BASIC MATHS

SECTION [A] : ALGEBRA

QUADRATIC EQUATION

An algebraic equation of second order (highest power of the variable is equal to 2) is called a quadratic equation. The equation $ax^2 + bx + c = 0$....(i) is the general form of quadratic equation where $a \neq 0$. The general solution of above equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If values of x be x_1 and x_2 then $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Here x_1 and x_2 are called roots of equation (i). We can easily see that

sum of roots $= x_1 + x_2 = -\frac{b}{a}$ *and* *product of roots* $= x_1 x_2 = \frac{c}{a}$

Ex. Find roots of equation $2x^2 - x - 3 = 0$.

Sol. Compare this equation with standard quadratic equation $ax^2 + bx + c = 0$,

we have a=2, b=—1, c=—3.

Now from $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; $x = \frac{-(1) \pm \sqrt{(-1)^2 - 4(2) - 3}}{2(2)}$

$x = \frac{1 \pm \sqrt{1+24}}{4} = \frac{1 \pm 5}{4} \Rightarrow x = \frac{6}{4}, x = \frac{-4}{4} \Rightarrow x = \frac{3}{2}$ or $x = -1$

Ex. In quadratic equation $ax^2 + bx + c = 0$, if discriminant $D = b^2 - 4ac$, then roots of quadratic equation are:

(A) real and distinct, if $D > 0$

(B) real and equal (repeated roots), if $D = 0$

(C) non-real (imaginary), if $D < 0$

(D) None of the above

Sol. (ABC)

BINOMIAL EXPRESSION

An algebraic expression containing two terms is called a binomial expression.

For example (a+b), $(a+b)^3$, $(2x—3y)^{-1}$, $\left(x + \frac{1}{y}\right)$ etc. are binomial expressions.

Binomial Theorem

$$(a+b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2 \times 1}a^{n-2}b^2 + ... \quad (1+x)^n = 1 + nx + \frac{n(n-1)}{2 \times 1}x^2 + ...$$

Binomial Approximation

If x is very small, then terms containing higher powers of x can be neglected so

$(1+x)^n = 1 + nx$

Ex. The mass m of a body moving with a velocity v is given by $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ where m_0 = rest mass of body = 10 kg and c = speed of light = 3×10^8 m/s. Find the value of m at $v = 3 \times 10^7$ m/s.

Sol. $m = m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 10\left[1-\left(\frac{3\times10^7}{3\times10^8}\right)^2\right]^{-1/2} = 10\left[1-\frac{1}{100}\right]^{-1/2}$

$$\approx 10\left[1-\left(-\frac{1}{2}\right)\left(\frac{1}{100}\right)\right] = 10+\frac{10}{200} \approx 10.05 \text{ kg}$$

LOGARITHM

Common formulae

(i) $\log mn = \log m + \log n$ **(ii)** $\log \frac{m}{n} = \log m - \log n$ **(iii)** $\log m^n = n \log m$ **(iv)** $\log_e m = 2.303 \log_{10} m$

COMPONENDO AND DIVIDENDO RULE

If $\frac{p}{q} = \frac{a}{b}$ then $\frac{p+q}{p-q} = \frac{a+b}{a-b}$

ARITHMETIC PROGRESSION (AP)

General form : a, a + d, a + 2d, ..., a + (n–1)d

Here a = first term, d = common difference

Sum of n terms $S_n = \frac{n}{2}[a+a+(n-1)d] = \frac{n}{2}[I^{st} \text{ term} + n^{th} \text{ term}]$

Ex. Find sum of first n natural numbers.

Sol. Let sum be S_n then $S_n = 1+2+3+....+n$; $S_n = \frac{n}{2}[1+n] = \left[\frac{n(n+1)}{2}\right]$

GEOMETRICAL PROGRESSION (GP)

General form : a, ar, ar^2,..., ar^{n-1} Here a = first term, r = common ratio

Sum of n terms $S_n = \frac{a(1-r^n)}{1-r}$ Sum of ∞ term $S_\infty = \frac{a}{1-r}$

Ex. Find $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....\text{upto } \infty$.

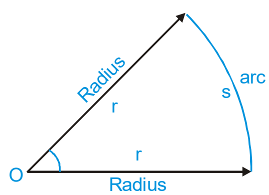
Sol. Here, a = 1, r = $\frac{1}{2}$ So, $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

SECTION [B] : TRIGONOMETRY

ANGLE

it is measure of change in direction.

$$\text{Angle } (\theta) = \frac{\text{Arc}(s)}{\text{Radius}(r)}$$



Angels measured in anticlockwise and clockwise direction are usually taken positive and negative respectively.

System of measurement of an angle

[A] Sexagesimal system :

In this system, angle is measured in degrees.

In this system , 1 right angle = 90°, 1° = 60' (arc minutes), 1' = 60" (arc seconds)

[B] Circular system :

In this system, angle is measured in radian.

if arc = radius then θ=1 rad

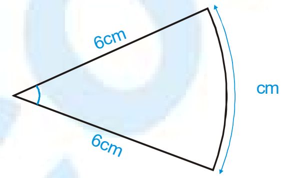
Relation between degrees and radian

$$2\pi \text{ rad} = 360° \Rightarrow \pi \text{ rad} = 180° \Rightarrow 1 \text{ rad} = \frac{180°}{\pi} = 57.3°$$

To convert from degree to radian multiply by $\frac{\pi}{180°}$

To convert from radian to degree multiply by $\frac{180°}{\pi}$

Ex. A circular arc of length π cm. Find angle subtended by it at the centre in radian and degree.

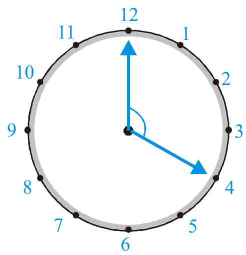


Sol. $\theta = \frac{s}{r} = \frac{\pi \text{ cm}}{6 \text{ cm}} = \frac{\pi}{6} \text{rad} = 30°$

Ex. When a clock shows 4 o'clock, how much angle do its minute and hour needles make?

(A) 120° **(B)** $\frac{\pi}{3}$ rad **(C)** $\frac{2\pi}{3}$ rad **(D)** 160°

Sol. **(AC)**



From diagram angle $\theta = 4 \times 30° = 120° = \frac{2\pi}{3}$ rad

Ex. The moon's distance from the earth is 360000 km and its diameter subtends an angle of 42' at the eye of the observer. The diameter of the moon in kilometers is

(A) 4400 **(B)** 1000 **(C)** 3600 **(D)** 8800

Sol. **(A)**

Here angle is very small so diameter ≈ arc

$$\theta = 42' = \left(42 \times \frac{1}{60}\right)^0 = 42 \times \frac{1}{60} \times \frac{\pi}{180} = \frac{7\pi}{1800} \text{ rad}$$

$$\text{Diameter} = R\theta = 360000 \times \frac{7}{1800} \times \frac{22}{7} = 4400 \text{ km}$$

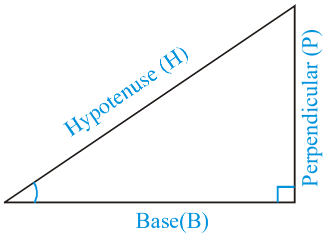
Trigonometric Ratios (T-ratios)

Following ratios of the sides of a right angled triangle are known as trigonometrical ratios.

$\sin\theta = \frac{P}{H}$ $\cos\theta = \frac{B}{H}$

$\tan\theta = \frac{P}{B}$ $\text{cosec}\theta = \frac{1}{\sin\theta} = \frac{H}{P}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{H}{B}$ $\cot\theta = \frac{1}{\tan\theta} = \frac{B}{P}$



Trigonometric Identities

In figure , $P^2 + B^2 = H^2$ Divide by H^2, $\left(\frac{P}{H}\right)^2 + \left(\frac{B}{H}\right)^2 = 1 \Rightarrow \sin^2\theta + \cos^2\theta = 1$

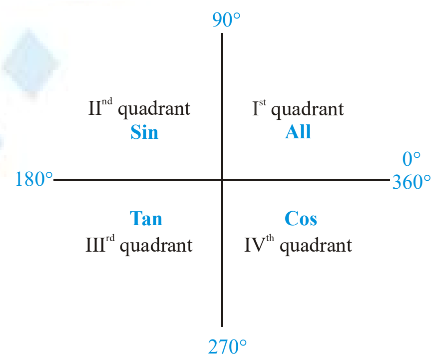
Divide by B^2, $\left(\frac{P}{B}\right)^2 + 1 = \left(\frac{H}{B}\right)^2 \Rightarrow 1 + \tan^2\theta = \sec^2\theta$

Divide by P^2, $1 + \left(\frac{B}{P}\right)^2 = \left(\frac{H}{P}\right)^2 \Rightarrow 1 + \cot^2\theta = \text{cosec}^2\theta$

Commonly Used Values of Trigonometric Functions

Angle(θ)	0°	30°	37°	45°	53°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{4}{5}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{4}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{3}{5}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	$\frac{3}{4}$	1	$\frac{4}{3}$	$\sqrt{3}$	∞

Four Quadrants and ASTC Rule



In first quadrant, all trigonometric ratios are positive.
In second quadrant, only sinθ and cosecθ are positive.
In third quadrant, only tanθ and cotθ are positive.
In fourth quadrant, only cosθ and secθ are positive

Remember as 'Add Sugar To Coffee' or 'After School To College'.

Trigonometrical Ratios of General Angles (Reduction Formulae)

(i) Trigonometric function of an angle $2n\pi + \theta$ where n=0, 1, 2, 3,.... will be remains same.

$\sin(2n\pi + \theta) = \sin\theta$ $\cos(2n\pi + \theta) = \cos\theta$ $\tan(2n\pi + \theta) = \tan\theta$

(ii) Trigonometric function of an angle $\left(\frac{n\pi}{2} + \theta\right)$ will remains same if n is even and sign of trigonometric function will be according to value of that function in quadrant.

$\sin(\pi - \theta) = +\sin\theta$ $\cos(\pi - \theta) = -\cos\theta$ $\tan(\pi - \theta) = -\tan\theta$

$\sin(\pi + \theta) = -\sin\theta$ $\cos(\pi + \theta) = -\cos\theta$ $\tan(\pi + \theta) = +\tan\theta$

$\sin(2\pi - \theta) = -\sin\theta$ $\cos(2\pi - \theta) = +\cos\theta$ $\tan(2\pi - \theta) = -\tan\theta$

(iii) Trigonometric function of an angle $\left(\frac{n\pi}{2} + \theta\right)$ will be changed into co-function if n is odd and sign of trigonometric function will be according to value of that function in quadrant.

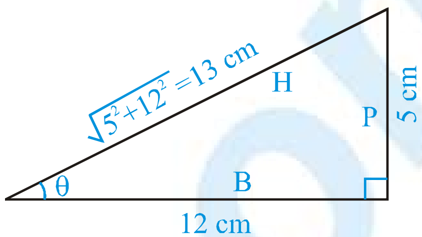
$\sin\left(\frac{\pi}{2} + \theta\right) = +\cos\theta$ $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ $\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$

$\sin\left(\frac{\pi}{2} - \theta\right) = +\cos\theta$ $\cos\left(\frac{\pi}{2} - \theta\right) = +\sin\theta$ $\tan\left(\frac{\pi}{2} - \theta\right) = +\cot\theta$

(iv) Trigonometric function of an angle $-\theta$ (negative angles)

$\sin(-\theta) = -\sin\theta \quad \cos(-\theta) = +\cos\theta \quad \tan(-\theta) = -\tan\theta$

Ex. The two shorter sides of right angled triangle are 5 cm and 12 cm. Let θ denote the angle opposite to the 5 cm side. Find sinθ, cosθ and tanθ.

