CENTRE OF MASS & COLLISION

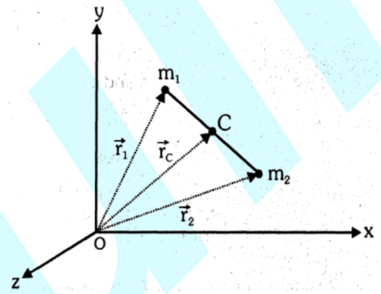
1. CENTRE OF MASS

For a system of particles centre of mass is that point at which its total mass is supposed to be concentrated. The centre of mass of an object is a point that represents the entire body and moves in the same way as a point mass having mass equal to that of the object, when subjected to the same external forces that act on the object.

1.1 Centre of mass of a system of discrete particles

- **Centre of Mass of a Two Particles System**

Consider two particles of masses m_1 and m_2 with position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively. Let their centre of mass C have position vector $\vec{r}_c$.



From definition, we have

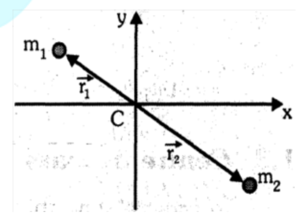
$$\vec{r}_c = \frac{\sum m_i \vec{r}_i}{M} \Rightarrow \vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

From the result obtained above we have,

$$x_c = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \quad \text{and} \quad y_c = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$$

If we assume origin to be at the centre of mass, then the vector $\vec{r}_c$ vanishes and we have

$$m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{0}.$$



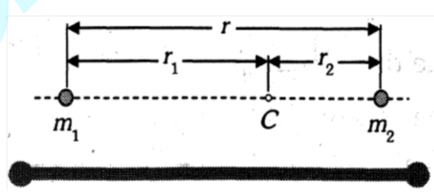
Since neither of the masses m_1 and m_2 can be negative, to satisfy the above equation, vectors $\vec{r}_1$ and $\vec{r}_2$ must have opposite signs. It is geometrically possible only when the centre of mass C lies between the two particles on the line joining them as shown in the figure.

If we substitute magnitudes r_1 and r_2 of vectors $\vec{r}_1$ and $\vec{r}_2$ in the above equation, we have

$$m_1r_1 = m_2r_2, \text{ or } \frac{r_1}{r_2} = \frac{m_1}{m_2}$$

We conclude that the centre of mass of the two particles system lies between the two particles on the line joining them which divides the distance between them in the inverse ratio of their respective masses.

Consider two particles of masses m_1 and m_2 at a distance r from each other. Their centre of mass C must lie in between them on the line joining them. Let the distances of these particles from the centre of mass be r_1 and r_2.



Since centre of mass of a two particles system lies between the two particles on the line joining them which divides the distance between them in the inverse ratio of masses of particles, we can write

$$\boxed{r_1 = \frac{m_2 r}{m_1 + m_2}} \quad \text{and} \quad \boxed{r_2 = \frac{m_1 r}{m_1 + m_2}}$$

Centre of mass (COM) of several Particles

if the co-ordinates of particles of masses m_1, m_2, ... are respectively

(x_1, y_1, z_1), (x_2, y_2, z_2).....

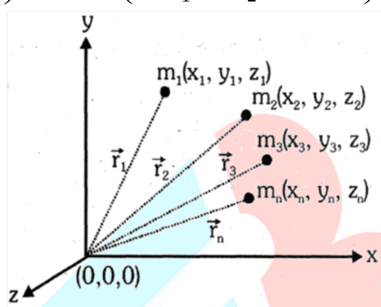
then position vector of their centre of mass is

$$\vec{R}_{CM} = x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}$$

$$= \frac{m_1(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + m_2(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) + m_3(x_3\hat{i} + y_3\hat{j} + z_3\hat{k}) + ...}{m_1 + m_2 + m_3 + ...}$$

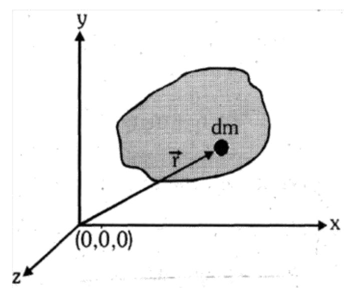
$$= \frac{(m_1x_1 + m_2x_2 + ...)\hat{i} + (m_1y_1 + m_2y_2 ...)\hat{j} + (m_1z_1 + m_2z_2 + ..)\hat{k}}{m_1 + m_2 + m_3 + ..}$$

