Continuity & Differentiability

1. Introduction

The word **'Continuous'** means without any break or gap. If the graph of a function has no break or gap or jump, then it is said to be **continuous**.

A function which is not continuous is called a **discontinuous function**.

In other words,

If there is slight (finite) change in the value of a function by slightly changing the value of x then function is continuous, otherwise discontinuous, while studying graphs of functions, we see that graphs of functions sin x, x, cos x, e^x etc. are continuous but greatest integer function [x] has break at every integral point, so it is not continuous. Similarly tan x, cot x, secx, 1/x etc. are also discontinuous function.

For examining continuity of a function at a point, we find its limit and value at that point, If these two exist and are equal, then function is continuous at that point.

2. Continuity of a Function at a Point

A function f(x) is said to be continuous at a point x = a if

(i) f (a) exists

(ii) $\lim_{x\to a} f(x)$ exists and finite

so $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

(iii) $\lim_{x\to a} f(x) = f(a)$.

or function f(x) is continuous at x = a.

If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$.

i.e. If right hand limit at 'a' = left hand limit at 'a'= value of the function at 'a'.

If $\lim_{x\to a} f(x)$ does not exist or $\lim_{x\to a} f(x) \neq f(a)$, then f(x) is said to be discontinuous at x= a.

3. Continuity from Left and Right

Function f(x) is said to be

(i) Left continuous at x= a if

$\lim_{x\to a-0} f(x) = f(a)$

(ii) right continuous at x = a if

$\lim_{x\to a+0} f(x) = f(a)$

Thus a function f(x) is continuous at a point x = a if it is left continuous as well as right continuous at x = a.

4. Continuity of a Function in an Interval

(a) A function f(x) is said to be continuous in an open interval (a,b) if it is continuous at every point in (a, b).

For example function y = sin x, y = cos x, $y = e^x$ are continuous in $(-\infty, \infty)$.

(b) A function f(x) is said to be continuous in the closed interval [a, b] if it is-

(i) Continuous at every point of the open interval (a, b).

(ii) Right continuous at x = a.

(iii) Left continuous at x = b.

5. Continuous Functions

A function is said to be continuous function if it is continuous at every point in its domain. Following are examples of some continuous function.

(i) f (x) = x (Identity function)

(ii) f(x) = C (Constant function)

(iii) $f(x) = x^2$

(iv) $f(x) = a_0x^n + a_1x^{n-1} + ... + a^n$ (Polynomial).

(v) f(x) = |x|, x+ |x|, x – |x|, x|x|

(vi) f(x) = sin x, f(x) = cos x

(vii) $f(x) = e^x$, $f(x) = a^x$, a> 0

(viii) f(x) = log x, $f(x) = \log_a x$, a> 0

(ix) f(x) = sinh x, cosh x, tanh x

(x) $f(x) = x^m \sin(1/x)$, m> 0

$f(x) = x^m \cos(1/x)$, m> 0

6. Discontinuous Functions

A function is said to be a discontinuous function if it is discontinuous at at least one point in its domain. Following are examples of some discontinuous function-

(i) f(x) = 1/x at x = 0

(ii) $f(x) = e^{1/x}$ at x = 0

(iii) f(x) = sin 1/x, f(x) = cos 1/x at x = 0

(iv) f(x) = [x] at every integer

(v) f(x) = x– [x] at every integer

(vi) f(x) = tan x, f(x) = sec x

when $x = (2n+1)\pi/2$, $n \in Z$.

(vii) $f(x) = \cot x$, $f(x) = \operatorname{cosec} x$ when $x = n\pi$, $n \in Z$.

(viii) $f(x) = \coth x$, $f(x) = \operatorname{cosech} x$ at $x = 0$.

7. Properties of Continuous Functions

The sum, difference, product, quotient (If Dr $\neq 0$) and composite of two continuous functions are always continuous functions. Thus if f(x) and g(x) are continuous functions then following are also continuous functions:

(a) $f(x) + g(x)$

(b) $f(x) - g(x)$

(c) $f(x) \cdot g(x)$

(d) $\lambda f(x)$, where λ is a constant

(e) $f(x)/g(x)$, if $g(x) \neq 0$

(f) $f[g(x)]$

Note :

The product of one continuous and one discontinuous function may or may not be continuous.

DIFFERENTIABILITY

8. Differentiability of a Function

A function f(x) is said to be differentiable at a point of its domain if it has a finite derivative at that point. Thus f(x) is differentiable at x = a

$\Rightarrow \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists finitely

$$= \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$f'(a-0) = f'(a+0)$

left- hand derivative = Right-hand derivative.

Generally derivative of f(x) at x = a is denoted by $f'(a)$. So $f'(a) = \frac{f(x)-f(a)}{x-a}$

Note : (i) Every differentiable function is necessarily continuous but every continuous function is not necessarily differentiable i.e. Differentiability $\Rightarrow$ continuity

but continuity $\not\Rightarrow$ differentiability

8.1 Differentiability in an interval

(a) A function f(x) is said to be differentiable in an open interval (a,b), if it is differentiable at every point of the interval.

(b) A function f(x) is differentiable in a closed interval [a,b] if it is –

(i) Differentiable at every point of interval (a,b)

(ii) Right derivative exists at x = a

(iii) Left derivative exists at x = b.

8.2 Differentiable function & their properties

A function is said to be a differentiable function if it is differentiable at every point of its domain.

(a) Example of some differentiable functions:–

(i) Every polynomial function

(ii) Exponential function : a^x, e^x, e^{-x}......

(iii) logarithmic functions : $\log_a x$, $\log_e x$,......

(iv) Trigonometrical functions : sin x, cos x,

(v) Hyperbolic functions : sinhx, coshx,......

(b) Examples of some non– differentiable functions:

(i) $|x|$ at x = 0

(ii) $x \pm |x|$ at x = 0

(iii) $[x]$, $x \pm [x]$ at every $n \in Z$

(iv) $x \sin\left(\frac{1}{x}\right)$, at x = 0

(v) $\cos\left(\frac{1}{x}\right)$, at x = 0

(c) The sum, difference, product, quoteint (Dr $\neq 0$) and composite of two differentiable functions is always a differentiable function.