RELATIONS AND FUNCTIONS

1.1 BASIC CONCEPTS AND IMPORTANT RESULTS

(a) Relation from A to B

If A, B are two (non-empty) sets, then any subset of A × B is called a relation from A to B.

If R is a relation from A to B, then

domain of R = $\{x : x \in A, (x, y) \in R$ for some $y \in B\}$ and

range of R = $\{y : y \in B, (x, y) \in R$ for some $x \in A\}$.

If R is a relation from A to B, then B is called codomain of R.

If A and B are finite sets containing m and n elements respectively, then the number of relations from A to B = $2^{mn.}$

(b) Relation on a set A

(i) If A is a (non-empty) set, then any subset of A × A is called a relation on A.

(ii) If A is a finite set containing m elements, then number of relations on A = 2^{m^2} .

1.2 Types of relations on A

Let R be a relation on a (non-empty) set A, then R is called a

(i) reflexive relation iff a R a i.e. $(a, a) \in R$ for all $a \in A$.

(ii) symmetric relation iff a R b, implies b R a i.e. $(a, b) \in R$ implies $(b, a) \in R$ for all $a, b \in A$.

(iii) transitive relation iff a R b, b R c implies a R c i.e. $(a, b) \in R$, $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

(iv) equivalence relation iff it is

(i) reflexive (ii) symmetric, and (iii) transitive.

(v) Let R be an equivalence relation on A. if $a \in A$, then the equivalence class of a is denoted by [a] and $[a] = \{x : x \in A,\ x R a\}$.

(vi) As $\phi \subset A \times A$, ϕ is the empty relation on A.

(vii) As $A \times A \subset A \times A$, A × A is the universal relation on A.

(viii) The relation $I_A = \{(a, a);$ for $a \in A\}$ is the identity relation on A.

1.3 Function

If X, Y be two (non-empty) sets, then a subset f of X × Y, written as $f : X \to Y$, is called a function (or mapping or map) from X to Y iff

(i) for each $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$ and

(ii) no two different ordered pairs of f have the same first component.

In other words, a function from X to Y is a rule (or correspondence) which asociates to each element x of X, a unique element of Y.

The unique element $y \in Y$ is called the image of the element x of X under the function f. It is denoted by f(x) i.e. y = f(x). The element y is also called the value of the function f at x.

Let f be a function from X to Y then X is the domain of the function and range of f = $\{f(x) :$ for all $x \in X\}$

If 'f' is a function from X to Y, then Y is called codomain of f.

1.4 Types of functions

1. **One-one function.** A function 'f' from X to Y is called one-one (or injective) iff different elements of X have different images in Y i.e. iff $x_1 \neq X_2 \Rightarrow f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in X$, or equivalently, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in X$.
2. **Many-one function.** A function 'f' from X to Y is called many-one iff two or more elements of X have same image in Y. In other words, a function 'f' from X to Y is called many-one iff it is not one-one.
3. **Onto function.** A function 'f' from X to Y is called onto (or surjective) iff each element of Y is the image of atleast one element of X i.e. iff codomain of f = range of f i.e. iff Y = f(X).
4. **Into function.** A function 'f' from X to Y is called into iff there exists atleast one element in Y which is not the image of any element of X i.e. iff range of f is a proper subset of codomain of f. In other words, a function 'f' from X to Y is called into iff it is not onto.
5. **One-one correspondnece.** A function 'f' from X to Y is called a one-one correspondence (or bijective) iff f is both one-one and onto.

Since we weill often be proving that certain functions are one-one or onto or both, so we outlines the techniques to be used :

(i) To prove that f is one-one, we must show that either $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in X$ or for all $x_1 \neq x_2 \in X \Rightarrow f(x_1) \neq f(x_2)$.

(ii) To prove that f is onto, we must show that either for every $y \in Y$, there exists atleast one element $x \in X$ such that $y = f(x)$ or $f(X) = Y$.

If A and B are non-empty) finite sets containing m and n elements respectively, then

(a) the number of functions from A to B = n^m.

(b) the number of one-one functions from A to B

$$= \begin{cases} {}^nP_m, \text{if } n \geq m \\ 0, \text{if } n < m. \end{cases}$$

(c) the number of onto functions from A to B

$$= \begin{cases} \sum_{r=1}^{n} (-1)^{n-r}\ {}^nC_r\ r^m, \text{if } m \geq n \\ 0, \text{if } m < n. \end{cases}$$

If $n = 2$ and $m \geq 2$, then the number of onto functions from A to B = $2^m - 2$.

If $n = 3$ and $m \geq 3$, then the number of onto function from A to B
$= 3^m - 3(2^m - 1)$.

(d) the number of one onto functions i.e. bijections from A to B

$$= \begin{cases} !m, \text{if } m = n \\ 0, \text{if } m \neq n. \end{cases}$$

6. Identity function. Let X be any (non-empty) set, then the function $f : X \to X$ defined by $f(x) = x$ for $x \in X$ is called the identity function on X. It is denoted by i or I_x.

7. Constant function. Let X, Y be two (non-empty) sets, then the function $f : X \to Y$ defined by $f(x) = y$ for $x \in X$ and for a fixed $y \in Y$ is called a constant function.

8. Equal functions

Two function f and g are called equal, written as $f = g$, iff

(i) domain of f = domain of g and

(ii) $f(x) = g(x)$ for al x in domain of f (or g).

Otherwise, the functions are called unequal and we write as $f \neq g$.