So, $x_{cm} = \left(\frac{m_1x_1 + m_2x_2 + ...}{m_1 + m_2 + m_3 + ...}\right)$, $y_{cm} = \left(\frac{m_1y_1 + m_2y_2 + ...}{m_1 + m_2 + ...}\right)$, $z_{cm} = \left(\frac{m_1z_1 + m_2z_2 + ...}{m_1 + m_2 + ...}\right)$



1.2 Centre of Mass of Continuous Distribution of Mass

If a system has continuous distribution of mass, treating the mass element dm at position $\vec{r}$ as a point mass and replacing summation by integration.

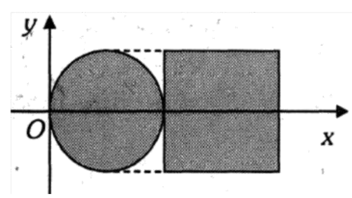


$$\vec{R}_{CM} = \frac{1}{M}\int \vec{r}\,dm \text{ ; where } m = \int dm$$

So that $x_{cm} = \frac{1}{M}\int x\,dm$, $y_{cm} = \frac{1}{M}\int y\,dm$ and $z_{cm} = \frac{1}{M}\int z\,dm$

1.3 Centre of mass of composite bodies

In order to find the centre of mass, the component bodies are assumed to be particle of masses equal to the corresponding bodies located at their respective centres of masses. Then we use the equation to find the coordinates of the centre of mass of the composite body.



To find the centre of mass of the composite body, we first have to calculate the masses of the bodies, because their mass distribution is given.

If we denote the surface mass density (mass per unit area) by σ then the masses of the bodies assumed to be uniform are

Mass of the disc m_d = Mass per unit area × Area = σ (A_d)

Mass of the square plate m_s = Mass per unit area × Area = σ (A_s)

Location of centre of mass of the disc = (x_d, y_d)

Location of centre of mass of the square plate = (x_p, y_p)

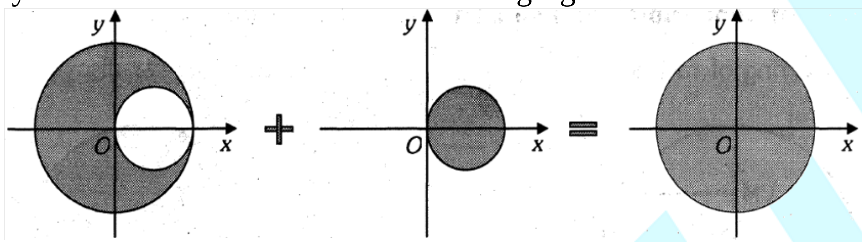
Using eq. corresponding to centre of mass, we obtain its coordinates (x_c, y_c) of the composite body.

$$x_c = \frac{m_d x_d + m_s x_s}{m_d + m_s} \quad \text{and} \quad y_c = \frac{m_d y_d + m_s y_s}{m_d + m_s}$$

$$= \frac{A_d x_d + A_s x_s}{A_d + A_s} \quad \text{and} \quad = \frac{A_d y_d + A_s y_s}{A_d + A_s}$$

1.4 Centre of mass of truncated bodies

To find the centre of mass of truncated bodies or bodies with cavities we can make use of superposition principle that is, if we restore the removed portion in the same place we obtain the original body. The idea is illustrated in the following figure.



The removed portion is added to the truncated body keeping their location unchanged relative to the coordinate frame.

If a portion of a body is taken out, the remaining portion may be considered as,

[Original mass (M) – mass of the removed part (m)] = {original mass (M)} + {–mass of the removed part (m)}

The formula changes to : $x_{cm} = \dfrac{Mx - mx'}{M - m}$; $y_{cm} = \dfrac{My - my'}{M - m}$; $z_{cm} = \dfrac{Mz - mz'}{M - m}$

Where x', y' and z' represent the coordinates of the centre of mass of the removed part.

1.5 Centre of gravity

Centre of gravity of a body is that point where it is assumed that the gravitational force of earth i.e. weight of its body acts on it.

