Probability

1. Definitions

1.1 Trial and Event :

An experiment is called a **trial** if it results in anyone of the possible outcomes and all the possible outcomes are called **events**.

For Example :-

(i) Participation of player in the game to win a game, is a trial but winning or losing is an event.

(ii) Tossing of a fair coin is a trial and turning up head or tail are events.

(iii) Throwing of a dice is a trial and occurrence of number 1 or 2 or 3 or 4 or 5 or 6 are events.

(iv) Drawing a card from a pack of playing cards is a trial and getting an ace or a queen is an event.

1.2 Exhaustive Events :

Total possible outcomes of an experiment are called its **exhaustive events**.

For Example :-

(i) Tossing a coin has 2 exhaustive cases i.e. either head or tail may come upward.

(ii) Throwing of a die has 6 exhaustive cases because any one of six digits 1, 2, 3, 4, 5, 6 may come upward.

(iii) The drawing of one ball from a bag which contains 4 black and 3 white balls result in ${}^{7}C_1$ events. Thus ${}^{7}C_1 = 7$ events are exhaustive.

(iv) Throwing of a pair of dice has 36 exhaustive cases because any one of six digits 1, 2, 3, 4, 5, 6 may come upward on any dice so total number of exhaustive cases $= 6 \times 6 = 36$.

(v) Tossing of two and three coins results in 4 and 8 exhaustive cases respectively because head or tail may come upward on any coin. So in case of two coins total number of cases $= 2 \times 2 = 4$ and in case of three coins total number of cases $= 2 \times 2 \times 2 = 8$

(vi) The drawing of three cards from a pack of 52 cards results in ${}^{52}C_3$ events. Thus ${}^{52}C_3 = 22100$ events are exhaustive.

(vii) The drawing of three balls from a bag containing 4 blue, 5 white and 4 red balls results in ${}^{13}C_3$ events.

Thus ${}^{13}C_3 = 286$ events are exhaustive

1.3 Favourable Events :

Those outcomes of a trial in which a given event may happen, are called **favourable cases** for that event.

For Example :-

(i) If a coin is tossed then favourable cases of getting H is 1.

(ii) If a dice is thrown then favourable case for getting 1 or 2 or 3 or 4 or 5 or 6, is 1.

(iii) If a ball is drawn from a bag containing 4 black and 3 white balls then favourable cases for drawn ball to be black are ${}^{4}C_1 = 4$

(iv) If two dice are thrown, then favourable cases of getting a sum of numbers as 9 are four i.e (4,5), (5,4), (3,6), (6,3).

(v) If three cards are drawn from a pack of 52 cards then favourable cases for all drawn cards be spade are ${}^{13}C_3$ i.e. 286 favourable events.

(vi) If three balls are drawn from a bag containing 3 blue, 4 white and 2 red balls then favourable cases for drawn balls to contain 2 white and 1 red ball are ${}^{4}C_2 \times {}^{2}C_1$ i.e. 12 favourable events. ($\because$ 2 white balls and 1 red ball will be drawn from 4 white balls and 2 red balls respectively.)

1.4 Equally likely events :

Two or more events are said to be **equally likely events** if they have same number of favourable cases.

For Example :-

(i) The result of drawing a card from a well shuffled pack of cards, any card may appear in a draw, so 52 different cases are equally likely.

(ii) In tossing of a coin, getting of 'H' or 'T' are two equally likely events.

(iii) In throwing of a dice, getting 1 or 2 or 3 or 4 or 5 or 6 are six equally likely events.

1.5 Mutually exclusive or disjoint events :

Two or more events are said to be **mutually exclusive**, if the occurrence of one prevents or precludes the occurrence of the others. In other words they cannot occur together.

For example :-

(i) In tossing of a coin, getting of 'H' or 'T' are two mutually exclusive events because then can not happen together.

(ii) In throwing of a dice, getting 1 or 2 or 3 or 4 or 5 or 6 are six mutually exclusive events.

(iii) In drawing a card from a pack of cards, getting a card of diamond or heart or club or spade are four mutually exclusive events.

1.6 Simple and Compound events :

If in any experiment only one event can happen at a time then it is called a **simple event**. If two or more events happen together then they constitute a **compound event.**

For Example :-

If we draw a card from a well shuffled pack of cards, then getting a queen of spade is a simple event and if two coins A and B are tossed together then getting 'H' from A and 'T' from B is a compound event.

1.7 Independent and Dependent events :

Two or more events are said to be **independent** if happening of one does not affect other events. On the other hand if happening of one event affects (partially or totally) other event, then they are said to be **dependent events.**

For Example :-

(i) If we toss two coins, then the occurrence of head on one coin does not influence the occurrence of head or tail on the other coin in any way. Hence these events are independent.

