Differential Equation

1. Introduction

In certain situations we notice that the relation between the rates of change of observable quantities is simpler than the relation between the quantities themselves. In such cases differential equations are taken as model for several problems in Engineering, Physical sciences, Biological sciences. In this chapter we shall study some basic concepts.

2. Differential Equation

An equation containing an independent variable, dependent variable and differential coefficients of dependent variable with respect to independent variable is called a **differential equation.**

For Example-

(i) $\frac{dy}{dx} = \sin x$

(ii) $\frac{dy}{dx} + xy = \cot x$

(iii) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

(iv) $\left(\frac{d^2y}{dx^2}\right)^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0$

(v) $\frac{d^2y}{dx^2} + \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = 0$

(vi) $\left(\frac{d^4y}{dx^4}\right)^3 - 4\frac{dy}{dx} + 4y = 5\cos 3x$

(vii) $x^2\frac{d^2y}{dx^2} + \sqrt{1+\left(\frac{dy}{dx}\right)^2} = 0$

2.1 Order of differential equation :

The order of a differential equation is the order of the highest derivative occurring in the differential equation. For example, the order of above differential equations are1,1,2,2,2,4,2 respectively.

2.2 Degree of differential equation :

The degree of a differential equation is the degree of the highest order derivative when differential coefficients are free from radical and fraction. For example the degree of above differential equations are 1, 1, 1, 2, 2, 3, 2 respectively.

3. Linear & Non-Linear Differential Equations

A differential equation in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together is called a **linear differential equation.**

The general and n[th] order differential equation is given below -

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_{n-1}\frac{dy}{dx} + P_n y = Q$$

where $P_0, P_1, P_2, \ldots, P_{n-1}$ and Q are either constants or functions of independent variable x.

Those equations which are not linear are called **non- linear differential equations.**

For example-

(i) $\frac{d^2y}{dx^2} + y = 0$ is a linear differential equation of order 2 and degree 1

(ii) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$ is a linear differential equation of order 2 and degree 1.

(iii) The differential equation $(x^2 + y^2)dx - 3xydy = 0$ is a non-linear differential equation because the exponent of dependent variable y is 2 and it involves the product of y and dy/dx.

(iv) The differential equation $\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2 + 5y = x$ is a non- linear differential equation, because differential coefficient has exponent 2.

4. Formation of Differential Equation

(i) Write down the given equation.

(ii) Differentiate it successively with respect to x that number of times equal to the arbitrary constants.

(iii) Hence on eliminating arbitrary constants results a differential equation which involves

$$x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots\ldots$$

5. Solution of Differential Equation

A solution of a differential equation is any function which when put into the equation changes it into an identity.

5.1 General Solution : The solution which contains a number of arbitrary constants equal to the order of the equation is called **general solution or complete integral or complete primitive of differential equation.**

5.2 Particular Solution : Solution obtained from the general solution by giving particular values to the constants are called particular solutions.

6. Methods of Solving A First Order First Degree Differential Equation

6.1 Differential equations of the form $\frac{dy}{dx} = f(x)$.

To solve this type of differential equations we integrate both sides to obtain the general solution as discussed below

$$\frac{dy}{dx} = f(x) \Rightarrow dy = f(x)\,dx.$$

Integrating both sides we obtain

$$\int dy = \int f(x)\,dx + c$$

$$\text{or } y = \int f(x)\,dx + c$$

6.2 Differential equations of the form dy/dx=f(x) g(y)

To solve this type of differential equation we integrate both sides to obtain the general solution as discussed below

$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx + c.$$

6.3 Differential equations of the form of

$$\frac{dy}{dx} = f(ax + by + c)$$

To solve this type of differential equations, we put

$$ax + by + c = v \text{ and } \frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$$

$$\therefore \frac{dv}{a + b\,f(v)} = dx$$

So solution is by integrating

$$\int \frac{dv}{a + b\,f(v)} = \int dx$$

6.4 Differential Equation of homogeneous type

An equation in x and y is said to be homogeneous if it can be put in the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where f(x, y) and g(x, y) are both homogeneous functions of the same degree in x & y.

So to solve the homogeneous differential equation $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$, substitute y = vx and

$$\text{so } \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\text{Thus } v + x\frac{dv}{dx} = f(v) \Rightarrow \frac{dx}{x} = \frac{dv}{f(v) - v}$$

$$\text{Therefore solution is } \int \frac{dx}{x} = \int \frac{dv}{f(v) - v} + c$$

6.5 Differential Equations reducible to homogeneous form

A differential equation of the form $\frac{dy}{dx} = \frac{a_1x_1 + b_1y + c_1}{a_2x + b_2y + c_2}$, where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ can be reduced to homogeneous form by adopting the following procedure-

put x = X + h, y = Y + k so that $\frac{dY}{dX} = \frac{dy}{dx}$

The equation then transformed to

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)}$$

Now choose h and k such that $a_1h+b_1k+c_1=0$ and $a_2h + b_2k + c_2 = 0$. Then for these values of h and k the equation becomes

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

This is a homogeneous equation which can be solved by putting Y = vX and then Y and X should be replaced by y – k and x – h.

Special case : If $dy/dx = \frac{ax + by + c}{a'x + b'y + c'}$ and $\frac{a}{a'} = \frac{b}{b'} = m$ (say) i.e. when coefficient of x and y in numerator and denominator are proportional, then the above equation can not be solved by the method discussed before because the values of h & k given by the equations will be indeterminate.

In order to solve such equations, we proceed as explained in the following example.

$$\text{Solve } dy/dx = \frac{3x - 6y + 7}{x - 2y + 4} = \frac{3(x - 2y) + 7}{x - 2y + 4}$$

$$\left(\text{obviously } \frac{a}{a'} = \frac{b}{b'} = 3\right)$$

$$\text{put } x - 2y = v \Rightarrow 1 - 2\frac{dy}{dx} = \frac{dv}{dx}$$

Now we can solve it.