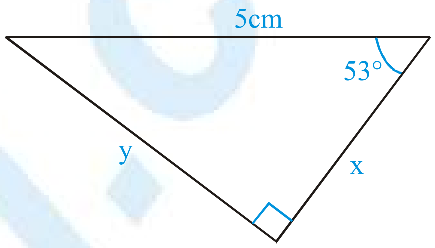


Sol. $\sin\theta = \frac{P}{H} = \frac{5\,cm}{13\,cm} = \frac{5}{13}$

$\cos\theta = \frac{B}{H} = \frac{12\,cm}{13\,cm} = \frac{12}{13}$

$\tan\theta = \frac{P}{B} = \frac{5\,cm}{12\,cm} = \frac{5}{12}$

Ex. Find x, y and perimeter of the triangle



Sol. $\frac{y}{5} = \sin 53° = \frac{4}{5} \Rightarrow y = 4$ cm

and $\frac{x}{5} = \cos 53° = \frac{3}{5} \Rightarrow x = 3$ cm

Perimeter of the triangle = x + y + 5 = 3 + 4 + 5 = 12 cm

Ex. Find the value of :

(i) sin30° + cos60° **(ii)** sin 0° – cos 0° **(iii)** tan 45°–tan 37° **(iv)** sin 390°
(v) cos 405° **(iv)** tan 420° **(viii)** sin 150° **(viii)** cos 120°
(ix) tan 135° **(x)** sin (330°) **(xi)** cos 300° **(xii)** sin(–30°)
(xiii) cos(–60°) **(xiv)** tan(–45°) **(xvi)** sin(–150°)

Sol. **(i)** $\sin 30° + \cos 60° = \frac{1}{2} + \frac{1}{2} = 1$

(ii) $\sin 0° - \cos 0° = 0 - 1 = -1$

(iii) $\tan 45° - \tan 37° = 1 - \frac{3}{4} = \frac{1}{4}$

(iv) $\sin 390° = \sin(360°+30°) = \sin 30° = \frac{1}{2}$

(v) $\cos 405° = \cos(360°+45°) = \cos 45° = \frac{1}{\sqrt{2}}$

(vi) $\tan 420° = \tan(360°+60°) = \tan 60° = \frac{1}{2}$

(vii) $\sin 150° = \sin(90°+60°) = \cos 60° = \frac{1}{2}$ or $\sin 150° = \sin(180°-30°) = \sin 30° = \frac{1}{2}$

(viii) $\cos 120° = \cos(180°-60°) = -\cos 60° = -\frac{1}{2}$

(ix) $\tan 135° = \tan(180°-45°) = -\tan 45° = -1$

(x) $\sin 330° = \sin(360°-30°) = -\sin 30° = -\frac{1}{2}$

(xi) $\cos 300° = \cos(360°-60°) = \cos 60° = \frac{1}{2}$

(xii) $\sin(-30°) = -\sin 30° = -\frac{1}{2}$

(xiii) $\cos(-60°) = +\cos 60° = \frac{1}{2}$

(xiv) $\tan(-45°) = -\tan 45° = -1$

(xv) $\sin(-150°) = -\sin(150°) = -\sin(180°-30°) = -\sin 30° = -\frac{1}{2}$

Ex. The values of $\sin\theta_1$, $\cos^2\theta_2$ and $\tan\theta_3$ are given as ½ , – ½ and 3 (not in order), for some angles θ_1, θ_2 and θ_3. Choose incorrect statement.

(A) The value of $\tan\theta_3$ could be –½

(B) The value of $\sin\theta_1$ can not be 3.

(C) The value of $\cos^2\theta_2$ can't be –½

(D) The value of $\cos^2\theta_2$ could be 3.

Sol. **(D)**

$-1 \le \sin\theta_1 \le 1$, $0 \le \cos^2\theta_2 \le 1$, $-\infty < \tan\theta_3 < \infty$

Addition/Subtraction Formulae for Trigonometrical Ratios

(i) sin (A+B)=sinA cosB + cosA sinB

(ii) sin(A–B)=sinA cosB–cosA sinB

(iii) cos(A+B)=cosA cosB –sinA sinB

(iv) cos(A–B)=cosA cosB + sinA sinB

Ex. By using above basic addition/ subtraction formulae, prove that

(i) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

(ii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

(iii) $\sin 2\theta = 2\sin\theta\cos\theta$

(iv) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

(v) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Sol. **(i)** $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A\cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

$$= \frac{\cos A\cos B\left(\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}\right)}{\cos A \cos B\left(1 - \frac{\sin A \sin B}{\cos A\cos B}\right)} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(ii) $\tan(A-B) = \dfrac{\sin(A-B)}{\cos(A-B)} = \dfrac{\sin A\cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$

$$= \frac{\cos A\cos B\left[\frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}\right]}{\cos A \cos B\left[1 + \frac{\sin A \sin B}{\cos A\cos B}\right]} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(iii) $\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$

(iv) $\cos 2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta = 1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

$= 1 - 2(1-\cos^2\theta) = 2\cos^2\theta - 1$

(v) $\tan 2\theta = \tan(\theta+\theta) = \dfrac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Ex. Find the value of

(i) sin 74° **(ii)** cos 106° **(iii)** sin 15° **(iv)** cos 75°

Sol. **(i)** $\sin 74° = \sin(2\times 37°) = 2\sin 37 \cos 37° = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$

(ii) $\cos 106° = \cos(2\times 53°) = \cos^2 53°\ \sin^2 53° = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9-16}{25} = -\frac{7}{25}$

(iii) $\sin 15° = \sin(45°-30°) = \sin 45°\cos 30° - \cos 45° \sin 30° = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(iv) $\cos 75° = \cos(45°+30°) = \cos 45°\cos 30° - \sin 45° \sin 30° = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Small Angle Approximation

If θ is small (say < 5°) then $\sin\theta \approx \theta$, $\cos\theta \approx 1$ and $\tan\theta \approx \theta$

Note : here θ must be in radian.

Ex. Find the approximate values of (i) sin 1° (ii) tan 2° (iii) cos 1°.

Sol. **(i)** $\sin 1° = \sin\left(1° \times \frac{\pi}{180°}\right) = \sin\frac{\pi}{180} \approx \frac{\pi}{180}$ **(ii)** $\tan 2° = \tan\left(2° \times \frac{\pi}{180°}\right) = \tan\frac{\pi}{90} \approx \frac{\pi}{90}$

(iii) $\cos 1° = \cos\left(1° \times \frac{\pi}{180°}\right) = \cos\frac{\pi}{180} = 1$

Maximum and Minimum Values of Some useful Trigonometric Functions

(i) $-1 \le \sin\theta \le 1$ **(ii)** $-1 \le \cos\theta \le 1$ **(iii)** $-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$

Ex. Find maximum and minimum values of y :

(i) y = 2 sinx **(ii)** y = 4–cosx **(iii)** y = 3sinx + 4 cosx

Sol. **(i)** y_{max} =2(1)=2 and y_{min}=2(–1)=–2 **(ii)** y_{max} = 4 – (–1)=4+1=5 and y_{min} = 4–(1)=3

(iii) $y_{max} = \sqrt{3^2+4^2} = 5$ and $y_{min} = -\sqrt{3^2+4^2} = -5$

Ex. A ball is projected with speed u at an angle θ to the horizontal. The range R of the projectile is given by

$$R = \frac{u^2 \sin 2\theta}{g}$$

for which value of θ will the range be maximum for a given speed of projection? (Here g = constant)

(A) $\frac{\pi}{2}$ rad **(B)** $\frac{\pi}{4}$ rad **(C)** $\frac{\pi}{3}$ rad **(D)** $\frac{\pi}{6}$ rad

Sol. **(B)**

As $\sin 2\theta \le 1$ so range will be maximum if $\sin 2\theta = 1$. Therefore $2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$ rad.

Ex. The position of a particle moving along x-axis varies with time t according to equation $x = \sqrt{3}\sin\omega t - \cos\omega t$ where ω is constants. Find the region in which the particle is confined.

Sol. Q $x = \sqrt{3}\sin\omega t - \cos\omega t$

$$\therefore x_{max} = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2 \text{ and } x_{min} = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = -2$$

Thus, the particle is confined in the region $-2 \le x \le 2$

SECTION [C] : CO-ORDINATE GEOMETRY

To specify the position of a point in space, we use right handed rectangular axes coordinate system. This system consists of (i) origin (ii) axis or axes. If point is known to be on a given line or in a particular direction, only one coordinate is necessary to specify its position, if it is in a plane, two coordinates are required, if it is in space three coordinates are needed.

ORIGIN

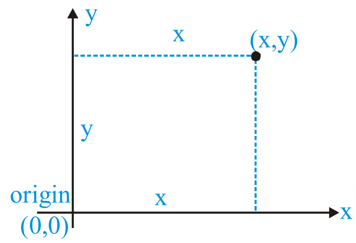
This is any fixed point which is convenient to you. All measurement are taken w.r.t. this fixed point.

Axis or Axes

Any fixed direction passing through origin and convenient to you can be taken as an axis. If the position of a point or position of all the points under consideration always happen to be in a particular direction, then only one axis is required. This is generally called the x-axis. If the positions of all the points under consideration are always in a plane, two perpendicular axes are required. These are generally called x and y-axis. If the points are distributed in a space, three perpendicular axes are taken which are called x, y and z-axis.

Position of a point in xy plane

The position of a point is specified by its distances from origin along (or parallel to) x and y-axis as shown in figure.



Here x-coordinate and y-coordinate is called abscissa and ordinate respectively.

Distance Formula

The distance between two points (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Ex. For point (2, 14) find abscissa and ordinates. Also find distance from y and x-axis.

Sol. Abscissa = x-coordinate = 2 = distance from y-axis.

Ordinate = y-coordinate = 14 = distance from x-axis.

Ex. Find value of a if distance between the points (–9 cm, a cm) and (3 cm, 3cm) is 13 cm.

Sol. By using distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \Rightarrow 13 = \sqrt{[13 - (-9)]^2 + [3 - a]^2}$

$\Rightarrow 13^2 = 12^2 + (3-a)^2$ $\Rightarrow (3-a)^2 = 13^2 - 12^2 = 5^2$ $\Rightarrow (3-a) = \pm 5$ $\Rightarrow a = -2$ cm or 8 cm

Ex. A dog wants to catch a cat. The dog follows the path whose equation is $y - x = 0$ while the cat follows the path whose equation is $x^2 + y^2 = 8$. The coordinates of possible points of catching the cat are :

(A) (2, –2) **(B)** (2, 2) **(C)** (–2, 2) **(D)** (–2, –2)

Sol. **(BD)**

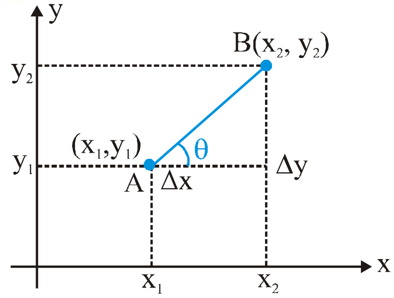
Let catching point be (x_1, y_1) then, $y_1 - x_1 = 0$ and $x_1^2 + y_1^2 = 8$

Therefore, $2x_1^2 = 8 \Rightarrow x_1^2 = 4 \Rightarrow x_1 = \pm 2$; So possible points are (2, 2) and (–2, –2).

Slope of a Line

The slope of a line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is denoted by m and is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta \quad \text{[If both axes have identical scales]}$$



Here θ is the angle made by line with positive x-axis. Slope of a line is a quantitative measure of inclination.

Ex. Distance between two points (8, – 4) and (0, a) is 10. All the values are in the same unit of length. Find the positive value of a.

Sol. **(2)**

From distance formula $(8-0)^2 + (-4-a)^2 = 100 \Rightarrow (4+a)^2 = 36 \Rightarrow a = 2$

SECTION [D] : CALCULUS

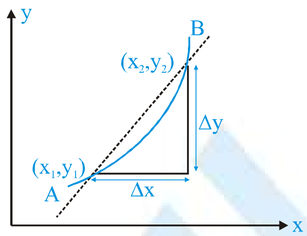
Calculus is the study of how things change. In this we study the relationship between continuously varying functions.

(A) DIFFERENTIAL CALCULUS

The purpose of differential calculus to study the nature (i.e., increase or decrease) and the amount of variation in a quantity when another quantity (on which first quantity depends) varies independently.

Average rate of change :

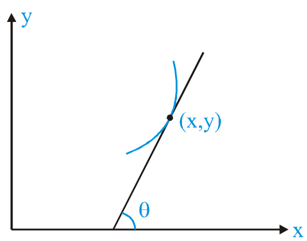
Let a function y = f(x) be plotted as shown in figure. Average rate of change in y w.r.t. x in interval $[x_1, x_2]$ is



Average rate of change $= \dfrac{\text{change in y}}{\text{change in x}} = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$ = slope of chord AB.

Instantaneous rate of change

It is defined as the rate of change in y with x at a particular value of x. It is measured graphically by the slope of the tangent drawn to the y-x graph at the point (x,y) and algebraically by the first derivative of function y =f(x).