In normal cases, if the acceleration due to gravity remains the same throughout the mass distribution then centre of gravity coincides with the centre of mass and both in turn coincide with the geometrical centre of the body.

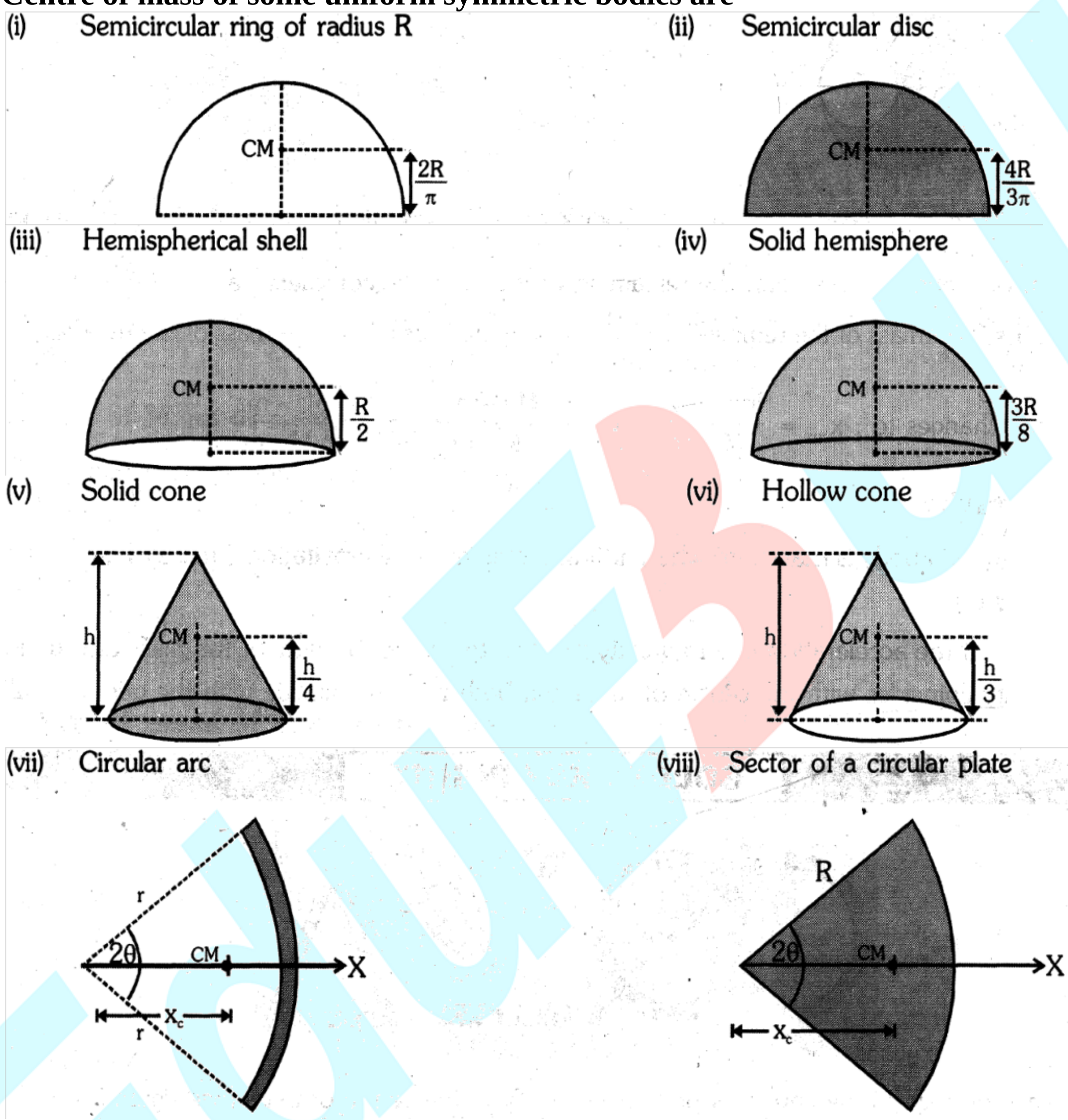
GOLDEN KEY POINTS

- There may or may not be any mass present physically at the centre of mass (See figure A, B, C, D)
- Centre of mass may be inside or outside a body (See figure A, B, C, D)
- Position of centre of mass depends on the shape of the body. (See figure A, B, C, D)
- For a given shape, it depends on the distribution of mass within the body and is closer to massive portion. (See figure A,C)
- For symmetrical bodies having homogeneous distribution of mass it coincides with the centre of symmetry or the geometrical centre. (See figure B, D).
- If we know the centre of mass of parts of the system· and their masses, we can get the combined centre of mass by treating the parts as particles placed at their respective centre of masses.
- It is independent of the co-ordinate system, e.g., the centre of mass of a ring is at its centre whatever be the co-ordinate system.
- If the origin of co-ordinate system is at the centre of mass, i.e., $\vec{R}_{CM} = \vec{0}$, then by definition,

$$\frac{1}{M}\sum m_1 \vec{r}_1 = 0 \Rightarrow \sum m_1 \vec{r}_1 = 0$$

- The sum of the moments of the masses of a system about its centre of mass is always zero.

Centre of mass of some uniform symmetric bodies are



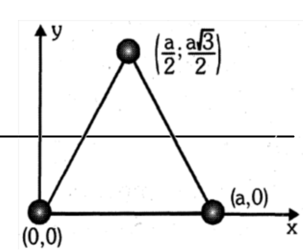
$x_c = \dfrac{r\sin\theta}{\theta}$ $\qquad\qquad$ $x_c = \dfrac{2r\sin\theta}{3\theta}$

Note : Here θ is in radians.

Illustrations

Illustration 1.

