Differentiation

1. Introduction

The rate of change of one quantity with respect to some another quantity has a great importance. For example the rate of change of displacement of a particle with respect to time is called its velocity and the rate of change of velocity is called its acceleration.

The rate of change of a quantity 'y' with respect to another quantity 'x' is called the **derivative** or **differential coefficient** of y with respect to x.

2. Differential Coefficient

Let y = f(x) be a continuous function of a variable quantity x, where x is independent and y is dependent variable quantity. Let δx be an arbitrary small change in the value of x and δy be the corresponding change in y then $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ if it exists, is called the derivative or differential coefficient of y with respect to x and it is denoted by

$\frac{dy}{dx}$. y', y_1 or Dy.

So, $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

The process of finding derivative of a function is called differentiation.

If we again differentiate (dy/dx) with respect to x then the new derivative so obtained is called second derivative of y with respect to x and it is denoted by $\left(\frac{d^2y}{dx^2}\right)$ or y'' or y_2 or D^2y. Similarly, we can find successive derivatives of y which may be denoted by

$$\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots\ldots, \frac{d^ny}{dx^n} \ldots\ldots$$

Note : (i) $\frac{\delta y}{\delta x}$ is a ratio of two quantities δy and δx where as $\frac{dy}{dx}$ is not a ratio, it is a single quantity i.e. $\frac{dy}{dx} \neq dy \div dx$

(ii) $\frac{dy}{dx}$ is $\frac{d}{dx}$(y) in which d/dx is simply a symbol of operation and not 'd' divided by dx.

3. Differential Coefficient of Some Standard Function

The following results can easily be established using the above definition of the derivative–

(i) $\frac{d}{dx}$ (constant) = 0

(ii) $\frac{d}{dx}$ (ax) = a

(iii) $\frac{d}{dx}(x^n) = nx^{n-1}$

(iv) $\frac{d}{dx} e^x = e^x$

(v) $\frac{d}{dx}(a^x) = a^x \log_e a$

(vi) $\frac{d}{dx}(\log_e x) = 1/x$

(vii) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$

(viii) $\frac{d}{dx}(\sin x) = \cos x$

(ix) $\frac{d}{dx}(\cos x) = -\sin x$

(x) $\frac{d}{dx}(\tan x) = \sec^2 x$

(xi) $\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$

(xii) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(xiii) $\frac{d}{dx}(\mathrm{cosec}\ x) = -\mathrm{cosec}\ x \cot x$

(xiv) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$

(xv) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$

(xvi) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xvii) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(xviii) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$ $|x| > 1$

(xix) $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

(xx) $\frac{d}{dx}(\sinh x) = \cosh x$

(xxi) $\frac{d}{dx}(\cosh x) = \sinh x$

(xxii) $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$

(xxiii) $\frac{d}{dx}(\coth x) = -\text{cosec h}^2 x$

(xxiv) $\frac{d}{dx}(\text{sech } x) = -\text{sech } x \tanh x$

(xxv) $\frac{d}{dx}(\text{cosech } x) = -\text{cosec hx} \coth x$

(xxvi) $\frac{d}{dx}(\sin h^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

(xxvii) $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$, $x > 1$

(xxviii) $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$

(xxix) $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{x^2-1}$, $|x| > 1$

(xxx) $\frac{d}{dx}(\text{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$, $(0 < x < 1)$

(xxxi) $\frac{d}{dx}(\text{cosech}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$, $x \neq 0$

(xxxii) $\frac{d}{dx}(e^{ax} \sin b x) = e^{ax}(a \sin b x + b \cos bx)$

$= \sqrt{a^2+b^2}\, e^{ax} \sin(bx + \tan^{-1} b/a)$

(xxxiii) $\frac{d}{dx}(e^{ax} \cos b x) = e^{ax}(a \cos b x - b \sin b x)$

$= \sqrt{a^2+b^2}\, e^{ax} \cos(bx + \tan^{-1} b/a)$

4. Some Theorems on Differentiation

Theorem I $\frac{d}{dx}[kf(x)] = k\, d/dx\, [f(x)]$, where k is a constant

Theorem II $\frac{d}{dx}[f_1(x) \pm f_2(x) \pm f_3(x) \pm ....]$

$= d/dx\,[f_1(x)] \pm d/dx\,[f_2(x)] \pm .....$

Theorem III $\frac{d}{dx}[f(x).g(x)]$

$= f(x)\, d/dx\,[g(x)] + g(x)\, d/dx\,[f(x)]$

Theorem IV

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)d/dx[f(x)] - f(x)d/dx[g(x)]}{[g(x)]^2}$$

Theorem V Derivative of the function of the function. If 'y' is a function of 't' and t' is a function of 'x' then

$$\frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx}$$

Theorem VI Derivative of parametric equations

If $x = \phi(t)$, $y = \psi(t)$ then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Theorem VII Derivative of a function with respect to another function If f(x) and g(x) are two functions of a variables x, then

$$\frac{d[f(x)]}{d[g(x)]} = \frac{d}{dx} f(x) / \frac{d}{dx}[g(x)]$$

Theorem VIII $\frac{dy}{dx} . \frac{dx}{dy} = 1$

5. Method of Differentiation

5.1 Differentiation of Implicit functions

If in an equation, x and y both occurs together i.e. f(x,y) = 0 and this equation can not be solved either for y or x, then y (or x) is called the implicit function of x (or y).