9. Zero function

Let R be the set of all real numbers, then the function $f_0 : R \to R$ defined by $f_0(x) = 0$ for all $x \in R$ is called the zero function on R.

1.5 Composition of real function

Let f, g be two real functions and let $D = \{x : x \in D_f, f(x) \in D_g\} \neq \phi$, then the composite of f and g, denoted by gof, is the function defined by

$$(gof)(x) = g(f(x)) \text{ with domain D.}$$

In particular, if $R_f \subset D_g$, then $D_{gof} = D_f$.

1.6 Some properties of composition of functions

1. The composition of functiosn is associative i.e. if
$f : A \to B$, $g : B \to C$, $h : C \to D$ are functions, then ho (gof) = (hog) of

2. Let $f : A \to B$, $g : B \to C$ be two functions.

(i) If f, g are both one-one, then gof is also one-one.

(ii) If f, g are both onto, then gof is also onto.

3. If $f : A \to B$ is a function and I_A, I_B are identity functions on A, B respectively, then

(i) I_B of = f (ii) of I_A = f.

1.7 Invertible functions

Let $f : X \to Y$ be one-one onto function and if $f(x) = y$ where $x \in X$, $y \in Y$, then $f^{-1} : Y \to X$ defined by $f^{-1}(y) = x$ is called an inverse function of f.

Clearly, we have

(i) domain of f^{-1} = range of f.

(ii) range of f^{-1} = domain of f.

(iii) $f^{-1}(y) = x$ iff $f(x) = y$ where $x \in X, y \in Y$.

From above it follows that if the function $f : X \to Y$ is one-one onto, then we can define a function $g : Y \to X$ by $g(y) = x$ if and only if $y = f(x)$.

We note that the composite function gof, fog exist and gof = I_X, fog = I_Y. Thus :

A function $f : X \to Y$ is called invertible if there exists function $g : Y \to X$ such that gof = I_X and fog = I_Y. The function g is called inverse of f and is denoted by f^{-1}.

In particulr, if $f : X \to X$ is a function and fof = I_x, then f is invertible and $f^{-1} = f$.

1.8 Some properties of invertible functions

1. Inverse of a bijective function is unique.
2. The inverse of a bijective function f is also bijective and $(f^{-1})^{-1} = f$.
3. If $f : A \to B$ is bijective, then
   (i) $f^{-1}$of = I_A (ii) fof^{-1} = I_B.
4. If $f : A \to B$ and $g : B \to C$ are both bijectives, then
   $gof : A \to C$ is also bijective and $(gof)^{-1} = f^{-1}og^{-1}$.
5. Let A be a non-empty set and $f : A \to A$, $g : A \to A$ be two functions such that gof = I_A = fog, then f and g are both bijectives and $g = f^{-1}$.

1.9 Binary operations

A binary opertion (or composition) '$*$' on a (non-empty) set A is a function $* : A \times A \to A$.

We denote $*(a, b)$ by $a * b$ for every ordered pair $(a, b) \in A \times A$.

In other words, a binary operation (or composition) on a (non-empty set) A is a rule that associates with every ordered pair of elements a, b (distinct or equal) of A, some unique element $a * b$ of A.

1.10 Types of operations

Let $*$ be a binary operation of a (non-empty) set A, then

(i) the operation is called commutative or abelian if and only if $a * b = b * a$ for all $a, b \in A$.

(ii) the operation is called associative if and only if $(a * b) * c = a * (b * c)$ for all $a, b, c \in A$.

(iii) an element $e \in A$, if it exists, is called identity element of the operation if and only if $e * a = a = a * e$ for all $a \in A$.

(iv) If $e \in A$ is identity element of the operation, then an element $a \in A$ s called invertible (or inversible) if and only if there exists an element $b \in A$ such that $a * b = e = b * a$. Element b is called inverse of a and is denoted by a^{-1}.

Identity element, if it exists, is unique.

Inverse of an element, if it exists, is unique.

Operation table

Let A be a finite non-empty set and '$*$' be an operation on A, then the operation (or composition) table is a square array indicating all possible products.

Entry in the ith row and jth column = (ith entry on the left) $*$ (jth entry at the top).

If A is (non-empty) finite set containing n elements, then the number of binary operations on A = n^{n^2}.

SOLVED PROBLEMS

Q.1 ***Show that the relation R in the set A of points in a plane given by R = {(P, Q) : distance of the point P from the origin is same as the distance of the point Q from the origin} is an equivalence relation. Further, show that the set of all points related to a point P ≠ (0, 0) is the circle passing through P with origin as centre.***

Sol. Let O be the origin.

R = {(P, Q) : | OP | = | OQ |, where O is the origin }

Since | OP | = | OP |,

therefore, $(P, P) \in R\ \forall\ P \in A$.

∴ R is reflexive.

Also $(P, Q) \in R$

⇒ | OP | = | OQ |

⇒ | OQ | = | OP |

⇒ $(Q, P) \in R$

⇒ R is symmetric.

Next let $(P, Q) \in R$ and $(Q, T) \in R$

⇒ | OP | = | OQ |

and | OQ | = | OT |

⇒ | OP | = | OT |

⇒ $(P, T) \in R$

∴ R is transitive.

Hence R is an equivalence relation.

Set of points related to P ≠ O

= $\{Q \in A : (Q, P) \in R\}$

= $\{Q \in A : | OQ | = | OP |\}$

= {Q ∈ A : Q lies on a circle through P with centre O}.

Q.2 ***Show that the relation R defined in the set A of all triangles as R = {(T_1, T_2) : T_1 is similar to T_2}, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related ?***

Sol. Here, $(T_1, T_2) \in R$ iff T_1 is similar to T_2.

