Permutation & Combinations

1. Fundamental Principles of Operation

When one or more operations can be accomplished by number of ways then there are two principles to find the total number of ways to accomplish one, two, or all of the operations without counting them as follows:

1.1 Fundamental Principle of Multiplication :

Let there are two parts A and B of an operation and if these two parts can be performed in m and n different number of ways respectively, then that operation can be completed in $m \times n$ ways.

1.2 Fundamental Principle of addition :

If there are two operations such that they can be done independently in m and n ways respectively, then any one of these two operations can be done by (m + n) number of ways.

2. Combinations

The different groups or selections of a given number of things by taking some or all at a time without paying any regard to their order, are called their **combinations.**

The number of combinations of n **different** things taken r at a time is denoted by

$$^nC_r \text{ or } C(n, r)$$

$$^nC_r = \frac{n!}{r!(n-r)!}$$

So $^nC_r = \dfrac{n(n-1)(n-2).....(n-r+1)}{r!}$

Particular cases : $^nC_r = \dfrac{^nP_r}{r!}$

$^nC_n = 1$

$^nC_0 = 1$

Some Important Results :

* $^nC_r = {^nC_{n-r}}$
* $^nC_x = {^nC_y} \Rightarrow x + y = n$
* $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$
* $^nC_r = \dfrac{n}{r}\,.\,{^{n-1}C_{r-1}}$
* $^nC_r = \dfrac{1}{r}(n-r+1)\,{^nC_{r-1}}$
* $^nC_1 = {^nC_{n-1}} = n$
* Greatest value of ^nC_r

  $= {^nC_{n/2}}$, when n is even

  $= {^nC_{(n-1)/2}}$ or $^nC_{(n+1)/2}$, when n is odd

2.1 Restricted Combinations :

The number of combinations of n different things taking r at a time

(a) When p particular things are always to be included = $\mathbf{^{n-p}C_{r-p}}$

(b) When p particular things are always to be excluded = $\mathbf{^{n-p}C_r}$

(c) When p particular things are always included and q particular things are always excluded

$= \mathbf{^{n-p-q}C_{r-p}}$

2.2 Total number of combinations in different cases :

(a) The number of combinations of n **different things** taking some or all (or atleast one) at a time $= {^nC_1} + {^nC_2} + ...+ {^nC_n}$

$= \mathbf{2^n - 1}$

(b) The number of ways to select some or all out of (p + q + r) things where p are alike of first kind, q are alike of second kind and r are alike of third kind is = **(p +1) (q +1) (r +1) – 1**

(c) The number of ways to select some or all out of (p + q + t) things where p are alike of first kind, q are alike of second kind and remaining t are different is = **(p +1) (q +1) $\mathbf{2^t}$ – 1**

3. Permutations

An arrangement of some given things taking some or all of them, is called a **permutation** of these things.

For Example, three different things a, b and c are given, then different arrangements which can be made by taking two things from the three given things are

ab, ac, bc, ba, ca, cb

Therefore, the number of permutations will be 6.

3.1 The number of permutations of n **different** things taken r at a time is ^nP_r, where

$$^nP_r = \frac{n!}{(n-r)!}$$

$$= n(n-1)(n-2)\,.....\,(n-r+1)$$

The number of permutations of n dissimilar things taken all at a time = $^nP_n = n!$

3.2 Permutations in which all things are not different :

The number of permutations of n things taken **all at a time** when p of them are alike and of one kind, q of them are alike and of second kind, r of them are alike and of third kind and all remaining being different is

$$\frac{n!}{p!q!r!}$$

3.3 Permutations in which things may be repeated :

The number of permutations of n **<u>different things</u>** taken r at a time when each thing can be used once, twice, ...upto r times in any permutation is $\mathbf{n^r}$.

In particular, in above case when n things are taken at a time then total number of permutation is $\mathbf{n^n}$.

3.4 Restricted Permutations : If in a permutation, some particular things are to be placed at some particular places or some particular things are always to be included or excluded, then it is called a **restricted permutation.** The following are some of the restricted permutations.

(a) The number of permutations of n dissimilar things taken r at a time, when m particular things always occupy definite places = $^{n-m}P_{r-m}$

(b) The number of permutations of n different things taken altogether when r particular things are to be placed at some **r** given places.

$$= {}^{n-r}P_{n-r} = (n-r)\,!$$

(c) The number of permutations of n different things taken r at a time, when m particular things are always to be excluded = $^{n-m}P_r$

(d) The number of permutations of n different things taken r at a time when m particular things are always to be included

$$= {}^{n-m}C_{r-m} \times r!$$

3.5 Permutation of numbers when given digits include zero :

If the given digits include 0, then two or more digit numbers formed with these digits cannot have 0 on the extreme left. In such cases we find the number of permutations in the following two ways.