Instantaneous rate of change $= \dfrac{dy}{dx}$ = slope of tangent = $\tan\theta$

First Derivatives of Commonly used Functions

(i) $y = \text{constant} \Rightarrow \dfrac{dy}{dx} = 0$ **(ii)** $y = x^n \Rightarrow \dfrac{dy}{dx} = nx^{n-1}$

(iii) $y = e^x \Rightarrow \dfrac{dy}{dx} = e^x$ **(iv)** $y = \ln x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x}$

(v) $y = \sin x \Rightarrow \dfrac{dy}{dx} = \cos x$ **(vi)** $y = \cos x \Rightarrow \dfrac{dy}{dx} = -\sin x$

(vii) $y = \tan x \Rightarrow \dfrac{dy}{dx} = \sec^2 x$ **(viii)** $y = \cot x \Rightarrow \dfrac{dy}{dx} = -\text{cosec}^2 x$

Method of Differentiation or Rules of Differentiation

(i) **Function multiplied by a constant i.e., y = kf(x)** $\Rightarrow \dfrac{dy}{dx} = kf'(x)$

Ex. Find derivatives of the following functions :

(i) $y = 2x^3$ **(ii)** $y = \dfrac{4}{x}$ **(iii)** $y = 3e^x$ **(iv)** $y = 6 \ln x$ **(v)** $y = 5 \sin x$

Sol. **(i)** $y = 2x^3 \Rightarrow \frac{dy}{dx} = 2\left[3x^{3-1}\right] = 6x^2$ **(ii)** $y = \frac{4}{x} = 4x^{-1} \Rightarrow \frac{dy}{dx} = 4\left[(-1)x^{-1-1}\right] = -\frac{4}{x^2}$

(iii) $y = 3e^x \Rightarrow \frac{dy}{dx} = 3e^x$ **(iv)** $y = 6\ln x \Rightarrow \frac{dy}{dx} = 6\left(\frac{1}{x}\right) = \frac{6}{x}$

(v) $y = 5\sin x \Rightarrow \frac{dy}{dx} = 5(\cos x) = 5\cos x$

(ii) Sum or Subtraction of Two functions i.e., $y = f(x) \pm g(x) \Rightarrow$ $\frac{dy}{dx} = f'(x) \pm g'(x)$

Ex. Find differentiation of y w.r.t x.
(i) $y = x^2 - 6x$ **(ii)** $y = x^5 + 2e^x$ **(iii)** $y = 4\ln x + \cos x$

Sol. **(i)** $\frac{dy}{dx} = 2x^{2-1} - 6(1) = 2x - 6$ **(ii)** $\frac{dy}{dx} = 5x^{5-1} + 2e^x = 5x^4 + 2e^x$ **(iii)** $\frac{dy}{dx} = 4\left(\frac{1}{x}\right) + (-\sin x) = \frac{4}{x} - \sin x$

(iii) Product of two functions : Product rule

$$y = f(x) \cdot g(x) \Rightarrow \mathbf{\frac{dy}{dx} = f(x) \times g'(x) + f'(x) \times g(x)}$$

Ex. Find first derivative of y w.r.t. x.
(i) $y = x^2 \sin x$ **(ii)** $y = 4(e^x)\cos x$

Sol. **(i)** $\frac{dy}{dx} = x^2(\cos x) + (2x)(\sin x) = x^2\cos x + 2x\sin x$

(ii) $\frac{dy}{dx} = 4\left[(e^x)(\cos x) + (e^x)(-\sin x)\right] = 4e^x\left[\cos x - \sin x\right]$

(iv) Division of two functions : Quotient rule

$$\mathbf{y = \frac{f(x)}{g(x)} \Rightarrow \frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$$

Ex. Find differentiation of y w.r.t. x. **(i)** $y = \frac{\sin x}{x}$ **(ii)** $y = \frac{4x^3}{e^x}$

Sol. **(i)** Here $f(x) = \sin x$, $g(x) = x$ So $f'(x) = \cos x$,, $g'(x) = 1$

Therefore $\frac{dy}{dx} = \frac{(\cos x)(x) - (\sin x)(1)}{x^2} = \frac{x\cos x - \sin x}{x^2}$

(ii) Here $f(x) = 4x^3$, $g(x) = e^x$ So $f'(x) = 12x^2, g'(x) = e^x$

Therefore, $\frac{dy}{dx} = \frac{12x^2(e^x) - 4x^3(e^x)}{(e^x)^2} = \frac{12x^2 - 4x^3}{e^x}$

(v) Function of Functions : Chain rule

Let f be a function of x, which in turn is a function of t. The first derivative of f w.r.t. t is equal to the product of $\frac{df}{dx}$ and $\frac{dx}{dt}$ Therefore $\frac{df}{dt} = \frac{df}{dx} \times \frac{dx}{dt}$

Ex. Find first derivative of y w.r.t. x.

(i) $y = e^{-x}$ **(ii)** $y = 4\sin 3x$ **(iii)** $y = 4e^{x^2-2x}$

Sol. (i) $y = e^{-x} = e^{z}$ where $z = -x$ So $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = (e^{z})(-1) = -e^{z} = -e^{-x}$

(ii) $y = 4\sin 3x = 4\sin z$ where $z = 3x$ So $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = 4(\cos z)(3) = 12\cos 3x$

(iii) $y = 4e^{x^2-2x} = 4e^{z}$ where $z = x^2 - 2x$ So $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = 4(z)(2x-2) = (8x-8)e^{x^2-2x}$

Ex. The position of a particle moving along x-axis varies with time t as $x=4t-t^2+1$. Find the time interval(s) during which the particle is moving along positive x-direction.

Sol. If the particle moves along positive x-direction, its x-coordinate must increase with time t.

x-coordinate will increase with time t if $\frac{dx}{dt} > 0$.

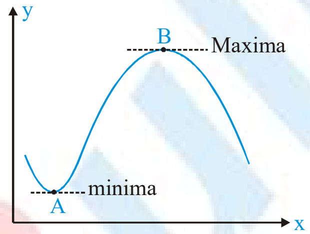
$$\frac{dx}{dt} = 4 - 2t$$

$$\frac{dx}{dt} > 0 \Rightarrow 4 - 2t > 0 \Rightarrow t < 2$$

Hence, the particle moves in positive x-direction during time-interval $0 < t < 2$.

MAXIMUM AND MINIMUM VALUE OF A FUNCTION

Higher order derivatives are used to find the maximum and minimum values of a function. At the points of maxima and minima, first derivative becomes zero.



At point 'A' (minima) : As we see in figure, in the neighborhood of A, slope is increases so $\frac{d^2y}{dx^2} > 0$.

Condition for minima : $\boxed{\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0}$

At point 'B' (maxima) : As we see in figure, in the neighborhood of B, slope is decreases so $\frac{d^2y}{dx^2} < 0$

Condition for maxima : $\boxed{\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} < 0}$

Ex. The minimum value of $y = 5x^2 - 2x + 1$ is

(A) $\frac{1}{5}$ (B) $\frac{2}{5}$ (C) $\frac{4}{5}$ (D) $\frac{3}{5}$

Sol. (C)

For maximum/minimum value $\frac{dy}{dx} = 0 \Rightarrow 5(2x) - 2(1) + 0 = 0 \Rightarrow x = \frac{1}{5}$

Now at $x=\frac{1}{5}$, $\frac{d^2y}{dx^2}=10$ which is positive so minima at $x = \frac{1}{5}$.

Therefore $y_{\min} = 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 1 = \frac{4}{5}$

Ex. The radius of a circular plate increase at the rate of 0.1 cm per second. At what rate does the area increase when the radius of plate is $\frac{5}{\pi}$ cm ?

(A) 1 cm^2/s **(B)** 0.1 cm^2/s **(C)** 0.5 cm^2/s **(D)** 2 cm^2/s

Sol. **(A)**

Area of disk, $A = \pi r^2$ (where r = radius of disk)

$\frac{dA}{dt} = \pi\left(2r\frac{dr}{dt}\right) = 2\pi r\frac{dr}{dt}$ So $\frac{dA}{dt} = 2\pi \times \frac{5}{\pi} \times 0.1 = 1\ cm^2/s$

Ex. A particle moves along the curve $12y = x^3$. Which coordinate changes at faster rate at x=10 ?

(A) x–coordinate **(B)** y–coordinate

(C) Both x and y–coordinate **(D)** Data insufficient

Sol. **(B)**

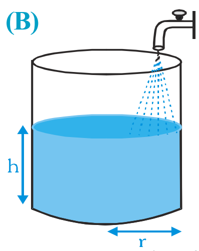
$12y = x^3 \Rightarrow 12dy = 3x^2dx \Rightarrow \frac{dy}{dt} = \left(\frac{x}{2}\right)^2\left(\frac{dx}{dt}\right)$

Therefore for $\left(\frac{x}{2}\right)^2 > 1$ or $x > 2$, y– coordinate changes at faster rate.

Ex. Water pours out at the rate of Q from a tap, into a cylindrical vessel of radius r. The rate at which the height of water level rises when the height is h, is

(A) $\frac{Q}{\pi rh}$ **(B)** $\frac{Q}{\pi r^2}$ **(C)** $\frac{Q}{2\pi r^2}$ **(D)** $\frac{Q}{\pi r^2 h}$

Sol. **(B)**



Volume : $V = \pi r^2 h$ $\therefore \frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$ But $\frac{dV}{dt} = Q$ so $\frac{dh}{dt} = \frac{Q}{\pi r^2}$

Ex. For conservative force field, $\vec{F} = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$

where F → Force, U → Potential energy and $\frac{\partial U}{\partial x}$ = Differentiation of U w.r.t. x keeping y and z constant and so on.

	Column – I		Column – II
(A)	For $U = x^2yz$, at (5,0,0)	(P)	$F_x = 0$
(B)	For $U = x^2 + yz$ at (5, 0,0)	(Q)	$F_y = 0$
(C)	For $U = x^2(y+z)$ at (5, 0,0)	(R)	$F_z = 0$
(D)	For $U = x^2y+z$ at (5,0,0)	(S)	$U = 0$

Sol. **(A) – P,Q,R,S ; (B) – Q,R ; (C) – P,S ; (D) – P,S**

For (A) : $\vec{F} = -2xyz\,\hat{i} - x^2z\hat{j} - x^2y\hat{k} \Rightarrow F_x=0, F_y=0, F_z=0, U=0$

For (B) : $\vec{F} = -2x\hat{i} - z\hat{j} - y\hat{k} \Rightarrow F_x \neq 0, F_y=0, F_z=0, U \neq 0$

For (C) : $\vec{F} = -2x(y+z)\hat{i} - x^2\hat{j} - x^2\hat{k} \Rightarrow F_x=0, F_y \neq 0, F_z \neq 0, U=0$

For (D) : $\vec{F} = -2xy\,\hat{i} - x^2\hat{j} - \hat{k} \Rightarrow F_x=0, F_y \neq 0, F_z \neq 0, U=0$

Ex. If surface area of a cube is changing at a rate of 5 m^2/s, find the rate of change of body diagonal at the moment when side length is 1 m.

(A) 5 m/s **(B)** $5\sqrt{3}$ m/s **(C)** $\frac{5}{2}\sqrt{3}$ m/s **(D)** $\frac{5}{4\sqrt{3}}$ m/s

Sol. **(D)**

Surface area of cube $S=6a^2$ (where a = side of cube)

Body diagonal $l = \sqrt{3}a$. Therefore $S=2l^2$

Differentiating it w.r.t. time $\frac{dS}{dt} = 2(2l)\frac{dl}{dt} \Rightarrow \frac{dl}{dt} = \frac{1}{4(\sqrt{3}a)}\frac{dS}{dt} = \frac{5}{4\sqrt{3}}$ m/s

(B) INTEGRAL CALCULUS

Integration is the reverse process of differentiation. By help of integration we can find a function whose derivative is known. Consider a function F**(x)** whose differentiation w.r.t. x is equal to f**(x)** then

$$\int f(x)dx = F(x)+c$$

here c is the constant of integration and this is called indefinite integration.

Few basic formulae of integration are :

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

(ii) $\int \frac{1}{x} dx = \ln x + c$

(iii) $\int \sin x dx = -\cos x + c$

(iv) $\int \cos x dx = \sin x + c$

(v) $\int \sec^2 x dx = \tan x + c$

(vi) $\int e^x dx = e^x + c$

(vii) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

(viii) $\int \frac{dx}{ax+b} = \frac{\ln(ax+b)}{a} + c$

(ix) $\int \sin(ax+b) dx = \frac{-\cos(ax+b)}{a} + c$

(x) $\int \cos(ax+b) dx = \frac{\sin(ax+b)}{a} + c$

(xi) $\int \sec^2(ax+b) dx = \frac{\tan(ax+b)}{a} + c$

(xii) $\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + c$

Ex. Integrate the following w.r.t. x.