As every triangle is similar to itself, therefore,

$(T, T) \in R\ \forall\ T \in A$

⇒ R is reflexive.

Again $(T_1, T_2) \in R$

⇒ T_1 is similar to T_2

⇒ T_2 is similar to T_1

⇒ $(T_2, T_1) \in R$

∴ R is symmetric.

Next, let $(T_1, T_2) \in R$ and also $(T_2, T_3) \in R$

⇒ T_1 is similar to T_2 and T_2 is similar to T_3

⇒ T_1 is similar to T_3

⇒ $(T_1, T_3) \in R$.

∴ R is transitive.

Hence, R is an equivalence relation.

As $\frac{6}{3} = \frac{8}{4} = \frac{10}{5}$, Therefore, the triangles T_1 and T_3 are similar, i.e. T_1 is related to T_3 and T_3 is related to T_1, i.e., $(T_1, T_3) \in R$.

Q.3 ***Show that the relation R defined in the set A of all polygons as R = {(P_1, P_2) : P_1 and P_2 have same number of sides}, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5 ?***

Sol. Since P and P have the same number of sides, therefore, $(P, P) \in R\ \forall\ P \in A$.

∴ R is reflexive.

Let $(P_1, P_2) \in R$

⇒ P_1 and P_2 have the same number of sides

⇒ P_2 and P_1 have the same number of sides

⇒ $(P_2, P_1) \in R$

∴ R is symmetric.

Let $(P_1, P_2) \in R$ and $(P_2, P_3) \in R$.

$\Rightarrow$ P_1 and P_2 have the same number of sides and also, P_2, P_3 have the same number of sides

$\Rightarrow$ P_1 and P_3 have the same number of sides

$\Rightarrow$ $(P_1, P_3) \in R$

$\Rightarrow$ R is transitive.

Hence R is an equivalence relation.

Here T is a triangle, therefore, $P \in A$ is related to T iff P and T have the same number of sides, i.e. $(P, T) \in R$ iff P and T have equal number of sides

or (P, T) $\in$ R iff P is a triangle.

Required set of all elements which are related to T is the set of all triangles in A.

Q.4 ***Let f : N $\rightarrow$ N be defined by***

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if n is odd} \\ \frac{n}{2}, & \text{if n is even} \end{cases} \quad \text{for all } n \in N.$$

State whether the function f is bijective, Justify your answer.

Sol. Given $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if n is odd} \\ \frac{n}{2}, & \text{if n is even} \end{cases}$

We note that

$f(1) = \frac{1+1}{2} = 1$

$f(2) = \frac{2}{2} = 1$

$f(3) = \frac{3+1}{2} = 2$

and $f(4) = \frac{4}{2} = 2$

In general $f(2m-1) = \frac{(2m-1)+1}{2} = m$

and $f(2m) = \frac{2m}{2} = m$ $\Rightarrow f(2m-1) = f(2m)$

where m is any +ve integer.

$\therefore$ f is not one-one.

However, f is onto as $R_f = N$.

($\because$ For any $n \in N$, there is $2n \in N$

such that $f(2n) = \frac{2n}{2} = n$)

So, f is onto but not one-one. Hence, f is not a bijection.

Q.5 ***Let A = R – {3} and B = R – {1}. Consider the function f : A $\rightarrow$ B defined by f(x) = $\left(\frac{x-2}{x-3}\right)$. Is f one-one and onto ? Justify your answer.***

Sol. Let $x_1, x_2 \in A = R - \{3\}$ be such that

$f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$

$\Rightarrow (x_1-2)(x_2-3) = (x_2-2)(x_1-3)$

$\Rightarrow x_1 x_2 - 2x_2 - 3x_1 + 6$

$= x_1 x_2 - 2x_1 - 3x_2 + 6$

$\Rightarrow x_2 = x_1$

$\therefore$ f is one-one.

Let $y \in B = R - \{1\}$, then f(x) = y

if $\frac{x-2}{x-3} = y, x \neq 3$

i.e., if $x - 2 = yx - 3y$

i.e., if $x - xy = 2 - 3y$

i.e., if $x(1-y) = 2 - 3y$

i.e., if $x = \frac{2-3y}{1-y} \in A$

$\left(\text{Note that } \frac{2-3y}{1-y} = \frac{3-3y-1}{1-y} = \frac{3(1-y)}{1-y} - \frac{1}{1-y} = 3 - \frac{1}{1-y} \neq 3\right)$

Thus, corresponding to each $y \in B$, there exists $\frac{2-3y}{1-y} \in A$ such that $f\left(\frac{2-3y}{1-y}\right) = y$.

Hence, f is onto.

Q.6 ***Let : R → R be defined as f(x) = x^4. Choose the correct answer.***

(A) f is one-one onto

(B) f is many-one onto

(C) f is one-one but not onto

(D) f is neither one-one nor onto.

Sol. **(D)**

Given $f(x) = x^4$

therefore, $f(-x) = (-x)^4 = x^4 = f(x)$

$\Rightarrow$ $f(-x) = f(x)\ \forall\ x \in R$

$\therefore$ f is not one-one.

Also, $x^4 \geq 0\ \forall\ x \in R$, therefore, $R_f = [0, \infty) \neq R$.

Hence, f is not onto.

Thus, f is neither one-one nor onto.

So, (D) is the correct alternative.