(a) (The number of digits which may be used at the extreme left) x (The number of ways in which the remaining places may be filled up)

(b) If given digits be n (including 0) then total number of m- digit numbers formed with them

$$= {}^{n}P_m - {}^{n-1}P_{m-1}.$$

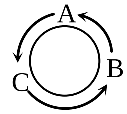
because $^{n-1}P_{m-1}$ is the number of such numbers which contain 0 at extreme left.

3.6 Circular Permutations

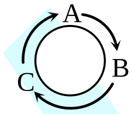
Till now we have calculated the number of linear permutation in which things are arranged in a row. Now we shall find the number of permutations in which things are arranged in a circular shape. Such permutations are named as **circular permutations**. Thus an arrangement of some given things round a circle is called their circular permutation.

It should be noted that in a circular permutation initial and final position of things can not be specified. Thus all linear permutations of some given things having the same order of elements will give the same circular permutation.

For example, there are 6 linear permutations of three letters A, B and C taken all at a time. These are ABC, ACB, BAC, BCA, CAB, CBA



(Anti- clockwise order)



(Clockwise order)

Since the arrangements ABC, BCA, CAB, are in the same order (clockwise order), therefore these three linear permutations are equal to one circular permutation.

From this example, it is clear that from a circular permutation of three things, there correspond three linear permutations. Thus, we conclude that if x be the number of circular permutations of 3 given things then the number of their linear permutations will be 3x.so

$$3x = 3! \Rightarrow x = \frac{3!}{3}.$$

In a similar way it can be seen that if x be the number of circular permutations of n different things taking r at a time, then

$$rx = {}^{n}P_r \Rightarrow x = {}^{n}P_r/r$$

Thus, we obtain the following results for the number of circular permutations.

3.6.1 Number of Circular Permutations :

(a) The number of Circular permutations of n **<u>different</u>** things taking r at a time $\frac{^nP_r}{r}$, when clockwise and anti-clockwise orders are treated as different.

(b) The number of circular permutations of n **<u>different</u>** things taking altogether

$\frac{^nP_n}{n}$, when clockwise and anti clockwise orders are treated as different.

(c) The number of Circular permutations of n **<u>different</u>** things taking r at a time $\frac{^nP_r}{2r}$, when the above two orders are treated as same.

(d) The number of circular permutations of n **<u>different</u>** things taking altogether

$\frac{^nP_n}{2n} = \frac{1}{2}(n-1)!$, when above two orders are treated as same.

3.6.2 Restricted Circular Permutations : When there is a restriction in a Circular permutation then first of all we shall perform the restricted part of the operation and then perform the remaining part treating it similar to a linear permutation.

4. Division into Groups

(a) The number of ways in which (p + q) things can be divided into two groups of p and q things is

$$^{p+q}C_p = {}^{p+q}C_q = \frac{(p+q)!}{p!\,q!}$$

Particular case : when p = q, then total number of combinations are

(i) $\frac{2p!}{(p!)^2}$ when groups are differentiable.

(ii) $\frac{2p!}{2!\,(p!)^2}$ when groups are not differentiable.

(b) The number of ways in which (p + q + r) things can be divided into three groups containing p, q and r things is

$$\frac{(p+q+r)!}{p!\,q!\,r!}$$

Particular case :

when p = q = r, then total number of combinations are

(i) $\frac{3p!}{(p!)^3}$ when groups are differentiable.

(ii) $\frac{3p!}{3!\,(p!)^3}$ when groups are not differentiable.

5. Permutations in Which the Operation of Selection is Necessary

There are questions of permutation in which we have to start with the operation of selection for the given number of things. After this we calculate the number of different arrangements for each of such selected group.

6. Dearrangement Theorem

Any change in the given order of the things is called a **Dearrangement** .