6.6 Linear differential equations

A differential equation is linear if the dependent variable (y) and its derivative appear only in first degree. The general form of a linear differential equation of first order is

$$\frac{dy}{dx} + Py = Q \qquad \text{...(1)}$$

Where P and Q are either constants or functions of x.

This type of differential equations are solved when they are multiplied by a factor, which is called **integrating factor**, because by multiplication of this factor the left hand side of the differential equation becomes exact differential of some function.

Multiplying both sides of (1) by $e^{\int P dx}$, we get

$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = Q\, e^{\int P dx}$ on integrating both sides with respect to x we get

$$y\, e^{\int P dx} = \int Q e^{\int P dx} dx + c$$

which is the required solution, where c is the constant and $e^{\int P dx}$ is called the integrating factor.

6.7 Equation reducible to linear form

(i) Bernoulli's Equation :

A differential equation of the form $\mathbf{dy/dx + Py = Qy^n}$, where P & Q are function of x alone is called Bernoullis equation. This form can be reduced to linear form by dividing y^n and then putting $y^{1-n} = v$

Dividing both sides by y^n, we get

$$y^{-n}\frac{dy}{dx} + P.y^{-n+1} = Q$$

putting $y^{-n+1} = v$ so that

$(1 - n)\, y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$, we get

$$\frac{dv}{dx} + (1 - n)\, P.\, v = (1 - n)\, Q.$$

which is a linear differential equation.

(ii) If the given equation is of the form $\frac{dy}{dx} + P.\ f(y) = Q.\ g(y)$, where P and Q are functions of x alone, we divide the equation by g(y), we get $\frac{1}{g(y)}\frac{dy}{dx} + P.\frac{f(y)}{g(y)} = Q$

Now substituted $\frac{f(y)}{g(y)} = v$ and solve.

6.8 If differential equation is of the form of $\frac{d^2y}{dx^2} = f(x)$

then its solution can be obtained by integrating it with respect to x twice.

Note : -

(1) General Form of Variables Separation

If we can write the differential equation in the form $f(f_1(x, y))\, d(f_1(x, y)) + \phi(f_2(x, y))\, d(f_2(x,y)) + \ldots = 0$, then each term can be easily integrated separately. For this the following results must be memorized.

(i) $d(x+y) = dx + dy$

(ii) $d(xy) = y\,dx + x\,dy$

(iii) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

(iv) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

(v) $d(\log xy) = \frac{y\,dx + x\,dy}{xy}$

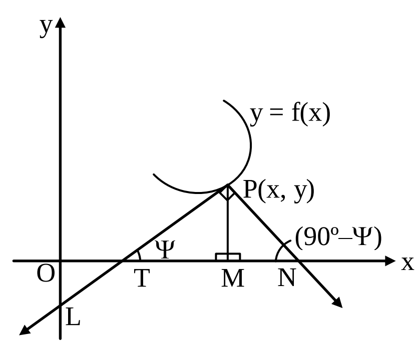
(vi) $d\left(\log\frac{y}{x}\right) = \frac{(x\,dy - y\,dx)}{xy}$

(vii) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}$

(viii) $d\left(\frac{1}{2}\sqrt{x^2 + y^2}\right) = \frac{x\,dx + y\,dx}{\sqrt{x^2 + y^2}}$

(2) Geometrical Applications

Let P(x, y) be any point on the curve y = f(x). Let the tangent and normal at P(x, y) to the curve meets x-axis at T and N



Now, draw perpendicular from P on x-axis.

$\therefore$ PM = y

If tangent at P makes angle ψ with positive direction of x-axis

$$\frac{dy}{dx} = \tan\psi$$

(a) **Length of subtangent** TM is defined as subtangent. In Δ PTM

$$TM = |y \cot\psi| = \left|\frac{y}{\tan\psi}\right| = \left|y\frac{dx}{dy}\right|$$

$$\therefore \text{ Length of subtangent} = \left|y\frac{dx}{dy}\right|$$

(b) **Length of subnormal** MN is defined as subnormal. In ΔPMN

$$MN = |y \cot(90^\circ - \psi)| = |y \tan\psi| = \left|y\frac{dy}{dx}\right|$$

$$\therefore \text{ Length of subnormal} = \left|y\frac{dy}{dx}\right|$$

(c) **Length of tangent** PT is defined as length as length of tangent. In ΔPMT

$$PT = |y \operatorname{cosec}\psi| = \left|y\sqrt{(1+\cot^2\psi}\right|$$

$$\therefore \text{ Length of tangent} = \left|y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\right|$$

(d) **Length of normal** PN is defined as length of normal. In ΔPMN

$$PN = |y \operatorname{cosec}(90^\circ - \psi)| = |y \sec\psi|$$

$$= \left|y\sqrt{(1+\tan^2\psi)}\right| = \left|y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\right|$$

$$\therefore \left|y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\right|$$

(e) **Intercepts made by the tangent on the coordinate axes**

The equation of tangent at P(x, y) is

$$Y - y = \frac{dy}{dx}(X - x) \qquad \dots \text{(i)}$$

Putting Y = 0 in (i), we get $X = x - y\frac{dx}{dy}$

Hence the length of intercept OT that the tangent cuts off from the x-axis is $x - y\frac{dx}{dy}$

Again the tangent meets y-axis then putting X = 0 in (i), we get

$$Y = y - x\frac{dy}{dx}$$

Hence the length of intercept OL that the tangent cuts off from the y-axis is

$$y - x\frac{dy}{dx}$$