Q.7 ***Find gof and fog, if***

(i) ***f(x) = | x | and g(x) = | 5x – 2 |***

(ii) ***f(x) = $8x^3$ and g(x) = $x^{\frac{1}{3}}$***

Sol. Here, $f(x) = |x|$, $g(x) = |5x - 2|$

$\therefore$ $D_f = D_g = R$ and $R_f = R_g = [0, \infty)$.

To find gof :

As $R_f \subset D_g$, therefore, $(\because\ [0, \infty) \subset R)$

gof is defined and $D_{gof} = D_f = R$.

Also. for all $x \in D_{gof} = R$,

$(gof)(x) = g(f(x)) = g(|x|) = |5|x| - 2|$

To find fog :

As $R_g \subset D_f$, therefore,

fog is defined and $D_{fog} = D_g = R$.

$(\because\ [0, \infty) \subset R)$

Also, for all $x \in D_{fog} = R$, $(fog)(x) = f(g(x))$

$= f(|5x - 2|) = \big||5x - 2|\big|$

$= |5x - 2|$

Note that $(fog)(x) = g(x),\ \forall\ x \in R$

$\Rightarrow$ fog = g.

(ii) Here, $f(x) = 8x^3$, $g(x) = x^{1/3}$

$\Rightarrow$ $D_f = D_g = R$

and $R_f = R_g = R$.

To find gof :

As $R_f = D_g$, therefore, gof is defined

and $D_{gof} = D_f = R$.

Also, for all $x \in R$. $(gof)(x) = g(f(x))$

$= g(8x^3) = (8x^3)^{1/3} = (2^3 x^3)^{1/3} = 2x$.

To find fog :

As $R_g = D_f$, therefore, fog is defined

and $D_{fog} = D_g = R$.

Also, for all $x \in R$. $(fog)(x) = f(g(x))$

$= f(x^{1/3}) = 8(x^{1/3})^3 = 8x$.

Q.8 ***If f(x) = $\frac{(4x+3)}{(6x-4)}$, $x \neq \frac{2}{3}$, show that fof(x) = x, for all $x \neq \frac{2}{3}$. What is the inverse of f ?***

Sol. Here, $f(x) = \frac{4x+3}{6x-4}$, $x \neq \frac{2}{3}$

$\therefore$ $D_f = R - \left\{\frac{2}{3}\right\}$.

To find R_f, let $y = f(x) = \frac{4x+3}{6x-4}$

$\Rightarrow$ $6xy - 4y = 4x + 3$

$\Rightarrow$ $x(6y - 4) = 4y + 3$

$\Rightarrow$ $x = \frac{4y+3}{6y-4}$ .....(1)

But $x \in R$, therefore, $6y - 4 \neq 0$ $\Rightarrow$ $y \neq 2/3$.

$\therefore$ $R_f = R - \left\{\frac{2}{3}\right\}$.

Since $R_f = D_f$, therefore, fof is meaningful

and $D_{fof} = D_f$

Also, for all $x \neq \frac{2}{3}$, $(fof)(x) = f(f(x))$

$$= \frac{4f(x)+3}{6f(x)-4} = \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$$

$$= \frac{16x+12+18x-12}{24x+18-24x+16} = x.$$

As $(fof)(x) = x = I(x)$ .....(2)

where I is the identity mapping of $R - \left\{\frac{2}{3}\right\}$.

therefore, $f^{-1} = f$.

Q.9 ***Consider f : $R_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.***

Sol. Let $x_1, x_2 \in D_f = R_+ = [0, \infty)$ be such that

$f(x_1) = f(x_2)$

$\Rightarrow \quad x_1^2 + 4 = x_2^2 + 4$

$\Rightarrow \quad x_1^2 = x_2^2$

$\Rightarrow \quad |x_1| = |x_2|$

$\Rightarrow \quad x_1 = x_2 \quad (\because \text{ both } x_1, x_2 \geq 0)$

$\therefore$ f is one-one.

Let $y \in R_f$, then $y = f(x)$, $x \in D_f = R_+$

$\Rightarrow \quad y = x^2 + 4$

$\Rightarrow \quad x = \sqrt{y-4} \qquad (\because \ x \geq 0)$

As x is real, therefore, $y - 4 \geq 0$

$\Rightarrow \quad y \geq 4 \quad \Rightarrow \quad R_f = [4, \infty)$.

$\therefore$ R_f = co-domain $\Rightarrow$ f is onto.

Also, corresponding to every $y \in R_f$ there exists $\sqrt{y-4} \in D_f$ such that $f\left(\sqrt{y-4}\right) = y$.

$\therefore$ f is both one-one and onto and hence invertible.

To find f^{-1} : let $y = f(x) \Rightarrow y = x^2 + 4$

$\Rightarrow \quad x^2 = y - 4 \quad \Rightarrow x = \sqrt{y-4} \quad (\because \ x \geq 0)$

$\Rightarrow \quad f^{-1}(y) = \sqrt{y-4}$

or $\quad f^{-1}(x) = \sqrt{x-4}$

for all $x \in D_{f^{-1}} = R_f = [4, \infty)$.