There bodies of equal masses are placed at (0, 0), (a, 0) and at $\left(\dfrac{a}{2}, \dfrac{a\sqrt{3}}{2}\right)$.



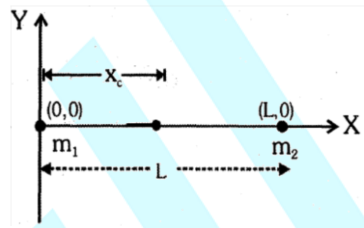
Find out the co-ordinates of centre of mass.

Solution:

$$x_{CM} = \frac{0\times m + a\times m + \frac{a}{2}\times m}{m+m+m} = \frac{a}{2},\quad y_{CM} = \frac{0\times m + 0\times m + \frac{a\sqrt{3}}{2}\times m}{m+m+m} = \frac{a\sqrt{3}}{6}$$

Illustration 2.

Calculate the position of, the centre of mass of a system consisting of two particles of masses m_1 and m_2 separated by a distance L, in relative to m_1.



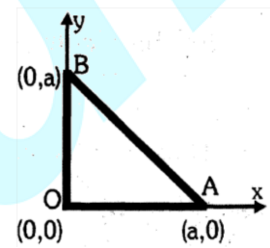
Solution:

Treating the line joining the two particles as x axis

$$x_{CM} = \frac{m_1\times 0 + m_2\times L}{m_1+m_2} = \frac{m_2 L}{m_1+m_2},\quad y_{CM} = 0 \quad z_{CM} = 0$$

Illustration 3.

Three rods of the same mass are place as shown in the figure. Calculate the coordinates of the centre of mass of the system.



Solution:

CM of rod OA is at $\left(\frac{a}{2},0\right)$, CM of rod OB is at $\left(0,\frac{a}{2}\right)$ and

CM of rod AB is at $\left(\frac{a}{2},\frac{a}{2}\right)$

For the system, $x_{cm} = \frac{m\times\frac{a}{2} + m\times 0 + m\times\frac{a}{2}}{m+m+m} = \frac{a}{3}$ $\Rightarrow$ $y_{cm} = \frac{m\times 0 + m\times\frac{a}{2} + m\times\frac{a}{2}}{m+m+m} = \frac{a}{3}$

Illustration 4.

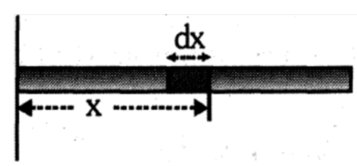
If the linear density of a rod of length L varies as $\lambda = A + Bx$, determine the position of its centre of mass. (where x is the distance from one of its ends)

Solution:

Let the X-axis be along the length of the rod with origin at one of its end as shown in figure. As the rod is along x-axis, so, $y_{CM} = 0$ and $z_{CM} = 0$ i.e., centre of mass will be on the rod.