(i) $4x^3$ **(ii)** $x - \frac{1}{x}$ **(iii)** $\frac{1}{2x+3}$ **(iv)** cos (4x+3) **(v)** $\cos^2 x$

Sol. (i) $\int 4x^3 dx = 4\left(\frac{x^{3+1}}{3+1}\right)+c = \frac{4x^4}{4}+c = x^4+c$

(ii) $\int\left(x-\frac{1}{x}\right)dx = \int x dx - \int \frac{1}{x}dx = \frac{x^2}{2} - \ln x + c$

(iii) $\int \frac{dx}{2x+3} = \frac{\ln(2x+3)}{2}+c$

(iv) $\int \cos(4x+3)dx = \frac{\sin(4x+3)}{4}+c$

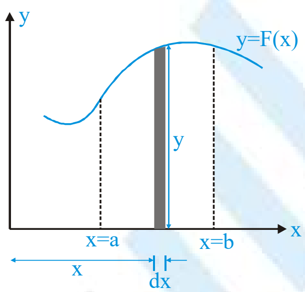
(v) $\int \cos^2 x dx = \int \frac{2\cos^2 x}{2}dx = \int \frac{(1+\cos 2x)}{2}dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx = \frac{x}{2}+\frac{1}{2}\frac{\sin 2x}{2}+c = \frac{x}{2}+\frac{1}{4}\sin 2x + c$

Definite Integration

When a function is integrated between a lower limit and an upper limit, it is called a definite integral. Consider a function F**(x)** whose differentiation w.r.t. x is equal to f**(x)**, in an interval $a \le x \le b$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Area under a curve and definite integration



Area of small shown element = ydx = f(x) dx

If we sum up all areas between x=a and x= b then $\int_a^b f(x)\,dx$ = shaded area between curve and x-axis.

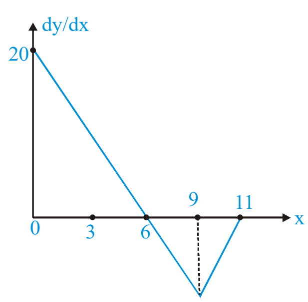
Ex. The integral $\int_1^5 x^2 dx$ is equal to

(A) $\frac{125}{3}$ (B) $\frac{124}{3}$ (C) $\frac{1}{3}$ (D) 45

Sol. (B)

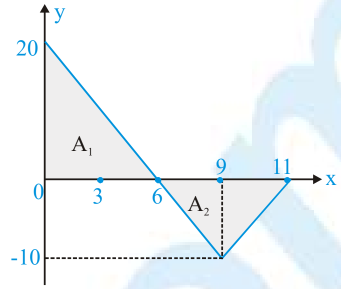
$$\int_1^5 x^2 dx = \left[\frac{x^3}{3}\right]_1^5 = \left[\frac{5^3}{3}-\frac{1^3}{3}\right] = \frac{125}{3}-\frac{1}{3} = \frac{124}{3}$$

Ex. The following curve represent rate of change of a variable y w.r.t x. The change in the value of y when x changes from 0 to 11 is:



(A) 60 (B) 25 (C) 35 (D) 85

Sol. As $dy = \left(\frac{dy}{dx}\right)dx$ So $\Delta y = \int dy = \int_0^{11}\left(\frac{dy}{dx}\right)dx$



Area under the curve

$$A_1 = \frac{1}{2} \times 6 \times 20 = 60$$

$$A_2 = -\frac{1}{2} \times (11-6)(10) = -25$$

$$\Delta y = A_1 + A_2 = 60 - 25 = 35$$

Average value of a continuous function in an interval :

Average value of a function y = f(x), over an interval $a \le x \le b$ is given by

$$y_{av} = \frac{\int_a^b y\,dx}{\int_a^b dx} = \frac{\int_a^b y\,dx}{b-a}$$

Ex. Determine the average value of y = 2x +3 in the interval $0 \le x \le 1$.

(A) 1 **(B)** 5 **(C)** 3 **(D)** 4

Sol. **(D)**

$$y_{av} = \frac{\int_0^1 y\,dx}{1-0} = \int_0^1 (2x+3)\,dx = \left[2\left(\frac{x^2}{2}\right) + 3x\right]_0^1 = 1^2 + 3(1) - 0^2 - 3(0) = 1 + 3 = 4$$

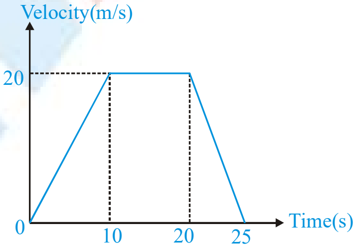
Ex. The average value of alternating current $I = I_0 \sin\omega t$ in time interval $\left[0, \frac{\pi}{\omega}\right]$ is

(A) $\frac{2I_0}{\pi}$ **(B)** $2I_0$ **(C)** $\frac{4I_0}{\pi}$ **(D)** $\frac{I_0}{\pi}$

Sol. **(A)**

$$I_{av} = \frac{\int_0^{\pi/\omega} I\,dt}{\frac{\pi}{\omega} - 0} = \frac{\omega}{\pi}\int_0^{\pi/\omega} I_0 \sin\omega t\,dt = \frac{\omega}{\pi}\left[\frac{I_0(-\cos\omega t)}{\omega}\right]_0^{\pi/\omega} = -\frac{\omega}{\pi}\frac{I_0}{\omega}[\cos\pi - \cos 0] = -\frac{I_0}{\pi}[-1-1] = \frac{2I_0}{\pi}$$

Ex. The velocity-time graph of a car moving along a straight road is shown in figure. The average velocity of the car is first 25 seconds is



(A) 20 m/s **(B)** 14 m/s **(C)** 10 m/s **(D)** 17.5 m/s

Sol. **(B)**

$$\text{Average velocity} = \frac{\int_0^{25} v\,dt}{25-0} = \frac{\text{Area of v-t graph between t=0 to t = 25 s}}{25} = \frac{1}{25}\left[\left(\frac{25+10}{2}\right)(20)\right] = 14 \text{ m/s}$$

SECTION [E] : FUNCTION AND GRAPH

FUNCTION

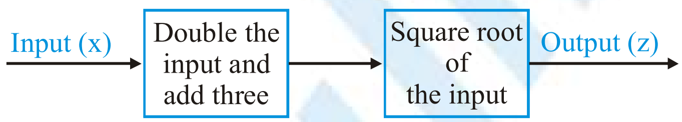
Physics involves study of natural phenomena and describes them in terms of several physical quantities. A mathematical formulation of interdependence of these physical quantities is necessary for a concise and precise description of the phenomena. These mathematical formulae are expressed in form of equations and known as function.

Thus, a function describing a physical process expresses an unknown physical quantity in terms of one or more known physical quantities. We call the unknown physical quantity as dependent variable and the known physical quantities as independent variables. For the sake of simplicity, we consider a function that involves a dependant variable y and only one independent variable x. It is denoted y=f(**x**) and is read as y equals to f of x. Here f(**x**) is the value of y for a given x. Following are some examples of functions.

$y=2x+1$, $y=2x^2+3x+1$, $y=\sin x$, $y=\ell n(2x+1)$

Knowledge of the dependant variable for different values of the independent variable, and how it changes when the independent variable varies in an interval is collectively known as behavior of the function.

Ex. In the given figure, each box represents a function machine. A function machine illustrates what it does with the input.



Which of the following statements are correct?

(A) $y=2x+3$ **(B)** $y=2(x+3)$ **(C)** $z=\sqrt{2x+3}$ **(D)** $z=\sqrt{2(x+3)}$

Sol. **(C)**

GRAPH OF A FUNCTION

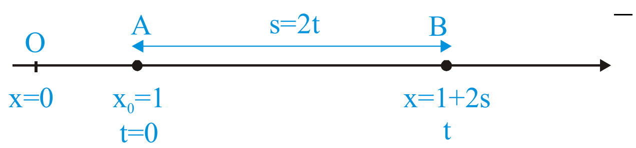
Graph is diagrammatic representation of a function and allows us to visualize it. To plot a graph the dependant variable (here y) is usually taken on the ordinate and the independent variable (here x) on the abscissa. Graph being an alternative way to represent a function does not require elaborate calculations and explicitly shows behavior of the function in a concerned interval.

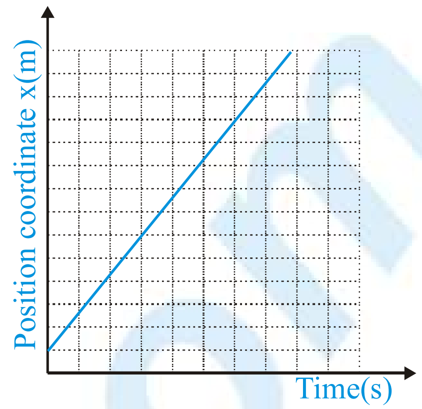
Ex. Consider a body moving with constant speed of 2 m/s in a straight line. When you start your stopwatch, you observe the body 1 m away from a fixed point on the line. Suggest suitable physical quantities, write a function and draw its graph describing motion of the body.

Sol. Distance x of the body from the given fixed point and time t measured by the stopwatch are the suitable variables. If we consider the fixed point as the origin, distance x is known as the position coordinate of the body.

In the following figure it is shown that the body is on point A at the instant $t = 0$ and after a time t it reaches another point B covering a distance, which equals to product of speed and time interval. Thus, distance s covered by the body in time t is given by the following equation.

$$s = 2t$$





With the help of the above figure, position coordinate x of the body at any time t is given by the following equation, which is the required function describing motion of the body.

$$x=2t+1$$

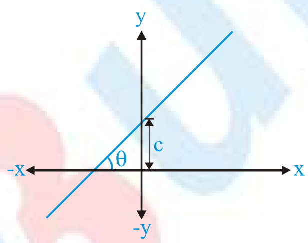
Graph of this equation is also shown in the adjoining figure.

GRAPHS OF SOME COMMONLY USED FUNCTIONS

Linear, parabolic, trigonometric and exponential functions are the most common in use.

(i) Straight line Equation and its Graph

When the dependant variable y varies linearly with the independent variable x, the relationship between them is represented by a linear equation of the type given below. The equation is also shown in graph by an arbitrary line.

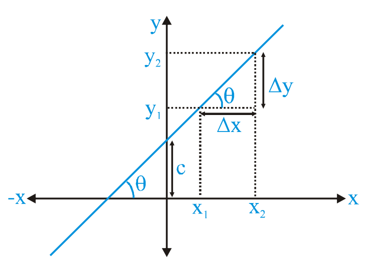


$$y = mx + c$$

Here m & c are known as slope of the line and intercept on the y-axis, respectively.

Slope

Slope of a line is a quantitative measure to express the inclination of the line. It is expressed by ratio of change in ordinate to change in abscissa.



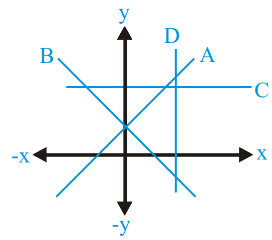
Slope of a line $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x}$ = slope of tangent

When the x and the y axes are scaled identically, slope equals to tangent of the angle, which line makes with the positive x-axis.

$m=\tan\theta$

Sometimes the slope is also called gradient and expressed by the term "Δy in ●" where ● is the length along the line $l = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

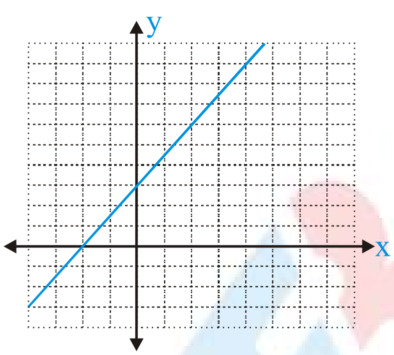
It is positive if y increases with increase in x, negative if y decreases with increase in x, zero if y remains unchanged with change in x and infinite if y changes but x remains unchanged. For these cases the line is inclined up, inclined down, parallel to x-axis and parallel to y-axis respectively as shown in the adjoining figure by lines A, B, C and D respectively.

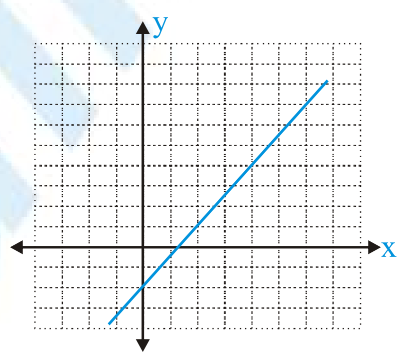


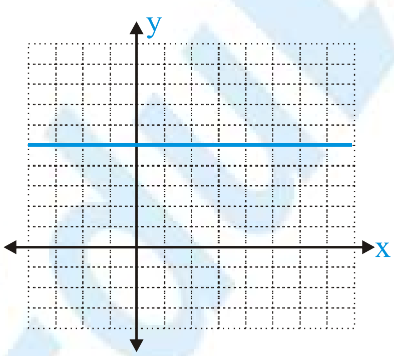
INTERCEPT

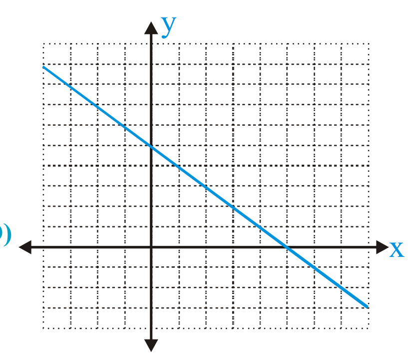
It equals to the value of ordinate y, where the line cuts the y-axis. It may be positive, negative or zero for lines crossing the positive y-axis, negative y-axis and passing through the origin respectively.