For example $x^3+y^3+3axy+c = 0$, $x^y+y^x = a^b$ etc.

Working rule for finding the derivative

First Method:

(i) Differentiate every term of f(x,y) = 0 with respect to x.

(ii) Collect the coefficients of dy/dx and obtain the value of dy/dx.

Second Method : If f (x,y) = constant, then

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

Where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are partial differential coefficients of f(x,y) with respect to x and y respectively.

Note : Partial differential coefficient of f (x,y) with respect to x means the ordinary differential coeffcient of f(x,y) with respect to x keeping y constant.

5.2 Differentiation of logarithmic functions

In differentiation of an expression or an equation is done after taking log on both sides, then it is called logarithmic differentiation. This method is useful for the function having following forms–

(i) When base and power both are the functions of x i.e. the function is of the form $[f(x)]^{g(x)}$.

$$y = [f(x)]^{g(x)}$$

$$\log y = g(x) \log [f(x)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx} g(x).\log [f(x)]$$

$$\frac{dy}{dx} = [f(x)]^{g(x)}. \left\{\frac{d}{dx}[g(x) \log f(x)]\right\}$$

5.3 Differentiation by trigonometrical substitutions

Some times before differentiation, we reduce the given function in a simple form using suitable trigonometrical or algebric transformations. This method saves a lot of energy and time. For this following formulae and substitutions should be remembered.

Formulae

(i) $\sin^{-1}x + \cos^{-1} x = \pi/2$

(ii) $\tan^{-1} x + \cot^{-1}x = \pi/2$

(iii) $\sec^{-1} x + \text{cosec}^{-1} x = \pi/2$

(iv) $\sin^{-1} x \pm \sin^{-1}y$

$$= \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$$

(v) $\cos^{-1}x \cos^{-1} y$

$$= \cos^{-1}\left[xy \mu \sqrt{(1-x^2)(1-y^2)}\right]$$

(vi) $\tan^{-1}x \pm \tan^{-1} y = \tan^{-1}\left[\frac{x \pm y}{1 \mu xy}\right]$

(vii) $2 \sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

(viii) $2 \cos^{-1} x = \cos^{-1} (2x^2 - 1)$

(ix) $2 \tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$$= \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

(x) $3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$

(xi) $3 \cos^{-1} x = \cos^{-1} (4x^3 -3x)$

(xii) $3 \tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$

(xiii) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$

$$= \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

(xiv) $\sin^{-1} (-x) = - \sin^{-1}x$

(xv) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(xvi) $\tan^{-1} (-x) = - \tan^{-1} x$ or $\pi - \tan^{-1} x$.

(xvii) $\pi/4 - \tan^{-1} x = \tan^{-1}\left(\frac{1-x}{1+x}\right)$

Some suitable substitutions

Function	**Substitution**
(i) $\sqrt{a^2 - x^2}$	$x = a \sin\theta$ or $a \cos\theta$
(ii) $\sqrt{x^2 + a^2}$	$x = a \tan\theta$ or $a \cot\theta$
(iii) $\sqrt{x^2 - a^2}$	$x = a \sec\theta$ or $a\ \text{cosec}\ \theta$
(vi) $\sqrt{\frac{a-x}{a+x}}$	$x = a \cos 2\theta$
(v) $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2 \cos 2\theta$
(vi) $\sqrt{ax - x^2}$	$x = a \sin^2\theta$
(vii) $\sqrt{\frac{x}{a+x}}$	$x = a \tan^2\theta$
(viii) $\sqrt{\frac{x}{a-x}}$	$x = a \sin^2\theta$
(ix) $\sqrt{(x-a)(x-b)}$	$x = a \sec^2\theta - b \tan^2\theta$
(x) $\sqrt{(x-a)(b-x)}$	$x = a \cos^2\theta + b \sin^2\theta$

5.4 Differentiation of infinite series

If y is given in the form of infinite series of x and we have to find out dy/dx then we remove one or more terms, it does not affect the series

(i) If y = $\sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+....\infty}}}$ then

$\Rightarrow$ y = $\sqrt{f(x)+y}$ $\Rightarrow y^2 = f(x) + y$

$$2y\frac{dy}{dx} = f'(x) + \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{f'(x)}{2y-1}$$

(ii) If $y = f(x)^{f(x)^{f(x)....\infty}}$ then $y = f(x)^y$.

$\therefore$ log y = y log [f(x)]

$$\frac{1}{y}\frac{dy}{dx} = \frac{y.f'(x)}{f(x)} + \log f(x).\left(\frac{dy}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1-y\log f(x)]}$$

(iii) If $y = f(x) + \cfrac{1}{f(x)+\cfrac{1}{f(x)+\cfrac{1}{f(x)....}}}$

then $\frac{dy}{dx} = \frac{yf'(x)}{2y-f(x)}$