(ii) Suppose a bag contains 5 white and 4 black balls. Two balls are drawn one by one. Then two events that first ball is white and second ball is black are independent if the first ball is replaced before drawing the second ball. If the first ball is not replaced then these two events will be dependent because second draw will have only 8 exhaustive cases.

Note :

Generally students find themselves in problem to distinguish between Independent and mutually exclusive events and get confused. These events have the following differences-

(i) Independent events are always taken from different experiments, while mutually exclusive events are from only one experiment.

(ii) Independent events can happen together but in mutually exclusive events one event may happen at one time.

(iii) Independent events are represented by the word "and" but mutually exclusive events are represented by the word "or".

1.8 Sample Space :

The set of all possible outcomes of a trial is called its **sample space**. It is generally denoted by S and each outcome of the trial is said to be a point of sample of S.

For example :-

(i) If a dice is thrown once, then its sample space S = {1, 2, 3, 4, 5, 6}

(ii) If two coins are tossed together then its sample space S = {HT, TH, HH, TT}.

2. Mathematical Definition of Probability

Let there are n exhaustive, mutually exclusive and equally likely cases for an event A and m of those are favourable to it, then probability of happening of the event A is defined by the ratio m/n which is denoted by P(A). Thus

$$P(A) = \frac{m}{n} = \frac{\text{No. of favourable cases to A}}{\text{No. of exhaustive cases to A}}$$

Note :

It is obvious that $0 \le m \le n$. If an event A is certain to happen, then m = n thus P (A) = 1.

If A is impossible to happen then m = 0 and so P (A) = 0. Hence we conclude that

$$0 \le P(A) \le 1$$

Further, if $\overline{A}$ denotes negative of A i.e. event that A doesn't happen, then for above cases m, n ; we shall have

$$P(\overline{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

$$\therefore P(A) + P(\overline{A}) = 1$$

3. Odds for an Event

If an event A happens in m number of cases and if total number of exhaustive cases are n then we can say that -

The probability of event A, $P(A) = \frac{m}{n}$

and $P(\overline{A}) = 1 - \frac{m}{n} = \frac{n-m}{n}$

$\therefore$ Odds in favour of

$$A = \frac{P(A)}{P(\overline{A})} = \frac{m/n}{(n-m)/n} = \frac{m}{n-m}$$

$\therefore$ Odds in against of

$$A = \frac{P(\overline{A})}{P(A)} = \frac{(n-m)/n}{m/n} = \frac{n-m}{m}$$

So Odds in favour of A = m : (n – m)

Odds in against of A= (n – m) : m

Notations :

(i) $P(A + B)$ or $P(A \cup B)$

= Probability of happening of A or B

= Probability of happening of the events A or B or both

= Probability of occurrence of at least one event A or B

(ii) P(AB) or $P(A \cap B)$ = Probability of happening of events A and B together.

(iii) P(A/B) = Conditional Probability of A when B has happened.

4. Addition theorem of Probability

Case I : When events are mutually exclusive:

If A and B are mutually exclusive events then $n(A \cap B) = 0 \Rightarrow P(A \cap B) = 0$

$\therefore\ P(A \cup B) = P(A) + P(B)$

For any three events A, B, C which are mutually exclusive then $P(A \cap B) = 0$, $P(B \cap C) = 0$, $P(C \cap A) = 0$ and $P(A \cap B \cap C) = 0$

$\therefore\ P(A \cup B \cup C) = P(A) + P(B) + P(C)$

The probability of happening of any one of several mutually exclusive events is equal to the sum of their probabilities, i.e. if $A_1, A_2 .... A_n$ are mutually exclusive events then

$$P(A_1 + A_2 + ... + A_n) = P(A_1) + P(A_2) + ... + P(A_n)$$

i.e. $P(\Sigma A_i) = \Sigma P(A_i)$

Case II : When events are not mutually exclusive.

If A & B are two events which are not mutually exclusive then.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or $P(A + B) = P(A) + P(B) - P(AB)$

For any three events A, B, C

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

or $P(A + B + C) = P(A) + P(B) + P(C)$

$- P(AB) - P(BC) - P(CA) + P(ABC)$

5. Conditional Probability

If A and B are dependent events, then the probability of B when A has happened is called **conditional probability** of B with respect to A and it is denoted by P (B/A). It may be seen that

$$P\left(\frac{B}{A}\right) = \frac{P(AB)}{P(A)}$$

6. Multiplication theorem of Probability

6.1 Case I : When events are independent :

If $A_1, A_2, \ldots, A_n$ are independent events, then

$P(A_1 . A_2 . \ldots A_n) = P(A_1)\,P(A_2) \ldots P(A_n)$.