Now consider an element of rod of length dx at a distance x from the origin, mass of this element $dm = \lambda dx = (A + Bx)dx$ so,

$$x_{CM} = \frac{\int_0^L x\,dm}{\int_0^L dm} = \frac{\int_0^L x(A+Bx)dx}{\int_0^L (A+Bx)dx} = \frac{\frac{AL^2}{2}+\frac{BL^3}{3}}{AL+\frac{BL^2}{2}} = \frac{L(3A+2BL)}{3(2A+BL)}$$



Note : (i) if the rod is of uniform density the $\lambda = A$ = constant & $B = 0$ the $x_{CM} = L/2$

(ii) If the density of rod varies linearly with x, then $\lambda = Bx$ and $A = 0$ the $x_{CM} = 2L/3$

Illustration 5.

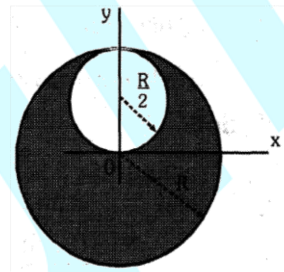
A disc of radius R is cut off from a uniform thin sheet of metal. A circular hole of radius $\frac{R}{2}$ is now cut out from the disc, with the hole being tangent to the rim of the disc. Find the distance of the centre of mass from the centre of the original disc.

Solution:

We treat the hole as a 'negative mass' object that is combined with the original uncut disc. (When the two are overlapped together, the hole region then has zero mass). By symmetry, the CM lies along the +y–axis in figure, so $x_{CM} = 0$. With the origin at the centre of the original circle whose mass is assumed to be m.

Mass of original uncut circle $m_1 = m$ & Location of CM = (0, 0)

Mass of hole of negative mass : $m_2 = \frac{m}{4}$; Location of CM = $\left(0, \frac{R}{2}\right)$



Thus, $y_{CM} = \frac{m_1y_1 + m_2y_2}{m_1 + m_2} = \frac{m(0) + \left(-\frac{m}{4}\right)\frac{R}{2}}{m + \left(-\frac{m}{4}\right)} = -\frac{R}{6}$

So the centre of mass is at the point $\left(0, -\frac{R}{6}\right)$.

Thus, the required distance is R/6.

Illustration 6.

Find the position vector of centre of mass of a system of three particles of masses 1 kg, 2 kg and 3 kg located at position vectors $\vec{r}_1 = \left(4\hat{i} + 2\hat{j} - 3\hat{k}\right)$m, $\vec{r}_2 = \left(\hat{i} - 4\hat{j} + 2\hat{k}\right)$m and $\vec{r}_3 = \left(2\hat{i} - 2\hat{j} + \hat{k}\right)$m respectively.

Solution.

From eq. corresponding to CM, we have

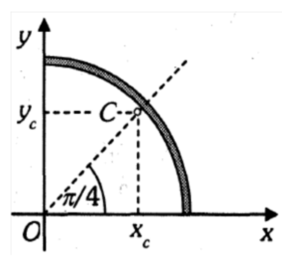
$$\vec{r}_c = \frac{\sum m_i \vec{r}_i}{M}$$

$$\vec{r}_c = \frac{1(4\hat{i} + 2\hat{j} - 3\hat{k}) + 2(\hat{i} - 4\hat{j} + 2\hat{k}) + 3(2\hat{i} - 2\hat{j} + \hat{k})}{1 + 2 + 3} = \left(2\hat{i} - 2\hat{j} + \frac{2}{3}\hat{k}\right)\text{m}$$

Illustration 7.

Find coordinates of center of mass of a quarter ring of radius r placed in the first quadrant of a Cartesian coordinate system, with centre at origin.

Solution:



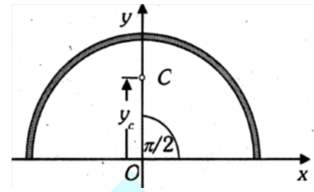
Making use of the result of circular arc, distance OC of the center of mass from the center is OC = $\frac{r\sin(\pi/4)}{\pi/4} = \frac{2\sqrt{2}r}{\pi}$.

Coordinates of the center of mass (x_c, y_c) are $\left(\frac{2r}{\pi}, \frac{2r}{\pi}\right)$

Illustration 8.