Ex. Write equations for the straight lines shown in the following graphs.

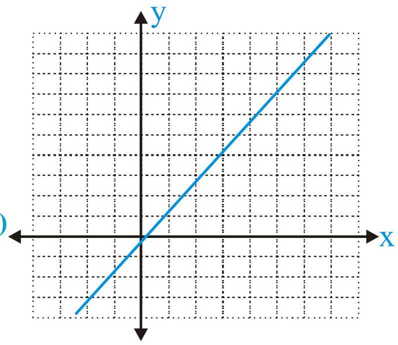
(A)


(B)


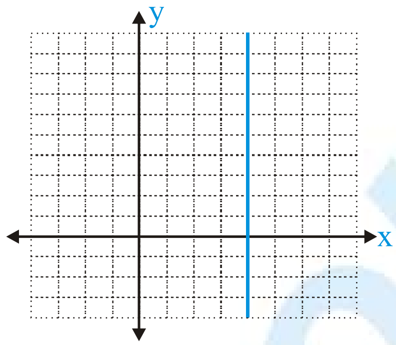
(C)


(D)


(E)

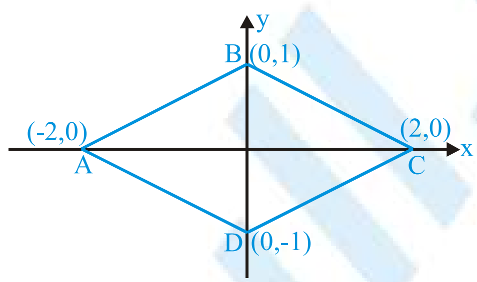


(F)



Sol. (A) $y=\frac{3}{2}x+3$; (B) $y=\frac{3}{2}x-2$; (C) y=5; (D) y=–x+5 (E) $y=\frac{3}{2}x$; (F) x=4

Ex. A parallelogram ABCD is shown in figure.



	Column I		Column II
(A)	Equation of side AB	(P)	$2y+x=2$
(B)	Equation of side BC	(Q)	$2y-x=2$
(C)	Equation of side CD	(R)	$2y+x=-2$
(D)	Equation of side DA	(S)	$2y-x=-2$
		(T)	$y+2x=2$

Sol. (A) Q (B) P (C) S (D) R

For side AB : $m=\dfrac{1-0}{0-(-2)}=\dfrac{1}{2}$, $c=1 \Rightarrow y=\dfrac{1}{2}x+1$

For side BC : $m=\dfrac{2-0}{0-1}=-2$, $c=1 \Rightarrow y=-2x+1$

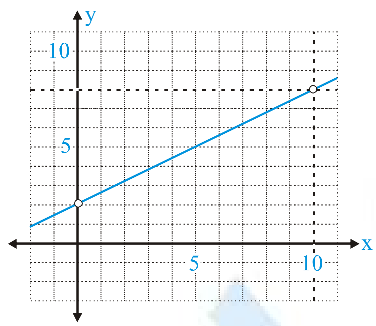
For side CD : $m=\dfrac{0-(-1)}{2-0}=\dfrac{1}{2}$, $c=-1 \Rightarrow y=\dfrac{1}{2}x-1$

For side DA : $m=\dfrac{-1-0}{0-(-2)}=-\dfrac{1}{2}$, $c=-1 \Rightarrow y=-\dfrac{1}{2}x-1$

Ex. A variable y increases from $y_1 = 2$ to $y_2 = 8$ linearly with another variable x in the interval $x_1 = 0$ to $x_2 = 10$. Express y as function of x and draw its graph.

Sol. Linear variation is represented by a linear equation of the form y=mx+c. To represent the function on graph we have to join two points whose coordinates are (x_1, y_1) and (x_2, y_2) i.e. (0, 2) and (10, 8).

Slope of the line is $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{8-2}{10-0} = \dfrac{3}{5}$

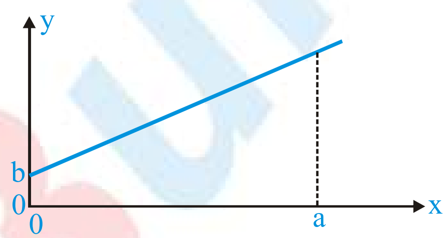


From the graph, intercept is c=2. Now the required equation is $y = \dfrac{3}{5}x + 2$

Ex. The graph shows a linear relation between variable y and x. Consider two quantities p and q defined by the equations .

$$p = \frac{y}{x}$$

$$q = \frac{y-b}{x}$$



As x changes from zero to a, which of the following statements are correct according to the graph?

(A) Quantity p increases and q decrease.
(B) Quantity p decrease and q increases.
(C) Quantity p decreases and q remain constant.
(D) Quantity p increases and q remain constant.

Sol. **(C)**

q is slope of the given line, which is a constant for a straight line.

p is slope of the line join origin and point on the line, which is decrease as x increases.

Ex. Frequency f of a simple pendulum depends on its length ● and acceleration g due to gravity according to the following equation

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

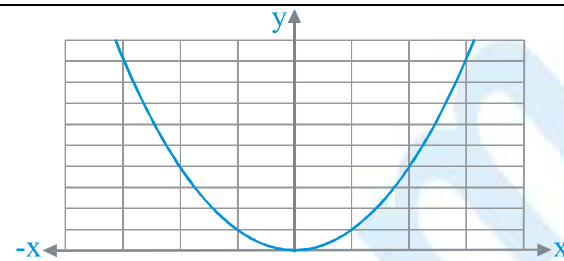
Graph between which of the following quantities is a straight line?

(A) f on the ordinate and l on the abscissa
(B) f on the ordinate and √λ on the abscissa
(C) f^2 on the ordinate and l on the abscissa
(D) f^2 on the ordinate and 1/λ on the abscissa

Ans. **(D)**

(ii) **Parabola equation and its graph**

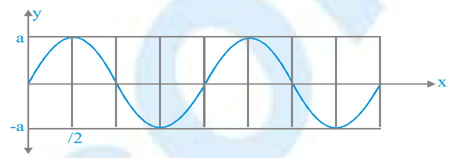
A function of the form $y=ax^2+bx+c$ is known as parabola.
The simplest parabola has the form $y=ax^2$.
Its graph is shown in the following figure.



GRAPH OF SOME TRIGONOMETRIC FUNCTIONS

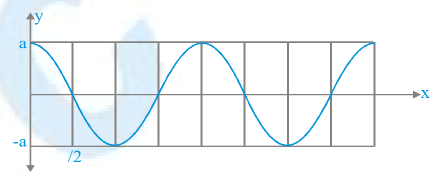
Among all the trigonometric functions, sinusoidal function, which includes sine and cosine both is most common in use.

Sine Function $y = a \sin x$



Here, a is known as the amplitude and equals to the maximum magnitude of y. In the adjoining figure graph of a sine function is shown, which has amplitude a units.

Cosine Function $y = a \cos x$

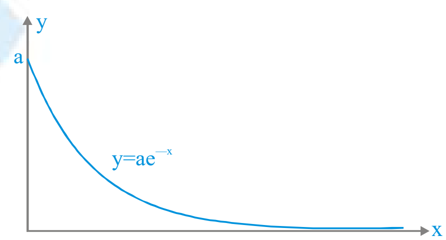


Here, a is known as the amplitude and equals to the maximum magnitude of y. In the adjoining figure graph of a cosine function is shown, which has amplitude a units.

(iii) **Exponential function and its graph**

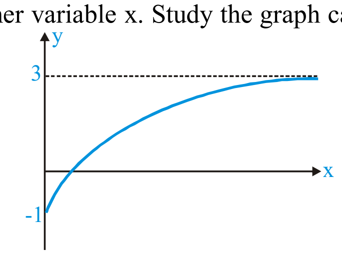
Behavior of several physical phenomena is described by exponential function to the base e. Here e is known as Euler's Number. $e = 2.718218$

Most commonly used exponential function has the form $y=ae^{-x}$. In the adjoining figure graph of this function is shown.



Ex. In the given figure is shown a variable y varying exponentially on another variable x. Study the graph carefully. Which of the following equations best suits the shown graph?

(A) $y = 3 - e^{-x}$ **(B)** $y = 1 - 4e^{-x}$
(C) $y = 1 - 3e^{-x}$ **(D)** $y = 3 - 4e^{-x}$

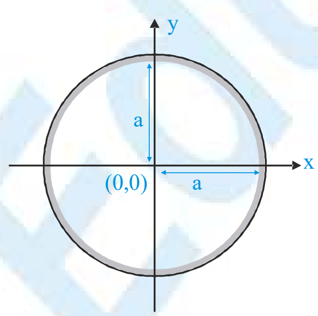


Sol. **(D)**

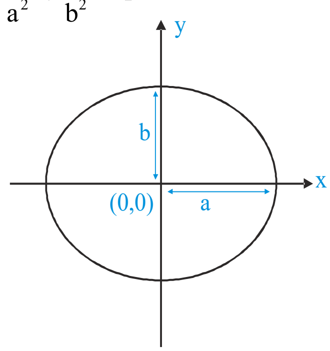
Shift the curve $(-4e^{-x})$ in positive y-direction by 3 units.

(iv) **Circle and Ellipse**

Circle : $x^2 + y^2 = a^2$



Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



VECTOR

VECTORS

A vector has both magnitude and sense of direction, and follows triangle law of vector addition.
For example, displacement, velocity, and force are vectors.

Vector quantities are usually denoted by putting an arrow over the corresponding letter, as $\vec{A}$ or $\vec{a}$. Sometimes in print work (books) vector quantities are usually denoted by boldface letters as **A** or **a**.

Magnitude of a vector $\vec{A}$ is a positive scalar and written as $|\vec{A}|$ or A.

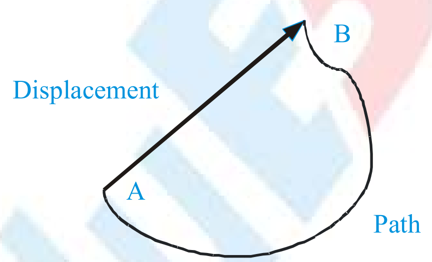
Unit Vector

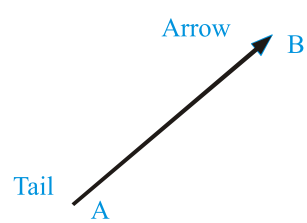
It is mathematical way to express direction of a vector and defined by the ratio of a vector to its magnitude. When a unit vector is multiplied with a scalar magnitude, we get a vector of corresponding magnitude in the direction of the unit vector. A unit vector is usually represented by putting a sign (ˆ) known as cap, hat or caret over a letter assigned to the unit vector. This letter may be the same as used for the vector, or its lower case letter, or some other symbol. For example, if we assign lower case letter a to unit vector in the direction of vector $\vec{A}$, the unit vector denoted by $\hat{a}$ is expressed by the following equation.

$\vec{A} = A\hat{a}$

Geometrical Representation of Vectors.

Geometrically a vector is represented by a directed straight-line segment drawn to a scale. Starting point of the directed line segment is known as tail and the end-point as arrow, head, or tip. The orientation of the line and the arrow collectively show the direction and the length of the line drawn to a scale shows the magnitude.
For example let a particle moves from point A to B following a curvilinear path shown in the figure. It displacement vector is straight line AB directed form A to B. If straight-line distance between A and B is 25 m, the directed line segment has to be drawn to suitable scale. If we assume the scale 1.0 cm = 10 m, the geometrical length of the displacement vector AB must be 2.5 cm.





Geometrical representation of Displacement Vector

Addition of Vectors: The Triangle Law

Use of geometry in solving problems involving vectors is of fundamental nature. The triangle law also uses principles of plane geometry. This law states:
The vectors to be added are drawn in such a manner that the tail of a vector coincides the tip of the preceding vector (in tip to tail fashion); their resultant is defined by the vector drawn from the tail of the first vector to the tip of the second vector. The two vectors to be added and their resultant are coplanar.

Consider vectors $\vec{A}$ and $\vec{B}$ shown in the figure-I. Using the triangle law, we obtain geometrical construction shown in the figure-II, where it is shown that two vectors and their sum $\vec{C} = \vec{A} + \vec{B}$ always make a closed triangle.

