}$	• Angular momentum $(\vec{L})$, $-\vec{L} = I\vec{\omega}$
• Translational kinetic energy, KE = $\frac{1}{2}mv^2$	• Rotational kinetic energy, E = $\frac{1}{2}I\omega^2$
• Work done W = $\vec{F}.\vec{s}$ (constant force)	• Work done W = $\vec{\tau}.\vec{\theta}$ (constant torque)
• W = $\int \vec{F}.d\vec{s}$ Variable force	• W = $\int \vec{\tau}.d\vec{\theta}$ Variable torque
• Power P = $\frac{dW}{dt} = \frac{\vec{F}.d\vec{s}}{dt} = \vec{F}.\vec{v}$	• Power P = $\frac{dW}{dt} = \frac{\vec{\tau}.d\vec{\theta}}{dt} = \vec{\tau}.\vec{\omega}$
• Work-energy theorem W = $\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$	• Work-energy theorem $\frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$
• Linear impulse It is the product of large force and the small time for which it acts $\vec{F} = \frac{\Delta\vec{p}}{\Delta t}$	• Angular impulse It is product of large torque for small time for which it acts $\vec{\tau} = \frac{\Delta\vec{L}}{\Delta t}$

Impulse-momentum theorem
$\Delta\vec{p} = \vec{F}\Delta t$ = Impulse

Angular Impulse-momentum theorem
$\Delta\vec{L} = \vec{\tau}\Delta t$ = Angular impulse

GOLDEN KEY POINTS

- For pure rolling there may or may not be friction on surface.
- On a fixed smooth inclined surface pure rolling cannot sustain.
- The displacement of point of contact with the surface is equal to zero in pure rolling so work done is equal to zero.

  Angular momentum is conserved about point of contact 'in pure rolling.

  For inclined plane :

  (i) Velocity of falling and sliding bodies along an inclined plane are equa1 and is more than that of a rolling body.

  (ii) Acceleration is maximum in case of falling and minimum in case of rolling.

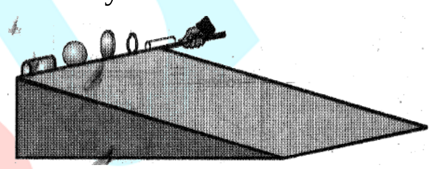
  (iii) Falling body reaches the bottom first while rolling body is the last to reach.

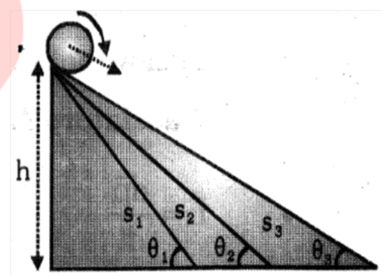
- If different bodies are allowed to roll down an inclined plane then body which has

  $\rightarrow$ least, $\frac{K^2}{R^2}$ will reach first

  $\rightarrow$ maximum, $\frac{K^2}{R^2}$ will reach last

  $\rightarrow$ equal, $\frac{K^2}{R^2}$ will reach together





- From the figure $\theta_3 < \theta_2 < \theta_1$ $\quad a_3 < a_2 < a_1$

  $t_3 > t_2 > t_l$ $\quad v_l = v_2 = v_3$

- When a body performs pure rolling then its motion is pure rotatory about an axis passing through the point of contact, parallel to the surface and perpendicular to the direction of motion. It is known as the **Instantaneous axis of rotation**.
- When a ring, disc, hollow sphere and a solid sphere roll on the same inclined plane then

  $v_S > v_D > v_H > v_R$

  $a_S > a_D > a_H > a_R$

  $t_S < t_D < t_H < t_R$

Illustrations

Illustration 54.

A thin hollow cylinder

(a) slides without rotating with a speed v. (b) rolls with the same speed without slipping.

Find the ratio of kinetic energies in the two cases.

Solution:

For thin hollow cylinder $\frac{K^2}{R^2} = 1$ [analogous to a as ring]

(a) $E_{trans.} = \frac{1}{2}Mv^2$

(b) $E_{rolling} = \frac{1}{2}Mv^2\left[1+\frac{K^2}{R^2}\right] = \frac{1}{2}Mv^2\,(1+1) = Mv^2$

$$\frac{E_{trans.}}{E_{rolling}} = \frac{\frac{1}{2}Mv^2}{Mv^2} = \frac{1}{2} = 1 : 2.$$

Illustration 55.