And coordinates of center of mass of a semicircular ring of radius r placed symmetric the y-axis of a Cartesian coordinate system.

Solution:



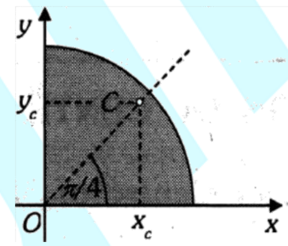
They-axis is the line of symmetry, therefore center of mass of the ring lies on it making x-coordinate zero.
Distance OC of center of mass from center is given by the result obtained for circular arc

$$OC = \frac{r\sin\theta}{\theta} \Rightarrow y_c = \frac{r\sin(\pi/2)}{\pi/2} = \frac{2r}{\pi}$$, So coordinates are $\left(0, \frac{2r}{\pi}\right)$

Illustration 9.

Find coordinates of center of mass of a quarter sector of a uniform disk of radius r placed in the first quadrant of a Cartesian coordinate system with centre at origin.

Solution:



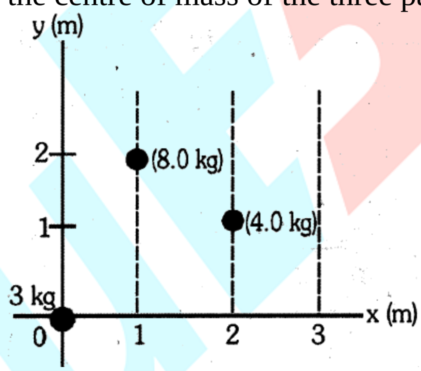
From the result obtained for sector of circular plate distance OC of the center of. mass form the center is

$$OC = \frac{2r\sin(\pi/4)}{3\pi/4} = \frac{4\sqrt{2r}}{3\pi}$$

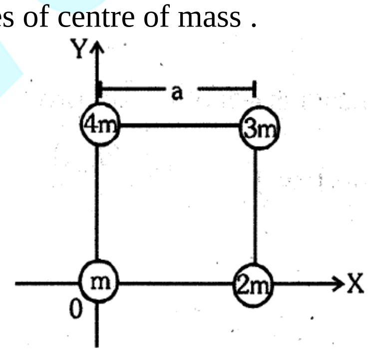
Coordinates of the center of mass (x_c, y_c) are $\left(\frac{4r}{3\pi}, \frac{4r}{3\pi}\right)$

BEGINNER BOX - 1

1. What are the co-ordinates of the centre of mass of the three particles system shown in figure?

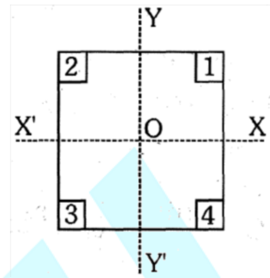


2. Four particles of masses m, 2m, 3m, 4m are placed at the comers of a square of side 'a' as shown in fig. Find out the co-ordinates of centre of mass .



3. A rigid body consists of a 3 kg mass connected to a 2 kg mass by a massless rod. The 3 kg mass is located at $\vec{r}_1 = (2\hat{i} + 5\hat{j})$ m and the 2 kg mass at $\vec{r}_2 = (4\hat{i} + 2\hat{j})$ m. Find the position and coordinates of the centre of mass.

4. Fig. shows a uniform square plate from which one or more of the four identical squares at the corners will be removed.



(a) Where is the centre of mass of the plate originally.
(b) Where is the C.M. after square 1 is removed.
(c) Where is the C.M. after squares 1 and 2 removed.
(d) Where is the C.M. after squares 1 and 3 are removed.
(e) Where is the C.M. after squares 1, 2 and 3 are removed.
(f) Where is the C.M. after all the four squares are removed.

Give your answers in terms of quadrants and axis.

5. Find the centre of mass of a uniform disc of radius 'a' from which a circular section of radius 'b' has been removed. The centre of the hole is at a distance c from the centre of the disc.