If we change order of vector $\vec{A}$ and $\vec{B}$, it shown in figure-III that sum given by equation $\vec{C} = \vec{B} + \vec{A}$ remain unchanged. Therefore, vector addition is commutative.

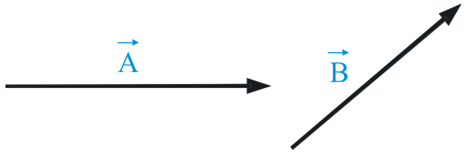


Fig. (i)

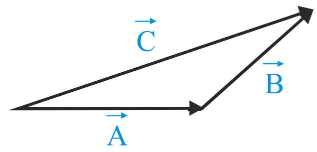


Fig. (ii)

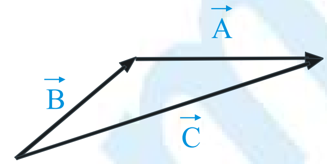
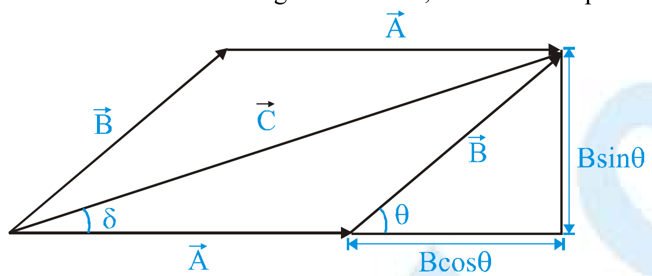


Fig. (iii)

Construction, which is combination of the figure-II and III, is in form of a parallelogram and is shown in figure-IV.

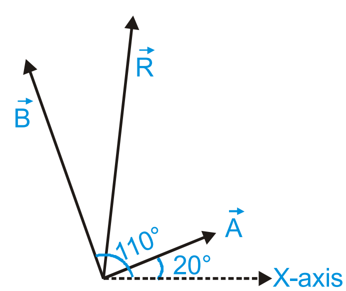


Geometry of the above figure suggests the following results.

$$C = \sqrt{A^2 + B^2 + 2AB\cos\theta}\ ;\ \delta = \tan^{-1}\left(\frac{B\sin\theta}{A + B\cos\theta}\right)$$

Ex. A vector $\vec{A}$ and $\vec{B}$ make angles of 20° and 110° respectively with the X–axis. The magnitudes of these vectors are 5m and 12m respectively. Find their resultant vector.

Sol. Angle between the $\vec{A}$ and $\vec{B}$ $= 110° - 20° = 90°$



$R = \sqrt{A^2 + B^2 + 2AB\cos 90^\circ} = \sqrt{5^2 + 12^2} = 13\text{m}$

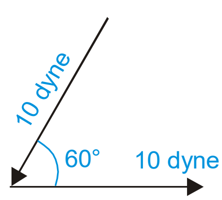
Let angle of $\vec{R}$ from $\vec{A}$ is α

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta} = \frac{12\sin 90^\circ}{5 + 12\cos 90^\circ} = \frac{12\times 1}{5 + 12\times 0} = \frac{12}{5}$$

or $\alpha = \tan^{-1}\left(\frac{12}{5}\right)$ with vector $\vec{A}$ or $(\alpha + 20°)$ with X–axis

Ex. Two forces each numerically equal to 10 dynes are acting as shown in the figure, then find resultant of these two vectors.

Sol. The angle θ between the two vectors is 120° and not 60°.



$\therefore\ R = \sqrt{(10)^2 + (10)^2 + 2(10)(10)(\cos 120^\circ)}$

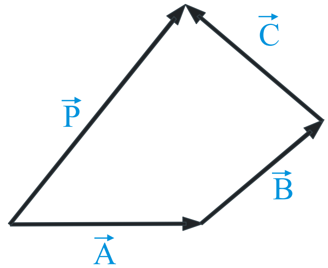
$= \sqrt{100 + 100 - 100} = 10$ dyne

Addition of more than two Vectors

The triangle law can be extended to define addition of more than two vectors. Accordingly, if vectors to be added are drawn in tip to tail fashion, resultant is defined by a vector drawn from the tail of the first vector to the tip of the last vector. This is also known as the polygon rule for vector addition.

Operation of addition of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ and their resultant $\vec{P}$ are shown in figure.

$$\vec{A} + \vec{B} + \vec{C} = \vec{P}$$

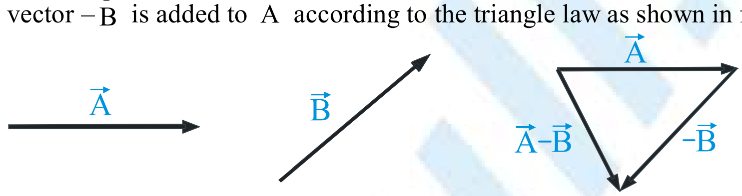


Here it is not necessary that three or more vectors and their resultant are coplanar. In fact, the vectors to be added and their resultant may be in different planes. However if all the vectors to be added are coplanar, their resultant must also be in the same plane containing the vectors.

Subtraction of Vectors

A vector opposite in direction but equal in magnitude to another vector $\vec{A}$ is known as negative vector of $\vec{A}$. It is written as $-\vec{A}$. Addition of a vector and its negative vector results a vector of zero magnitude, which is known as a null vector. A null vector is denoted by arrowed zero $(\vec{0})$.

The idea of negative vector explains operation of subtraction as addition of negative vector. Accordingly to subtract a vector from another consider vectors $\vec{A}$ and $\vec{B}$ shown in the figure. To subtract $\vec{B}$ from $\vec{A}$, the negative vector $-\vec{B}$ is added to $\vec{A}$ according to the triangle law as shown in figure-II.



Multiplying by a number

Multiplication by a positive number changes magnitude of the vector but not the direction and multiplication by a negative number changes magnitude and reverses direction.

Thus multiplying a vector by a number n makes magnitude of the vector n times. $n\vec{A} = (nA)\hat{a}$

Here $\hat{a}$ denotes the unit vector in the direction of vector $\vec{A}$.

Resolution of a Vector into Components

Following laws of vector addition, a vector can be represented as a sum of two (in two-dimensional space) or three (in three-dimensional space) vectors each along predetermined directions. These directions are called axes and parts of the original vector along these axes are called components of the vector.

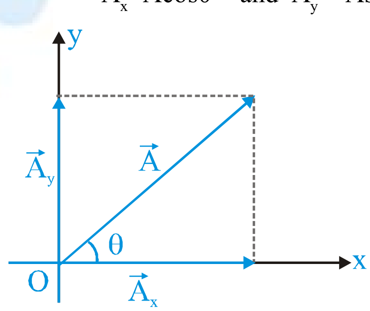
Cartesian components in two dimensions

If a vector is resolved into its components along mutually perpendicular directions, the components are called Cartesian or rectangular components.

In figure is shown, a vector $\vec{A}$ resolved into its Cartesian components $\vec{A}_x$ and $\vec{A}_y$ along the x and y-axis. Magnitudes A_x and A_y of these components are given by the following equation.

$$A_x = A\cos\theta \quad \text{and} \quad A_y = A\sin\theta$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$



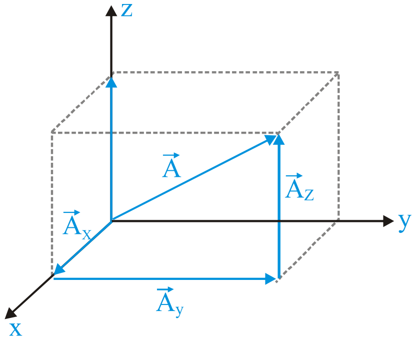
$$A = \sqrt{A_x^2 + A_y^2}$$

Here $\hat{i}$ and $\hat{j}$ are the unit vectors for x and y coordinates respectively.

Mathematical operations e.g. addition, subtraction, differentiation and integration can be performed independently on these components. This is why in most of the problems use of Cartesian components becomes desirable.

Cartesian components in three dimensions

A vector $\vec{A}$ resolved into its three Cartesian components one along each of the directions x, y, and z-axis is shown in the figure.



$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\ ;\ A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Equal Vectors

Two vectors of equal magnitudes and same directions are known as equal vectors. Their x, y and z components in the same coordinates system must be equal.

If two vectors $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$ are equal vectors, we have

$\vec{a} = \vec{b} \Rightarrow a_x=b_x,\ a_y=b_y$ and $a_z=b_z$

Parallel Vectors

Two parallel vectors must have the same direction and may have unequal magnitudes. Their x, y and z components in the same coordinate system bear the same ratio.

Consider two vectors $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, if they are parallel, we have

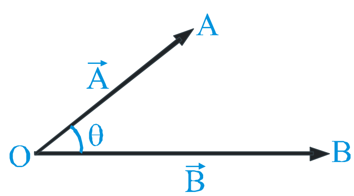
$$\vec{a} \cdot \vec{b} \Rightarrow \frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}$$

Product of Vectors

In all physical situation, whose description involve product of two vectors, only two categories are observed. One category where product is also a vector involves multiplication of magnitudes of two vectors and sine of the angle between them, while the other category where product is a scalar involves multiplication of magnitudes of two vectors and cosine of the angle between them. Accordingly, we define two kinds of product operation. The former category is known as vector or cross product and the latter category as scalar or dot product.

(i) Scalar or dot product of two vectors

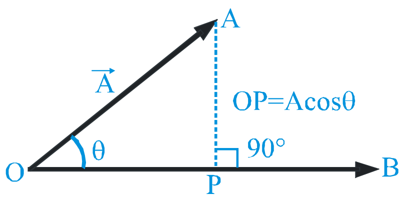
The scalar product of two vectors $\vec{A}$ and $\vec{B}$ equals to the product of their magnitudes and the cosine of the angle θ between them.

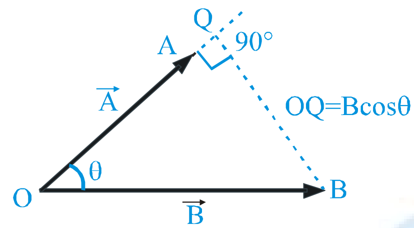


$\vec{A} \cdot \vec{B} = AB\cos\theta = OA \cdot OB \cdot \cos\theta$

The above equation can also be written in the following ways.

$\vec{A}\cdot\vec{B} = (A\cos\theta)B = OP\cdot OB$ $\qquad$ $\vec{A}\cdot\vec{B} = A(B\cos\theta) = OA\cdot OQ$





Above two equations and figures, suggest a scalar product as product of magnitude of the one vector and magnitude of the component of another vector in the direction of the former vector.

ETOOS KEY POINTS

(i) Dot product of two vectors is commutative: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

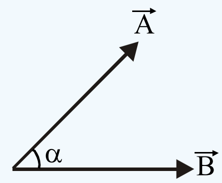
(ii) If two vectors are perpendicular, their dot product is zero. $\vec{A}\cdot\vec{B} = 0$, if $\vec{A}\perp\vec{B}$

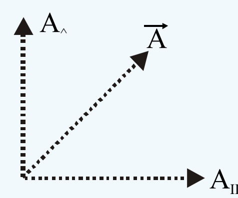
(iii) Dot product of a vector by itself is known as self-product. $\vec{A}\cdot\vec{A} = A^2 \Rightarrow A = \sqrt{\vec{A}\cdot\vec{A}}$

(iv) The angle between the vectors $\theta = \cos^{-1}\left(\dfrac{\vec{A}\cdot\vec{B}}{AB}\right)$

(v) **(A)** Component of $\vec{A}$ in direction of $\vec{B}$

$$\vec{A}_{\parallel} = (|\vec{A}|\cos\theta)\hat{B} = |\vec{A}|\left(\frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|}\right)\hat{B} = \left(\frac{\vec{A}.\vec{B}}{|\vec{B}|}\right)\hat{B} = (\vec{A}.\hat{B})\hat{B}$$





(B) Component of $\vec{A}$ perpendicular to $\vec{B}$: $\vec{A}_{\perp} = \vec{A} - \vec{A}_{II}$

(vi) Dot product of Cartesian unit vectors: $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

(vii) If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, their dot product is given by

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Ex. If $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$, then find the angle between $\vec{A}$ and $\vec{B}$.