Q.10 ***Consider f : $R_+ \to [-5, \infty)$ given by***

$$f(x) = 9x^2 + 6x - 5.$$

Show that f is invertible with

$$f^{-1}(y) = \left(\frac{\left(\sqrt{y+6}\right) - 1}{3}\right).$$

Sol. Given $D_f = R_+ = [0, \infty)$

Let $x_1, x_2 \in D_f$ be such that

$f(x_1) = (x_2)$

$\Rightarrow \quad 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow \quad 9\left(x_1^2 - x_2^2\right) + 6(x_1 - x_2) = 0$

$\Rightarrow \quad 9(x_1 - x_2)(x_1 + x_2) + 6(x_1 - x_2) = 0$

$\Rightarrow \quad (x_1 - x_2)\{9(x_1 + x_2) + 6\} = 0$

$\Rightarrow \quad x_1 - x_2 = 0 \quad \Rightarrow \quad x_1 = x_2$

$(\because x_1, x_2 \geq 0, \ \therefore \ 9(x_1 + x_2) + 6 \neq 0$ as $x_1, x_2 \geq 0)$

$\therefore$ f is one-one.

Let $y \in R_f$, then $y = f(x)$, $x \in D_f$

$\Rightarrow \quad y = 9x^2 + 6x - 5, \ x \in D_f$

$\Rightarrow \quad 9x^2 + 5x - (5 + y) = 0$

$$\Rightarrow \quad x = \frac{-6 \pm \sqrt{36 + 36(5+y)}}{18}$$

$$= \frac{-6 \pm \sqrt{36}\sqrt{1+5+y}}{18}$$

$$= \frac{-6 \pm 6\sqrt{6+y}}{18} = \frac{-1 \pm \sqrt{y+6}}{3}$$

As $x \geq 0$, therefore, $x = \dfrac{-1-\sqrt{y+6}}{3}$ is not possible.

$$\therefore \quad x = \frac{-1+\sqrt{y+6}}{3}$$

......(1)

Also, $\quad x \geq 0 \quad \Rightarrow \dfrac{-1+\sqrt{y+6}}{3} \geq 0$

$\Rightarrow \quad \sqrt{y+6} \geq 1 \quad \Rightarrow \quad y \geq -5$

$\therefore \quad R_f = [-5, \infty)$ = co-domain

$\Rightarrow$ f is onto.

Thus, f is both one-one onto and hence invertible.

To find f^{-1}. Let $y = f(x)$

$\Rightarrow \quad y = 9x^2 + 6x - 5$

$$\Rightarrow \quad x = \frac{-1+\sqrt{y+6}}{3} \Rightarrow f^{-1}(y) = \frac{-1+\sqrt{y+6}}{3}$$

or $\quad f^{-1}(x) = \dfrac{-1+\sqrt{x+6}}{3}$,

$x \in D_{f^{-1}} = R_f = [-5, \infty)$.

Q.11 ***Let f : X → Y be an invertible function. Show that f has unique inverse.***

Sol. Supppose that g_1 and g_2 are two inverses of f, then for all y ∈ Y,

$(fog_1)(y) = f(g_1(y)) = y = I_Y(y)$

and $(fog_2)(y) = f(g_2(y)) = y = I_Y(y)$

⇒ $(fog_1)(y) = (fog_2)(y)$ for all y ∈ Y

⇒ $f(g_1(y)) = f(g_2(y))$ for all y ∈ Y

⇒ $g_1(y) = g_2(y)$ (∵ f is one-one)

⇒ $g_1 = g_2$

∴ inverse of f is unique.

Q.12 ***Consider f : [1, 2, 3] → [a, b, c] given by f(1) = a, f(2) = b and f(3) = c. Find f^{-1} and show that $(f^{-1})^{-1} = f$.***

Sol. Given f(1) = a, f(2) = b, f(3) = c,

i.e., f = {(1, a), (2, b), (3, c)}.

⇒ R_f = co-domain and also f is 1 – 1

⇒ f is invertible and $1 = f^{-1}(a)$,

$2 = f^{-1}(b)$, $3 = f^{-1}(c)$

⇒ f^{-1} = {a, b, c} → {1, 2, 3} is both one-one and onto.

⇒ $(f^{-1})^{-1}$ exists

and $(f^{-1})^{-1}$ = {(1, a), (2, b), (3, c)}

(∵ f^{-1} = {(a, 1), (b, 2), (c, 3)})

⇒ $(f^{-1})^{-1} = f$.

Q.13 ***Let f : X → Y be in invertible function. Show that the inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.***

Sol. f : X → Y is invertible

⇒ f is one-one and onto and f^{-1} : Y → X is defined as $f^{-1}(y) = (x)$

iff y = f(x) ∀ x ∈ X, y ∈ Y.

Let $y_1, y_2 \in Y$ be such that

$f^{-1}(y_1) = f^{-1}(y_2)$

⇒ $f(f^{-1}(y_1)) = f(f^{-1}(y_2)) = (fof^{-1})(y_1)$

⇒ $= (fof^{-1})(y_2)$

⇒ $I_Y(y_1) = I_Y(y_2)$

⇒ $y_1 = y_2$

⇒ f^{-1} is one-one.

Also for each x ∈ X, there exists y = f(x) ∈ Y such that

$f^{-1}(y) = x$ (∵ y = f(x))

Thus, f^{-1} is both one-one and onto and hence invertible.

let $g = (f^{-1})^{-1}$

then $gof^{-1} = I_Y$ and $f^{-1}og = I_X$

For all x ∈ X, $I_X(x) = x$

⇒ $(f^{-1}og)(x) = x$

⇒ $f^{-1}(g(x)) = x$

⇒ $f\{f^{-1}(g(x))\} = f(x)$

⇒ $(fof^{-1})(g(x)) = f(x)$

⇒ $g(x) = f(x)$ (∵ $fof^{-1} = I_Y$)

⇒ g = f

⇒ $(f^{-1})^{-1} = f$.