When a sphere of moment of inertia 'I' rolls down an inclined plane then find the percentage of rotational kinetic energy of the total energy.

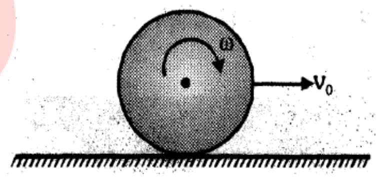
Solution:

$$\frac{E_r}{E_r}\times 100 = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}Mv^2+\frac{1}{2}I\omega^2}\times 100 = \frac{\frac{2}{5}MR^2\times\frac{v^2}{R^2}}{Mv^2\left[1+\frac{K^2}{R^2}\right]}\times 100 = \frac{\frac{2}{5}}{1+\frac{2}{5}}\times 100 = 28.6\%$$

Illustration 56.

A solid sphere rolls without slipping on a rough surface and the centre of mass has a constant speed v_0. If the mass of the sphere is m and its radius is R, then find the angular momentum of the sphere about the point of contact.

Solution:



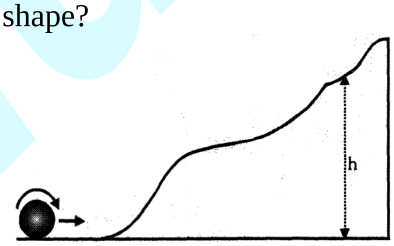
$\because\ \vec{L}_P = \vec{L}_{cm} + \vec{r}\times\vec{p}_{cm} = I_{cm}\vec{\omega} + \vec{R}\times m\vec{v}_{cm}$; here $v_{cm} = v_0$

Since sphere is in pure rolling motion hence $w = v_0 / R$

$$\Rightarrow \vec{L}_p = \left(\frac{2}{5}MR^2\,\frac{v_0}{R}\right)\left(-\hat{k}\right) + Mv_0R\left(-\hat{k}\right) = \frac{7}{5}Mv_0R\left(-\hat{k}\right)$$

Illustration 57.

A body of mass M and radius r, rolling with velocity v on a smooth horizontal floor, rolls up a rough irregular inclined plane up to a vertical height $(3v^2/4g)$. Compute the moment of inertia of the body and comment on its shape?



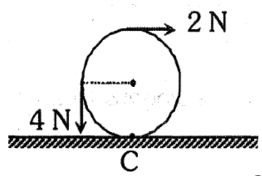
Solution:

The total kinetic energy of the body $E = E_t + E_r = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \Rightarrow E = \frac{1}{2}Mv^2[1 + (I/Mr^2)]$[as $v = r\omega$]

When it rolls up on an irregular inclined plane of height $h = (3v^2/vg)$, its KE is fully converted into PE, so by conservation of mechanical energy $\frac{1}{2}Mv^2\left[1+\frac{I}{Mr^2}\right] = Mg\left[\frac{3v^2}{4g}\right]$ which on simplification gives $I = (1/2)\,Mr^2$. This result clearly indicates that the body is either a disc or a cylinder.

Illustration 58.

A uniform solid disc of mass 1 kg and radius 1m is kept on a rough horizontal surface. Two force of magnitudes 2 N and 4 N have been applied on the disc as shown in the figure. If there is no slipping then the linear acceleration of the centre of mass of the disc is :-



(A) 4 m/s^2 (B) 2 m/s^2 (C) 1 m/s^2 (D) zero

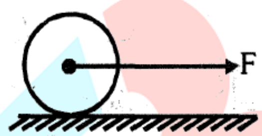
Solution:

Taking torque about contact point C we have, $\tau = 4 \times R - 2 \times 2R = 0$, $F_{net} = 0$

Since $ma_{cm} = F_{net}$ $\therefore$ $a_{cm} = 0$.

Illustration 59.

A horizontal force F acts on a sphere of mass M at its centre as shown. Coefficient of friction between the ground and the sphere is μ. What is maximum value of F, for which there is no slipping ?