Sol. ✈ $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$ $\quad\therefore \sqrt{A^2+B^2+2AB\cos\theta} = \sqrt{A^2+B^2-2AB\cos\theta}$

or $A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 - 2AB\cos\theta$ or $\cos\theta = 0$ $\quad\therefore \theta = 90°$

Ex. If $\vec{A} = 4\hat{i} + n\hat{j} - 2\hat{k}$ and $\vec{B} = 2\hat{i} + 3\hat{j} + \hat{k}$, then find the value of n so that $\vec{A}\perp\vec{B}$

Sol. Dot product of two mutually perpendicular vectors is zero $\vec{A}\cdot\vec{B} = 0$

$\therefore (4\hat{i} + n\hat{j} - 2\hat{k}).(2\hat{i} + 3\hat{j} + \hat{k}) = 0 \Rightarrow (4\times2) + (n\times3) + (-2\times1) = 0$

$\Rightarrow 3n = -6 \Rightarrow n = -2$

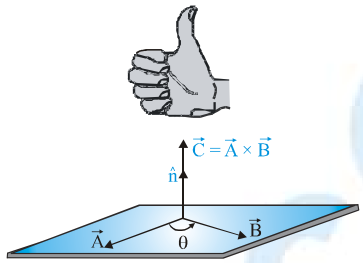
(ii) Vector or cross product of two vectors

The vector product $\vec{C}$ of two vectors $\vec{A}$ and $\vec{B}$ is defined as

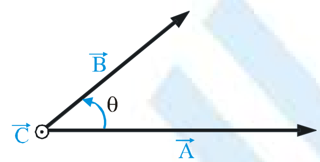
(A) Its magnitude is the product of magnitudes of $\vec{A}$ and $\vec{B}$ and of the sine of angle θ between vectors $\vec{A}$ and $\vec{B}$.

(B) Its direction is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$ and is decided by right hand rule by curling fingers in the direction from the first vector towards the second vector. In figure, where it is represented by $\hat{n}$.

$$\vec{C} = \vec{A} \times \vec{B} = (AB\sin\theta) \cdot \hat{n}$$



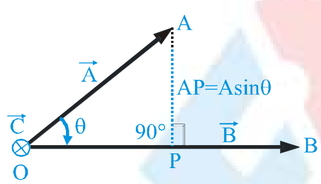
On paper vectors perpendicularly out and into the plane of paper are represented by encircled dot $\odot$ and encircled cross $\otimes$ signs respectively. Following this convention, cross product $\vec{C} = \vec{A} \times \vec{B}$ is shown in the figure.

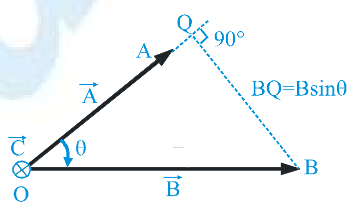


To have different symbols for scalar and vector products, symbols dot $(\cdot)$ and cross $(\times)$ respectively are written between the vectors undergoing these operations.

Cross product $\vec{C} = \vec{A} \times \vec{B}$, can also be written in the following ways.

$\vec{C} = \vec{A} \times \vec{B} = A(B\sin\theta)\hat{n}$ $\qquad$ $\vec{C} = \vec{A} \times \vec{B} = (A\sin\theta)B\hat{n}$





The above two equations and figures explain that the magnitude of vector or cross product is the product of magnitude of one vector and magnitude of the component of the other vector in the direction perpendicular to the first one.

Ex. Find a unit vector perpendicular to both the vectors $(2\hat{i} + 3\hat{j} + \hat{k})$ and $(\hat{i} - \hat{j} + 2\hat{k})$.

Sol. Let $\vec{A} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{B} = \hat{i} - \hat{j} + 2\hat{k}$

unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ is $\hat{n} = \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix} == \hat{i}(6+1) - \hat{j}(4-1) + \hat{k}(-2-3) = 7\hat{i} - 3\hat{j} - 5\hat{k}$$

$\therefore |\vec{A} \times \vec{B}| = \sqrt{7^2 + (-3)^2 + (-5)^2} = \sqrt{83}$ $\qquad$ unit $\qquad$ $\therefore \hat{n} = \dfrac{1}{\sqrt{83}}(7\hat{i} - 3\hat{j} - 5k)$

ETOOS KEY POINTS

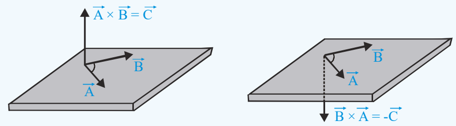
(i) Vector product of two vectors is always a vector perpendicular to the plane containing the two vectors, i.e., orthogonal

(perpendicular) to both the vectors $\vec{A}$ and $\vec{B}$.

Unit vector perpendicular to $\vec{A}$ and $\vec{B}$ is $\hat{n} = \pm\dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$

(ii) Vector product of two vectors is not commutative i.e. cross products $\vec{A}\times\vec{B}$ and $\vec{B}\times\vec{A}$ have equal magnitudes but opposite directions as shown in the figure.

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$



(iii) The vector product is distributive when the order of the vectors is strictly maintained,

i.e. $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$

(iv) Angle θ between two vectors $\vec{A}$ and $\vec{B}$ is given by $\theta = \sin^{-1}\left[\dfrac{|\vec{A}\times\vec{B}|}{|\vec{A}|\,|\vec{B}|}\right]$

(v) The self cross product, i.e., product of a vector by itself is a zero vector or a null vector.

$\vec{A}\times\vec{A} = (AA\sin 0°)\hat{n} = \vec{0} = \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k}$

(vi) In case of orthogonal unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$; according to right hand thumb rule

$\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$ and $\hat{j}\times\hat{i} = -\hat{k}, \hat{k}\times\hat{j} = -\hat{i}, \hat{i}\times\hat{k} = -\hat{j}$

(v) If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, their cross-products is given by

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}(A_yB_z - A_zB_y) - \hat{j}(A_xB_z - A_zB_x) + \hat{k}(A_xB_y - A_yB_x)$$

(vi) If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are coplanar, then $\vec{A}.(\vec{B}\times\vec{C}) = 0$.

Rate of change of a vector with time

It is derivative of a vector function with respect to time. Cartesian components of a time dependent vector, if given as function of time as $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, the time rate of change can be calculated according to equation

$$\frac{d\vec{r}(t)}{dt} = \frac{dx(t)\hat{i}}{dt} + \frac{dy(t)\hat{j}}{dt} + \frac{dz(t)\hat{k}}{dt}$$

Methods of differentiation of vector functions

Methods of differentiation of scalar functions are also applicable to differentiation of vector functions.

i. $\dfrac{d}{dt}\left(\vec{F}\pm\vec{G}\right) = \dfrac{d\vec{F}}{dt} \pm \dfrac{d\vec{G}}{dt}$

ii. $\dfrac{d}{dt}\left(\vec{F}\cdot\vec{G}\right) = \dfrac{d\vec{F}}{dt}\cdot\vec{G} + \vec{F}\cdot\dfrac{d\vec{G}}{dt}$

iii. $\dfrac{d}{dt}\left(X\vec{F}\right) = \dfrac{dX}{dt}\vec{F} + X\dfrac{d\vec{F}}{dt}$ Here X is a scalar function of time.

iv. $\dfrac{d}{dt}\left(\vec{F}\times\vec{G}\right) = \dfrac{d\vec{F}}{dt}\times\vec{G} + \vec{F}\times\dfrac{d\vec{G}}{dt}$ Order of the vector functions $\vec{F}$ and $\vec{G}$ must be retained.

• Etoos Tips & Formulas •

BASIC MATHEMATICS

1. Quadratic Equation

Roots of $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Sum of roots $= x_1 + x_2 = -\dfrac{b}{a}$; Product of roots $= x_1 x_2 = \dfrac{c}{a}$

2. Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 + ...$$

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2}x^2 - \frac{n(n-1)(n-2)}{6}x^3 + ...$$

If $x << 1$ then $(1+x)^n \approx 1 + nx$ & $(1-x)^n \approx 1 - nx$

3. Logarithm

(i) $\log mn = \log m + \log n$ (ii) $\log \dfrac{m}{n} = \log m - \log n$ **(iii)** $\log m^n = n \log m$ **(iv)** $\log_e m = 2.303 \log_{10} m$

4. Componendo and Dividendo Rule

If $\dfrac{p}{q} = \dfrac{a}{b}$ then $\dfrac{p+q}{p-q} = \dfrac{a+b}{a-b}$

5. Arithmetic progression (AP)

a, a + d, a +2d, ..., a + (n–1)d here d = common difference

Sum of n terms $S_n = \dfrac{n}{2}[2a+(n-1)d]$

Note : (i) $1+2+3+4+5 .... n = \dfrac{n(n+1)}{2}$

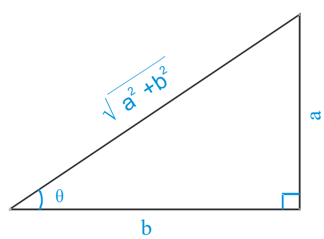
(ii) $1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

6. Geometrical Progression (GP)

a, ar, ar^2,..., ar^{n-1} here r = common ratio

Sum of n terms $S_n = \dfrac{a(1-r^n)}{1-r}$ Sum of ∞ term $S_\infty = \dfrac{a}{1-r}$

7. Trigonometry



2π rad = 360° $\Rightarrow$ 1 rad = 57.3°

(i) $\sin\theta = \dfrac{\text{perpendicular}}{\text{hypotenuse}}$ (ii) $\cos\theta = \dfrac{\text{base}}{\text{hypotenuse}}$ (iii) $\tan\theta = \dfrac{\text{perpendicular}}{\text{base}}$

(iv) $\text{cosec}\,\theta = \dfrac{\text{hypotenuse}}{\text{perpendiculaer}}$ (v) $\sec\theta = \dfrac{\text{hypotenuse}}{\text{base}}$ (vi) $\cot\theta = \dfrac{\text{base}}{\text{perpendicular}}$

(vii) $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$ (viii) $\cos\theta = \frac{b}{\sqrt{a^2+b^2}}$ (ix) $\tan\theta = \frac{a}{b}$

(x) $\text{cosec}\,\theta = \frac{1}{\sin\theta}$ (xi) $\sec\theta = \frac{1}{\cos\theta}$ (xii) $\cot\theta = \frac{1}{\tan\theta}$

(xiii) $\sin^2\theta + \cos^2\theta = 1$ (xiv) $1+\tan^2\theta = \sec^2\theta$ (xv) $1+\cot^2\theta = \text{cosec}^2\theta$

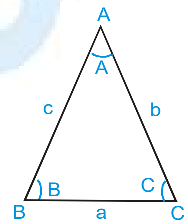
(xvi) $\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B$ (xvii) $\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$

(xvii) $\tan(A\pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$ (xviii) $\sin 2A = 2\sin A\cos A$

(xix) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$ (xx) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

8. **sine law**

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



9. **cosine law**

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad \cos B = \frac{c^2+a^2-b^2}{2ca}, \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

10. **For small θ**

$\sin\theta \approx \theta$ $\cos\theta \approx 1$ $\tan\theta \approx \theta$ $\sin\theta \approx \tan\theta$

11. **Differentiation**

(i) $y = x^n \to \frac{dy}{dx} = nx^{n-1}$ (ii) $y = \ln x \to \frac{dy}{dx} = \frac{1}{x}$ (iii) $y = \sin x \to \frac{dy}{dx} = \cos x$

(iv) $y = \cos x \to \frac{dy}{dx} = -\sin x$ (v) $y = e^{\alpha x+\beta} \to \frac{dy}{dx} = \alpha e^{\alpha x+\beta}$ (vi) $y = uv \to \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

(vii) $y = f(g(x)) \Rightarrow \frac{df(g(x))}{dg(x)} \times \frac{dg(x)}{dx}$ (viii) $y = k = \text{constant} \Rightarrow \frac{dy}{dx} = 0$ (ix) $y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

12. **Integration**

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$ (ii) $\int \frac{1}{x} dx = \ln x + C$ (iii) $\int \sin x\,dx = -\cos x + C$

(iv) $\int \cos x\,dx = \sin x + C$ (v) $\int e^{\alpha x+\beta} dx = \frac{1}{\alpha} e^{\alpha x+\beta} + C$ (vi) $\int (\alpha x+\beta)^n dx = \frac{(\alpha x+\beta)^{n+1}}{\alpha(n+1)} + C$

13. **Maxima & Minima of a function y = f(x)**

(i) For maximum value $\frac{dy}{dx} = 0$ & $\frac{d^2y}{dx^2} - ve$

(ii) For minimum value $\frac{dy}{dx} = 0$ & $\frac{d^2y}{dx^2} + ve$

14. Average of a varying quantity

If y = f(x) then $< y >= \bar{y} \dfrac{\int_{x_1}^{x_2} y dx}{\int_{x_1}^{x_2} dx} = \dfrac{\int_{x_1}^{x_1} y dx}{x_2 - x_1}$

15. Formulae for determination of area

(i) Area of a square = $(\text{side})^2$

(i)i Area of a rectangle = length × breadth

(iii) Area of a triangle = $\frac{1}{2}$×base×height

(iv) Area of a trapezoid = $\frac{1}{2}$×(distance between parallel sides) × (sum of parallel sides)

(v) Area enclosed by a circle = πr^2 (r = radius)

(vi) Surface area of a sphere = $4\pi r^2$ (r = radius)

(vii) Area of parallelogram = base × height

(viii) Area of curved surface of cylinder = $2\pi r l$ (r = radius and ● = length)

(ix) Area of whole surface of cylinder = 2πr (r + ●) (● = length)

(x) Area of ellipse = π ab (a & b are semi major axis respectively)

(xi) Surface area of a cube = $6(\text{side})^2$

(xii) Total surface area of a cone = $\pi r^2 + \pi r$●, where $\pi r l = \pi r\sqrt{r^2 + h^2}$ = lateral area

16. Formulae for determination of volume

(i) Volume of rectangle slab = length × breadth × height

(ii) Volume of a cubee = $(\text{side})^3$

(iii) Volume of a sphere = $\frac{4}{3}\pi r^3$ (r = radius)

(iv) Volume of cylinder = $\pi r^2 l$ (r = radius and ● = length)

(v) Volume of a cone = $\frac{1}{3}\pi r^2 h$ (r = radius and h =height)

17. To convert an angle from degree to radian, we have to multiply it by $\frac{\pi}{180°}$ and to convert an angle from radian to degree, we have to multiply it by $\frac{180°}{\pi}$.