Q.14 ***Let f : W → W be defined as f(n) = n – 1, if n is odd and f(n) = n + 1, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.***

Sol. Let m, n be any two distinct elements of W. We shall show that f(m) ≠ f(n).

Case (i). When both m and n are even, then

f(m) = m + 1, f(n) = n + 1

and m ≠ n ⇒ m + 1 ≠ n + 1

⇒ f(m) ≠ f(n).

Case (ii). When both m and n are odd, then

f(m) = m – 1, f(n) = n – 1

and m ≠ n ⇒ m – 1 ≠ n – 1

⇒ f(m) ≠ f(n).

Case (iii). When m is odd and n is even, then f(m) = m – 1 is even and f(n) = n + 1 is odd so that f(m) ≠ f(n) in this case also. Similarly, if m is even and n is odd,

f(m) ≠ f(n).

So, in all cases m ≠ n ⇒ f(m) ≠ f(n).

Hence f is a one-one function.

Further, we show that f is onto.

If $n \in W$ is any element, then

$f(n-1) = n$ if n is odd ($\because$ n – 1 is even)

and $f(n+1) = n$ if n is even ($\because$ n + 1 is odd)

So, every element of W is the f-image of some element in W. Hence f is onto.

Thus f is both one-one and onto, i.e., f is a one-one correspondence.

Consequently, f is invertible. We have seen above that

$f(n-1) = n$ if n is odd

and $f(n+1) = n$ if n is even

$\Rightarrow$ $n - 1 = f^{-1}(n)$ if n is odd

Q.15 ***Show that the function $f : R \to \{x \in R : -1 < x < 1\}$ defined by $f(x) = \dfrac{x}{1+|x|}$, $x \in R$ is one one and onto function.***

Sol. Given $f(x) = \dfrac{x}{1+|x|} = \begin{cases} \dfrac{x}{1+x}, & \text{if } x \ge 0 \\ \dfrac{x}{1-x}, & \text{if } x < 0 \end{cases}$.

Clearly, $D_f = R$ as $1 + |x| \ne 0 \ \forall \ x \in R$.

T.P. f is 1 – 1 : Let $x_1, x_2 \in D_f = R$ such that $x_1 \ne x_2$. For cases arise :

Case I. If $x_1 \ge 0$ and $x_2 < 0$,

then $f(x_1) = \dfrac{x_1}{1+x_1} \ge 0$

and $f(x_2) = \dfrac{x_2}{1+x_2} < 0$

$\Rightarrow$ $f(x_1) \ne f(x_2)$

Case II. If $x_1 < 0$ and $x_2 \ge 0$,

then as in case I,

$f(x_1) < 0$ and $f(x_2) \ge 0$

$\Rightarrow$ $f(x_1) \ne f(x_2)$.

Case III. If $x_1 \ge 0$ and $x_2 \ge 0$,

then $x_1 \ne x_2$

$\Rightarrow$ $1 + x_1 \ne 1 + x_2$

$\Rightarrow$ $\dfrac{1}{1+x_1} \ne \dfrac{1}{1+x_2} \Rightarrow \dfrac{-1}{1+x_1} \ne \dfrac{-1}{1+x_2}$

$\Rightarrow$ $1 - \dfrac{1}{1+x_1} \ne 1 - \dfrac{1}{1+x_2} \Rightarrow \dfrac{x_1}{1+x_1} \ne \dfrac{x_2}{1+x_2}$

$\Rightarrow$ $f(x_1) \ne f(x_2)$.

Q.16 ***Given a non empty set X, consider P(X) which is the set of all subsets of X.***

Define the relation R in P(X) as follows :

For subsets A, B in P(X), ARB if and only if $A \subset B$. Is R an equivalence relation on P(X) ? Justify your answer.

Sol. Since $A \subset A \ \forall \ A \in P(X)$, therefore, R is reflexive.

Also, for $A, B, C \in P(X)$, A R B and B R C

$\Rightarrow$ $A \subset B$ and $B \subset C$

$\Rightarrow$ $A \subset C \Rightarrow A R C$

$\therefore$ R is transitive.

However, R is not symmetric as $A \subset B$ need not imply $B \subset A$.

So, A R B does not imply BRA.

Hence R is not an equivalence relation.

Q.17 ***Given a non-empty set X, consider the binary operation $* : P(X) \times P(X) \to P(X)$ given by $A * B = A \cup B \ \forall$ A, B in P(X), where P(X) is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in P(X) with respect to the operation *.***

Sol. Let $E \in P(X)$ be an identity element, then

$A * E = E * A = A$ for all $A \in P(X)$

$\Rightarrow$ $A \cap E = E \cap A = A$ for all $A \in P(X)$

$\Rightarrow$ $X \cap E = X$ as $X \in P(X)$

$\Rightarrow$ $X \subset E$

Also $E \subset X$ as $E \in P(X)$

$\therefore$ $E = X$

Thus, X is the identity element.

let $A \in P(X)$ be invertible, then there exists $B \in P(X)$ such that $A * B = B * A = X$, then identity element.

$\Rightarrow$ $A \cap B = B \cap A = X$

$\Rightarrow$ $X \subset A$ and also $X \subset B$

Also, $A, B \subset X$ as $A, B \in P(X)$

$\therefore$ $A = X = B$

$\therefore$ X is the only invertible element and

$X^{-1} = B = X$.

Q.18 ***Find the number of all onto functions from the set {1, 2, 3, ..., n) to itself.***

Sol. Let A = {1, 2, 3, ..., n}.