So if A and B are two independent events then happening of B will have no effect on A. So $P(A/B) = P(A)$ and $P(B/A) = P(B)$, then

$P(A \cap B) = P(A) . P(B)$ OR

$P(AB) = P(A) . P(B)$

Case II : When events are not independent :

The probability of simultaneous happening of two events A and B is equal to the probability of A multiplied by the conditional probability of B with respect to A (or probability of B multiplied by the conditional probability of A with respect to B).i.e.

$P(A \cap B) = P(A) . P(B/A)$ or $P(B) . P(A/B)$

OR

$P(AB) = P(A) . P(B/A)$ or $P(B) . P(A/B)$

6.2 Probability of at least one of the n Independent events :

If $p_1, p_2, p_3, \ldots p_n$ are the probabilities of n independent events $A_1, A_2, A_3 \ldots A_n$ then the probability of happening of at least one of these event is

$$1 - [(1-p_1)(1-p_2) \ldots (1-p_n)]$$

$$P(A_1 + A_2 + A_3 + \ldots + A_n) = 1 - P(\overline{A}_1)$$
$$P(\overline{A}_2)\,P(\overline{A}_3) \ldots P(\overline{A}_n)$$

7. Binomial distribution for Repeated Trials

Let an experiment is repeated **n** times and probability of happening of any event called success is **p** and not happening the event called failure is q = 1– p then by binomial theorem.

$(q + p)^n = q^n + {}^nC_1\, q^{n-1}\, p + \ldots + {}^nC_r\, q^{n-r}\, p^r + \ldots + p^n$

Now probability of

(a) Occurrence of the event exactly r times

$= {}^nC_r\, q^{n-r} p^r$

(b) Occurrence of the event at least r times

$= {}^nC_r\, q^{n-r}\, p^r + \ldots + p^n$

(c) Occurrence of the event at the most r times

$= q^n + {}^nC_1\, q^{n-1} p + \ldots + {}^nC_r\, q^{n-r}\, p^r$

8. Boole's Inequality

(a) For any two events A and B.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore\ P(A \cup B) \le P(A) + P(B)$

$\{\Theta\ P(A \cap B) \ge 0\}$

(b) For any three events A,B,C

$P(A \cup B \cup C) \le P(A) + P(B) + P(C)$

(c) In general for any n events $A_1, A_2, \ldots A_n$

$P(A_1 \cup A_2 \cup \ldots \cup A_n) \le P(A_1) + P(A_2)$

$+ \ldots + P(A_n)$

9. Some Important Results

(a) Let A and B be two events, then

(i) $P(A) + P(\overline{A}) = 1$

(ii) $P(A + B) = 1 - P(\overline{A}\ \overline{B})$

(iii) $P(A/B) = \dfrac{P(AB)}{P(B)}$

(iv) $P(A + B) = P(AB) + P(\overline{A}B) + P(A\overline{B})$

(v) $A \subset B \Rightarrow P(A) \le P(B)$

(vi) $P(\overline{A}B) = P(B) - P(AB)$

(vii) $P(AB) \le P(A)\,P(B) \le P(A+B)$

$\le P(A) + P(B)$

(viii) $P(AB) = P(A) + P(B) - P(A+B)$

(ix) P (Exactly one event)

$= P(A\overline{B}) + P(\overline{A}B) = P(A) + P(B) - 2p(AB)$

$= P(A+B) - P(AB)$

(x) P(neither A nor B) = $P(\overline{A}\,\overline{B}) = 1 - P(A+B)$

(xi) $P(\overline{A} + \overline{B}) = 1 - P(AB)$

(b) Number of exhaustive cases of tossing n coins simultaneously (or of tossing a coin n times) = $\mathbf{2^n}$

(c) Number of exhaustive cases of throwing n dice simultaneously (or throwing one dice n times) = $\mathbf{6^n}$

(d) Playing Cards :

(i) Total : 52 (26 red, 26 black)

(ii) Four suits : Heart, Diamond, Spade, Club - 13 cards each

(iii) Court Cards : 12 (4 Kings, 4 queens, 4 jacks)

(iv) Honour Cards : 16 (4 aces, 4 kings, 4 queens, 4 jacks)

(e) Probability regarding n letters and their envelopes:

If n letters corresponding to n envelopes are placed in the envelopes at random, then

(i) Probability that all letters are in right envelopes = $\frac{1}{n!}$

(ii) Probability that all letters are not in right envelopes = $1 - \frac{1}{n!}$

(iii) Probability that no letter is in right envelope

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - ..... + (-1)^n \frac{1}{n!}$$

(iv) Probability that exactly r letters are in right envelopes

$$= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - ....+ (-1)^{n-r}\frac{1}{(n-r)!}\right]$$