18. By help of differentiation, if y is given, we can find $\frac{dy}{dx}$ and by help of integration, if $\frac{dy}{dx}$ is given, we can find y.

19. The maximum and minimum values of function A cosθ + B sinθ are

$\sqrt{A^2 + B^2}$ and $-\sqrt{A^2 + B^2}$ respectively.

(i) $(a+b)^2 = a^2 + b^2 + 2ab$

(ii) $(a-b)^2 = a^2 + b^2 - 2ab$

(iii) $(a+b)(a-b) = a^2 - b^2$

(iv) $(a+b)^3 = a^3 + b^3 + 3ab\,(a+b)$

(v) $(a-b)^3 = a^3 - b^3 - 3ab\,(a-b)$

• Etoos Tips & Formulas •

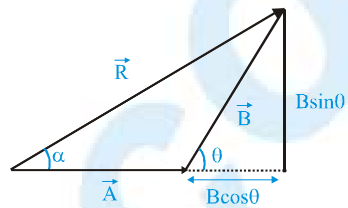
VECTOR

1. Vector Quantities

A physical quantity which requires magnitude and a particular direction, when it is expressed.

2. Triangle law of Vector addition $\vec{R} = \vec{A} + \vec{B}$

$$R = \sqrt{A^2 + B^2 + 2AB\cos 90^\circ}$$

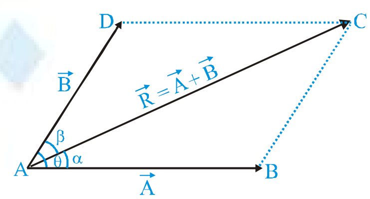


$$\Rightarrow \quad \tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

If A = B then $R = 2A\cos\frac{\theta}{2}$ & $\alpha = \frac{\theta}{2}$

$R_{max} = A + B$ for $\theta = 0^\circ$; $R_{min} = A - B$ for $\theta = 180^\circ$

3. Parallelogram law of Addition of Two Vectors

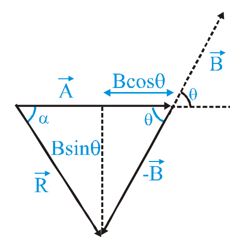


If two vectors are represented by two adjacent sides of a parallelogram which are directed away from their common point then their sum (i.e. resultant vector) is given by the diagonal of the parallelogram passing away through that common point.

$\vec{AB} + \vec{AD} = \vec{AC} = \vec{R}$ or $\vec{A} + \vec{B} = \vec{R} \Rightarrow R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta} \quad \text{and} \quad \tan\beta = \frac{A\sin\theta}{B + A\cos\theta}$$

4. Vector subtraction



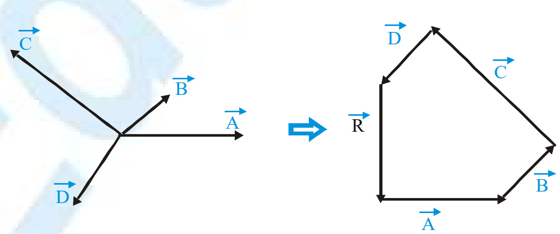
$\vec{R} = \vec{A} - \vec{B} \Rightarrow \vec{R} = \vec{A} + \left(-\vec{B}\right)$

$R = \sqrt{A^2 + B^2 - 2AB\cos 90^\circ}$, $\tan\alpha = \frac{B\sin\theta}{A - B\cos\theta}$

If A = B then $R = 2A\sin\frac{\theta}{2}$

5. Addition of More than Two Vectors (Law of Polygon)

If some vectors are represented by sides of a polygon in same order, then their resultant vector is represented by the closing side of polygon in the opposite order.



$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$

6. Rectangular component of a 3-D vector

(a) $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

Angle made with x - axis $\quad \cos\alpha = \dfrac{A_z}{A} = \dfrac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = 1$

Angle made with y - axis $\quad \cos\beta = \dfrac{A_y}{A} = \dfrac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = m$

Angle made with z - axis $\quad \cos\gamma = \dfrac{A_z}{A} = \dfrac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = n$

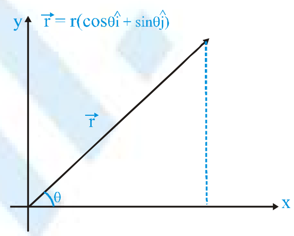
(b) ●, m, n are called direction cosines.

$$1^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{A_x^2 + A_y^2 + A_z^2}{\left(\sqrt{A_x^2 + A_y^2 + A_z^2}\right)^2} = 1$$

or $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

7. General Vector in x-y plane

$\vec{r} = x\hat{i} + y\hat{j} = r\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right)$



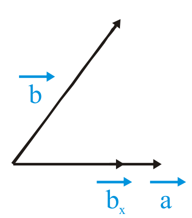
8. Scalar product (Dot Product)

(a) $\vec{A}\cdot\vec{B} = AB\cos\theta \Rightarrow$ Angle between two vectors $\theta = \cos^{-1}\left(\dfrac{\vec{A}.\vec{B}}{AB}\right)$

(b) If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ then

$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$ and angle between $\vec{A}$ & $\vec{B}$ is given by

$$\cos\theta = \frac{\vec{A}.\vec{B}}{AB} = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$



(c) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

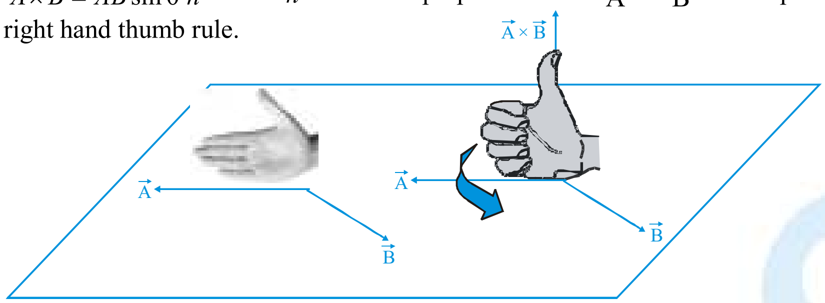
$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

(d) Component of vector $\vec{b}$ along vector $\vec{a}, \vec{b}_{\parallel} = \left(\vec{b}.\vec{a}\right)\vec{a}$

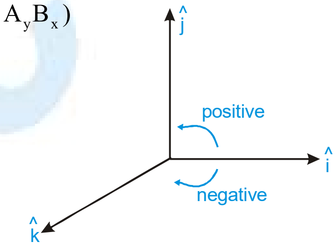
(e) Component of $\vec{b}$ perpendicular to $\vec{a}$, $\vec{b}_{\perp} = \vec{b} - \vec{b}_{\parallel} = \vec{b} - \left(\vec{b}.\vec{a}\right)\vec{a}$

9. Cross Product (Vector product)

(a) $\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$ where $\hat{n}$ is a vector perpendicular to $\vec{A}$ & $\vec{B}$ or their plane and its direction given by right hand thumb rule.



(b) $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}(A_yB_z - A_zB_y) - \hat{j}(A_xB_z - A_zB_x) + \hat{k}(A_xB_y - A_yB_x)$



(c) $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

(d) $(\vec{A}\times\vec{B}).\vec{A} = (\vec{A}\times\vec{B}).\vec{B} = 0$

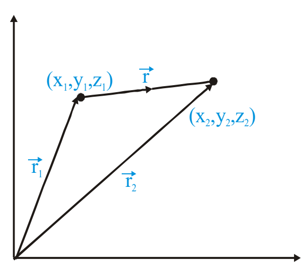
(e) $\hat{i}\times\hat{i} = 0, \hat{j}\times\hat{j} = 0, \hat{k}\times\hat{k} = 0$

(f) $\hat{i}\times\hat{j} = \hat{k}\ ;\ \hat{j}\times\hat{k} = \hat{i}$ $\quad\hat{k}\times\hat{i} = \hat{j}\ ;\ \hat{j}\times\hat{i} = -\hat{k}$ $\quad\hat{k}\times\hat{j} = -\hat{i}\ ;\ \hat{i}\times\hat{k} = -\hat{j}$

10. Differentiation

(i) $\frac{d}{dt}\left(\vec{A}.\vec{B}\right) = \frac{d\vec{A}}{dt} + \vec{A}.\frac{d\vec{B}}{dt}$ (ii) $\frac{d}{dt}\left(\vec{A}\times\vec{B}\right) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}.\frac{d\vec{B}}{dt}$

11. When a particle moved from (x1, y1, z1) to (x2, y2, z2) then its displacement vector.

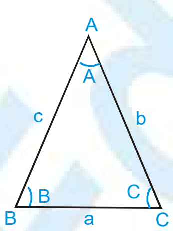


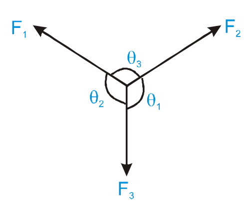
$$\vec{r} = \vec{r}_2 - \vec{r}_1 = \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right)$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

Magnitude $\quad r = |\vec{r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

12. Lami's theorem





$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\text{Sin } C}{c} \qquad \frac{F_1}{\sin\theta_1} = \frac{F_2}{\sin\theta_2} = \frac{F_3}{\sin\theta_3}$$

(a) Area of triangle, Area $= \frac{|\vec{A}\times\vec{B}|}{2} = \frac{1}{2}AB\sin\theta$

(b) Area of parallelogram, Area $= |\vec{A}\times\vec{B}| = AB\sin\theta$

(c) For Parallel vectors, $\vec{A}\times\vec{B} = \vec{0}$

(d) For perpendicular vectors, $\vec{A}.\vec{B} = \vec{0}$

(e) For coplanar vectors, $\vec{A}.\vec{B} = \vec{0}$

(f) For coplanar vectors, $\vec{A}.(\vec{B}\times\vec{C}) = 0$

13. Examples of dot products :

(a) Work, $W = \vec{F}.\vec{d} = Fd\cos\theta$ where F → force, d → displacement

(b) Power, $P = \vec{F}.\vec{v} = Fv\cos\theta$ where F → force, v → velocity

(c) Electric flux, $\phi_E = \vec{E}.\vec{A} = EA\cos\theta$ where E → electric flux, A → Area

(d) Magnetic flux, $\phi_B = \vec{B}.\vec{A} = BA\cos\theta$ where B → magnetic field, A → Area

(e) Potential energy of dipole in uniform field, $U = -\vec{p}.\vec{E}$ where p → dipole moment, where E → Electric field

14. Examples of cross products :

(a) Torque $\vec{\tau} = \vec{r}\times\vec{F}$ where r → position vector, F → force

(b) Angular momentum $\vec{J} = \vec{r}\times\vec{p}$ where r → position vector, p → linear momentum

(c) Linear velocity $\vec{v} = \vec{\omega}\times\vec{r}$ where r → position vector, ω → angular velocity

(d) Torque on dipole placed in electric field $\vec{\tau} = \vec{p}\times\vec{E}$

where p → dipole moment, E → Electric Field

(e) Tensor : A quantity that has different values in different directions is called tensor. Ex. Moment of Inertia

In fact tensors are merely a generalisation of scalars and vectors ; a scalar is a zero rank tensor, and a vector is a first rank tensor.

15. Electric current is not a vector as it does not obey the law of vector addition.]

16. A unit vector has no unit.

17. To a vector only a vector of same type can be added and the resultant is a vector of the same type.

18. A scalar or a vector can never be divided by a vector.