If $f : A \to A$ is an onto function, then range of f = A.

$\Rightarrow$ f is one-one.

($\because$ If f is not one-one then range of f will be a proper subset of A)

Thus, f is both one-one and onto

i.e., f is a one-one correspondence. Number of such functions is the same as the number of arrangements of numbers taken all at a time,

i.e., ${}^nP_n = n!$. Hence n! functions from A to A are onto

Q.19 ***Consider the binary operations $* : R \times R \to R$ and $o : R \times R \to R$ defined as $a * b = |a - b|$ and $a \, o \, b = a$, $\forall$ a, b $\in$ R. Show that * is commutative but not associative, o is associative but not commuatvie. Further, show that $\forall$ a, b c $\in$ R, a * (b o c) = (a * b) o (a * b). Does o distribute over *? Justify your answer.***

Sol. For all a, b $\in$ R, $a * b = | a - b |$

$= | -(b - a) | = | b - a | = b * a$

$\Rightarrow$ $a * b = b * a$

$\Rightarrow$ '*' is commutative.

Also, for all a, b, c $\in$ R,

$(a * b) * c = | a - b | * c = || a - b | - c |$

and $a * (b * c) = a * | b - c | = || a - | b - c ||$

$\Rightarrow$ $(a * b) * c \neq a * (b * c)$

($\because$ $|| a - b | - c | \neq |a - | b - c||$ in general, as an example $|| 3 - 5 | - 7 | = 5$ but $| 3 - | 5 - 7 || = 1$)

$\therefore$ '*' is not associative.

Again aob = a and boa = b

$\Rightarrow$ aob $\neq$ boa

$\Rightarrow$ o is not commutative.

However, for all a, b, c $\in$ R,

(aob) oc = aoc = a

and ao (boc) = aob = a

$\Rightarrow$ (aob) oc = ao (boc)

$\Rightarrow$ 'o' is associative.

Further, we find that for all a, b, c $\in$ R,

$a * (boc) = a * b = | a - b |$

and $(a * b) \, o \, (a * c) = | a - b | \, o \, | a - c |$

$= | a - b |$

$\Rightarrow$ $a * (boc) = (a * b) \, o \, (a * c)$

Again, $ao \, (b * c) = ao \, | b - c | = a$

and $(aob) * (aoc) = a * a = | a - a | = 0$

$\Rightarrow$ $ao \, (b * c) \neq (aob) * (aoc)$

($\because$ $a \neq 0$ in general)

$\therefore$ 'o' is not distributive over '*'.

Q.20 ***Given a non-empty set X, let $* : P(X) \times P(X) \to P(X)$ be defined as $A * B = (A - B) \cup (B - A)$, $\forall$ A, B $\in$ P(X). Show that the empty set f is the identity for the operation * and all the elements A of P(X) are invertible with $A^{-1} = A$.***

Sol. Let E be an identity element, then

$A * E = E * A = A$ for all $A \in P(X)$

$\Rightarrow$ $(A - E) \cup (E - A) = A$ for all $A \in P(X)$

Taking $A = \phi$, we get

$(\phi - E) \cup (E - \phi) = \phi$

$\Rightarrow$ $\phi \cup E = \phi$ $\Rightarrow$ $E = \phi$

Observe that $A * \phi = \phi * A = (A - \phi) \cup (\phi - A)$

$= A$ for all $A \in P(X)$.

So, ϕ is the identity element.

Let $A \in P(X)$ be invertible, then there is $B \in P(X)$ such that $A * B = B * A = \phi$.

$\Rightarrow$ $(A - B) \cup (B - A) = \phi$

$\Rightarrow$ $A - B = \phi$ and also $B - A = \phi$

$\Rightarrow$ $A \subset B$ and $B \subset A$

$\Rightarrow$ $A = B$

Thus for all $A \in P(X)$, $A * A = \phi$.

$\Rightarrow$ A is invertible and $A^{-1} = A$.

Q.1 In the set N of all natural numbers, let a relation R be defined by

R = {(x, y) : x $\in$ N, y $\in$ N, x – y is divisible by 5}

prove that R is an equivalence relation.

Q.2 Let 'm' be a given positive integer. Prove that the

EXERCISE – I UNSOLVED PROBLEMS

relation, 'congruence modulo m' on the set z of all integers defined by $a \equiv b \pmod{m} \Leftrightarrow (a - b)$ is divisible by m is an equivalence relation.

Q.3 Prove that the relation R in the set of integers Z defined by

$xRy \Leftrightarrow x^y = y^x \ \forall\ x, y \in Z$ is an equivalence relation.

Q.4 Let N be the set of all natural numbers and R be the relation on N × N defined by

(i) $(a, b)\ R\ (c, d) \Leftrightarrow ad = bc$

(ii) $(a, b)\ R\ (c, d) \Leftrightarrow ad\,(b + c) = bc\,(a + d)$

prove that R. is an equivalence relation in each case.

Q.5 If $f : X \to Y$ and $A, B \subseteq X$, then prove that

(i) $f(A \cup B) = f(A) \cup f(B)$

(i) $f(A \cap B) \subseteq f(A) \cap f(B)$

Q.6 If $f : X \to Y$ and $A, B \subseteq Y$, then prove that

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

(iii) $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$

Q.7 Prove that only one-one onto function has inverse function.

Q.8 Prove that the product of any function with the identity function is the function itself.

Q.9 Prove that the product of any invertible function f with its inverse f^{-1} is an identity function.

Q.10 Prove that composite of functions is associative.