(a) If n items are arranged in a row, then the number of ways in which they can be rearranged so that no one of them occupies the place assigned to it is

$$n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+.....(-1)^n\frac{1}{n!}\right]$$

(b) If n things are arranged at n places then the number of ways to rearrange exactly r things at right places is

$$\frac{n!}{r!}\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+.....(-1)^{n-r}\frac{1}{(n-r)!}\right]$$

7. Multinomial Theorem & Its Applications

7.1 Multinomial Theorem :

The expansion of $[x_1 + x_2 + x_3 + ... + x_n]^r$ where n & r are integers ($0 < r \le n$) is a homogenous expression in $x_1, x_2, x_3, ... x_n$ and given as below :

$[x_1 + x_2 + x_3 + ... + x_n]^r$

$$= \sum\left(\frac{r!}{\lambda_1!\lambda_2!\lambda_3!....\lambda_n!}\right)x_1^{\lambda_2}x_2^{\lambda_2}x_3^{\lambda_3}....x_n^{\lambda_n}$$

(where n & r are integers $0 \le r \le n$ and

$\lambda_1, \lambda_2, ..., \lambda_n$ are non negative integers)

Such that $\lambda_1 + \lambda_2 + ... + \lambda_n = r$

(valid only if $x_1, x_2, x_3, ... x_n$ are independent of each other)

coefficient of $x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3}$.....= total number of arrangements of r objects out of which λ_1 number of x_1's are identical λ_2 number of x_2's are identical and so on .....

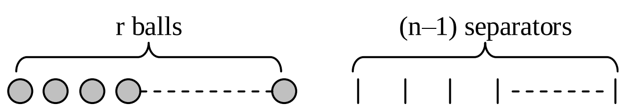
$$= \left(\frac{(\lambda_1+\lambda_2+\lambda_3+.....\lambda_n)!}{\lambda_1!\lambda_2!\lambda_3!....\lambda_4!}\right) = \frac{r!}{\lambda_1!\,\lambda_2!\,\lambda_3!....\lambda_n!}$$

7.2 Number of distinct terms :

Since $(x_1 + x_2 + x_3 + ... + x_n)^r$ is multiplication of $(x_1 + x_2 + x_3+......+ x_n)$, r times & will be a homogeneous expansion of r^{th} degree in $x_1, x_2,......x_n$ So in each term sum of powers of variables must be r

So number of distinct terms will be total number of non-negative integral solution of equation is $\lambda_1 + \lambda_2 + \lambda_3+......+\lambda_n= r$

= Number of ways of distributing r identical objects among n persons



= number of arrangements of r identical balls & n – 1 identical separators.

$$= \frac{(n-1+r)!}{(n-1)!\,r!} = {}^{n+r-1}C_r = {}^{n+r-1}C_{n-1}$$

7.3 Application of multinomial theorem

If we want to distribute n identical objects in r different groups under the condition that empty groups are not allowed. $a_1 + a_2 + a_3 +.....+ a_r=n$

Boundary conditions are $1 \le a_1, a_2......a_r \le n$

(As each box contains at least one object)

Number of ways

$= \text{coefficient of } x^n \text{ in } (x^1 + x^2 + ...+x^n)^r$

$= \text{coefficient of } x^{n-r} \text{ in } (1 + x + x^2 + ...+x^{n-1})^r$

$= {}^{(n-r)+r-1}C_{r-1} = {}^{n-1}C_{r-1}$

8. Divisibility of Numbers

The following chart shows the conditions of divisibility of numbers by 2,3,4,5,6,8,9,25

Divisible by	**Condition**
2	whose last digit is even (2, 4, 6, 8, 0)
3	sum of whose digits is divisible by 3
4	whose last two digits number is divisible by 4
5	whose last digit is either 0 or 5
6	which is divisible by both 2 and 3
8	whose last three digits number is divisible by 8
9	sum of whose digits is divisible by 9
25	whose last two digits are divisible by 25

9. Sum of Numbers

(a) For given n different digits a_1, a_2, a_3 ......,a_n the sum of the digits in the unit place of all numbers formed (if numbers are not repeated) is

$\mathbf{(a_1 + a_2 + a_3 + ...+ a_n)\ (n-1)!}$

i.e. (sum of the digits) (n-1)!

(b) Sum of the total numbers which can be formed with given n different digits a_1, a_2, a_3 .....a_n is **$(a_1 + a_2 + a_3 + ...+ a_n)(n-1)!.(111 ...n \text{ ties})$**

10. Some Important Results About Points

If there are n points in a plane of which m (< n) are collinear, then

(a) Total number of different straight lines obtained by joining these n points is

$$\mathbf{{}^nC_2 - {}^mC_2 + 1}$$

(b) Total number of different triangles formed by joining these n points is

$$\mathbf{{}^nC_3 - {}^mC_3}$$

(c) Number of diagonals in polygon of n sides is

$$\mathbf{{}^nC_2 - n} \quad \text{i.e.} \quad \frac{n(n-3)}{2}$$

(d) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then total number of parallelograms so formed is

$$\mathbf{{}^mC_2} \times \mathbf{{}^nC_2} \quad \text{i.e.} \quad \frac{mn(m-1)(n-1)}{4}$$