Solution:

For linear motion $F - f = Ma$

and for rotational motion $\tau = I\alpha$

$\Rightarrow f.R = \frac{2}{5}MR^2 \cdot \frac{a}{R} \Rightarrow f = \frac{2}{5}Ma$ or $Ma = \frac{2}{5}f$

$\therefore F - f = \frac{2}{5}f$ or $f = \frac{2F}{7}$ $\because$ $f \le \mu Mg$ $\therefore \frac{2F}{7} \le \mu Mg$ so $F \le \frac{7}{2}\mu Mg$.

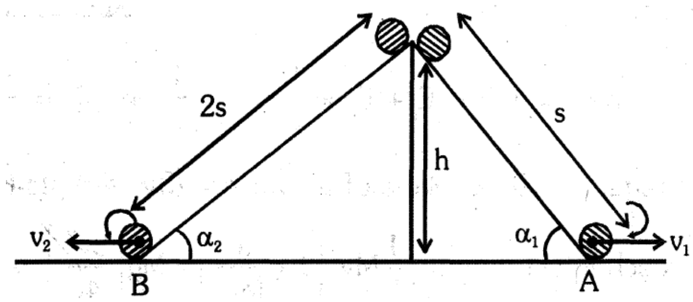
BEGINNER'S BOX - 6

1. A ring, a disc and a solid sphere are simultaneously released to roll down from top or a rough inclined plane of height h. Write the order in which these three bodies will reach the bottom ?

2. A hollow cylinder is rolling down an inclined plane which is inclined at an angle of 30° to the horizontal. Find its speed after travelling a distance of 10 m.

3. A wheel is rotating about a fixed axis. Find the moment of inertia of the wheel about the axis of rotation, when its angular speed is 30 radians/s and its kinetic energy is 360 joules.

4. Two uniform identical discs roll down on two inclined planes of length s and 2s respectively as shown in the figure. Find the ratio of velocities of the two discs at the points A and B of the inclined planes ?



5. Translational kinetic energy of a rolling hollow sphere is .... percent of its total energy.

ANSWERS

BEGINNER'S BOX - 1

1. $\theta = 22$ rad., $\alpha = 2.0$ rad/s^2
2. $\omega = 22$ rad/s; $\theta = 24$ radians
3. $\frac{A^2}{2B}$
4. (a) 120 rad/s; (b) (i) $-\frac{90}{\pi}$ rad/s^2; (ii) 24π m
5. (A) 6. (A) 7. (B) 8. (A)

BEGINNER'S BOX - 2

1. 48 kg-m^2 2. $1 : \sqrt{2}$ 3. 1 : 4 4. 3 Kg-m^2 5. $\frac{6I}{5}$
6. Moment of inertia of a hollow cylinder will be larger as compare to a disc because most of the mass of the former is located away from the axis of rotation as compared to the latter.
7. Spoke do not carry much mass. Most of the mass is located at the rim. This gives cycle wheel more moment of inertia for same mass.
8. $I_B = 64I_A$ 9. $\frac{\mu l^4}{6}$ 10. $I = \frac{a^2}{4}(m_2+m_3)$ 11. $\frac{3}{2}MR^2$
12. It will be again $= \frac{2}{5}MR^2$, because the axis in question is also diameter of the sphere.

BEGINNER'S BOX - 3

1. $\vec{\tau} = 11\hat{i} - 7\hat{j} - 5\hat{k}$
2. About point B moment of inertia is less than A so it is easier to rotated about point B
3. (B) 4. (D) 5. (B) 6. (D) 7. (D)
8. (A)

BEGINNER'S BOX - 4

1. (D)
2. The angular momentum of particle about origin will be zero because given straight line passes through origin, so line of action of velocity (momentum) also passes through this point.
3. (D) 4. 4×10^{-3} J ; 8×10^{-4} J-s

BEGINNER'S BOX - 5

1. $\sqrt{\frac{\omega_2}{\omega_1}}$ 2. $\frac{L}{4}$ 3. $\frac{2}{3}ml^2\pi^2f^2$ 4. 131.6 J 5. 0.45 J

6. Angular velocity will decrease. **7.** $T = \frac{\pi}{10}$

8. (a) 4.95×10^4 N–m (b) 4.95×10^6 J

BEGINNER'S BOX - 6

1. Solid sphere, disc and then ring **2.** 7 m/s **3.** 0.8 kg-m^2 **4.** $\frac{v_1}{v_2} = 1$

5. 60