Q.11 Let $f : A \to B$ and $g : B \to A$ such that gof is an identity function on A and fog is an identity function on B. Then, $g = f^{-1}$

Q.12 Let $f : A \to B$ and $g : B \to C$ be one-one onto functions. Then gof is also one-one onto and $(gof)^{-1} = f^{-1}og^{-1}$

Q.13 Let $f : N \to N$ be defined by

$$f(x) = \begin{cases} n+1, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even} \end{cases}$$

show that f is a bijection.

Q.14 Show that the function

$f : R - \{3\} \to R - \{1\}$ given by

$$f(x) = \frac{x-2}{x-3} \text{ is a bijection.}$$

Q.15 (i) If $f : A \to B$ and $g : B \to C$ are one-one functions. Show that gof is a one-one function.

(ii) If $f : A \to B$ and $g : B \to C$ are onto functions. Show that gof is an onto function.

Q.16 Prove that the composition of two bijections is a bijection i.e. if f and g are two bijections, then gof is also a bijections.

Q.17 If $f : R \to (-1, 1)$ defined by $f(x) = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$ is invertible find f^{-1}.

Q.18 Let S = N × N and '*' be an operation on S defined by (a, b) * (c, d) = (ac, bd) for all a, b, c, d $\in$ N. Determine whether '*' is a binary operation on S. If yes, check the commutativity and associativity.

Q.19 Let Q be the set of all rational numbers, define an operation on * Q – {–1} by a * b = a + b + ab. show that

(i) '*' is a binary operation on Q – {–1}

(ii) '*' is commutative

(iii) '*' is associative

(iv) zero is the identity element in Q – {–1} for *

(v) $a^{-1} = \left(\dfrac{-a}{1+a}\right)$. where $a \in Q - \{-1\}$

Q.20 Let $S = R_0 \times R$, where Ro denote the set of all non-zero real numbers. A binary operation '*' is defined on s as follows. (a, b) * (c, d) = (ac, bc + d) for all (a, b) (c, d) $\in R_0 \times R$

(i) Find the identity element in S.

(ii) Find the invertible element in S.

Q1. Show that the relation R defined by R = {(a, b) : a – b is divisible by 3; a, b $\in$ N is an equivalence relation.

Q2. Let T be the set of all triangles in a plane with R as a relation in T given by $R = \{(T_1, T_2) : T_1 \cong T_2\}$. Show that R is an equivalence relation.

EXERCISE – II BOARD PROBLEMS

Q3. Show that the relation R defined by (a, b) R(c, d) $\Rightarrow$ a + d = b + c on the set N × N is an equivalence relation.

Q4. (i) Is the binary operation $*$, defined on the set N, given by $a * b = \frac{a+b}{2}$ for all a, b $\in$ Q, commutative ?
(ii) Is the above binary operation $*$ associative ?

Q5. Let $*$ be a binary operation defined by $a * b = 3a + 4b - 2$. Find $4 * 5$.

Q6. If f(x) is an invertible function, find the inverse of $f(x) = \frac{3x-2}{5}$.

Q7. Let $*$ be a binary operation defined by $a * b = 2a + b - 3$. Find $3 * 4$.

Q8. If f(x) = x + 7 and g(x) = x – 7, find (fog) (7).

Q9. Let f : N $\rightarrow$ N be defined by $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if n is odd} \\ \frac{n}{2}, & \text{if n is even} \end{cases}$ for all n $\in$ N
Find whether the function f is bijective.

Q10. Let $*$ be a binary operation on N given by $a * b$ = HCF (a, b), a, b $\in$ N. Write the value of $22 * 4$.

Q11. Show that the relation S defined on the set N × N by
(a, b) S (c, d) $\Rightarrow$ a + d = b + c is an equivalence relation.

Q12. If f : R $\rightarrow$ R be defined by $f(x) = (3 - x^3)^{1/3}$, then find fof(x).

Q13. A binary operation $*$ on the set {0, 1, 2, 3, 4, 5} is defined as : $a * b = \begin{cases} a+b, & \text{if } a+b < 6 \\ a+b-6, & \text{if } a+b \ge 6 \end{cases}$
Show that zero is the identity for this operation and each element 'a' of the set is invertible with 6 – a, being the inverse of 'a'.

Q14. Let f : R $\rightarrow$ R be defined as f(x) = 10x + 7. Find the function g : R $\rightarrow$ R such that gof = fog = I_R.

Q15. Show that f : N $\rightarrow$ N, given by $f(x) = \begin{cases} x+1, & \text{if x is odd} \\ x-1, & \text{if x is even} \end{cases}$ is both one - one and onto :

OR

Consider the binary operations * : R × R $\rightarrow$ R and o : R × R $\rightarrow$ R defined as a * b = | a – b | and aob = a for all a, b $\in$ R. Show that '*' is commutative but not associative, 'o' is associative but not commutative

Q.16 Consider f : $R_+ \rightarrow [4, \infty]$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y - 4}$, where R_+ is the set of all non-negative real numbers.

EXERICISE – 1 (UNSOLVED PROBLEMS)

Answers

7. $\frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right)$ **20.** **(i)** (1, 0) **(ii)** $\left(\frac{1}{a}, \frac{-b}{a}\right)$

EXERCISE – 2 (BOARD PROBLEMS)

4. (i) yes (ii) yes **5.** 30 **6.** $f^{-1}(x) = \frac{5x+2}{3}$ **7.** 7 **8.** 7 **9.** not bijective **10.** 2

12. x **14.** $g(x) = \frac{x-7}{10}$