2. MOTION OF CENTRE OF MASS

2.1 Motion of Centre of Mass

As for a system of particles, position of centre of mass is given by $\vec{r}_{CM} = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ...}{m_1 + m_2 + m_3 + ...}$

So $\dfrac{d}{dt}(\vec{r}_{CM}) = \dfrac{m_1\dfrac{d\vec{r}_2}{dt} + m_2\dfrac{d\vec{r}_2}{dt}m_3\dfrac{d\vec{r}_3}{dt} + ...}{m_1 + m_2 + m_3 + ...}$

$\Rightarrow$ velocity of centre of mass $\vec{v}_{CM} = \dfrac{d\vec{R}_{CM}}{dt} = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2 + ...}{m_1 + m_2 + ...}$

Similarly acceleration $\vec{a}_{CM} = \dfrac{d}{dt}(\vec{v}_{CM}) = \dfrac{m_1\vec{a}_1 + m_2\vec{a}_2 + ...}{m_1 + m_2 + ...}$

We can write $M\vec{v}_{CM} = m_1\vec{v}_1 + m_2\vec{v}_2 + ... = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + ...$ $[\because \vec{p} = m\vec{v}]$

$M\vec{v}_{CM}\ [\because \Sigma\vec{p}_i = \vec{p}_{CM}]$

Linear momentum of a system of particles is equal to the product of mass of the system with velocity of its centre of mass. From Newton's second law $\vec{F}_{ext.} = \dfrac{d(M\vec{v}_{CM})}{dt}$

If no external force acts on a system the velocity of its centre of mass remains constant, i.e., velocity of centre of mass is unaffected by internal forces.

3. APPLICATION OF METHODS OF IMPULSE AND MOMENTUM TO ASYSTEM OF PARTICLES

In a phenomenon, when a system changes its configuration, some or all of its particles change their respective locations and momenta. Sum of linear momenta of all the particles equals to the linear momentum due to translation of centre of mass.

Impulse momentum theorem : Impulse = Change in momentum

i.e., $\int \vec{F}dt = \Delta\vec{p} = \vec{p}_{final} - \vec{p}_{intial}$

3.1 Conservation of Linear momentum

Total linear momentum of a system of particles remains conserved in a time interval in which impulse of external forces is zero.

Total momentum of a system of particles cannot change under the action of internal forces and if net impulse of the external forces in a time interval is zero, the total momentum of the system in that time interval will remain conserved.

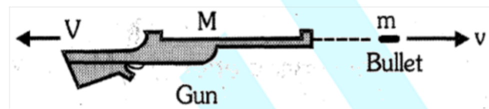
$\vec{p}_{final} = \vec{p}_{intial}$

The above statement is known as the principle of conservation of momentum.

Since force, impulse and momentum are vectors, component of momentum of a system in a particular direction is conserved, if net impulse of all external forces in that direction vanishes.

No external force ⇒ Stationary mass relative to an inertial frame remains at rest

Example : Firing a Bullet from a Gun :



If the bullet is the system, the force exerted by trigger will be external and so the linear momentum of the bullet will change from 0 to mv. This is not the violation of the law of conservation of linear momentum as linear momentum is conserved only in the absence of external force .

If the bullet and gun is the system, then the force exerted by trigger will be internal so.

total momentum of the system $\vec{p}_s = \vec{p}_B + \vec{p}_G$ = constant. .... (i)

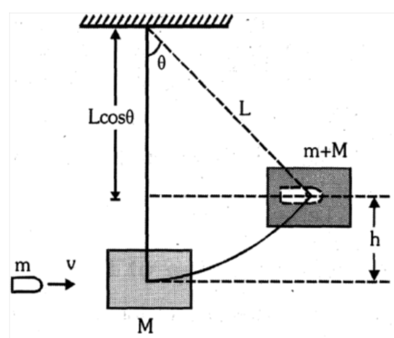
Now, as initially both bullet and gun are at rest so $\vec{p}_B + \vec{p}_G = \vec{0}$. From this it is evident that : $\vec{p}_G = -\vec{p}_B$, i.e., if bullet acquires forward momentum, the gun will acquired equal and opposite (backward) momentum.

From (i) $m\vec{v} + M\vec{V} = \vec{0}$, i.e., $\vec{V} = \frac{m}{M}\vec{v}$ i.e., if the bullet moves forward, the gun 'recoils' or 'kicks backwards'. Heavier the gun lesser will be the recoil velocity V.

Kinetic energy K = $\frac{p^2}{2m}$ and $|\vec{p}_B| = |\vec{p}_G|$ = p. Kinetic energy of gun K_G = $\frac{p^2}{2M}$,

Kinetic energy of bullet K_B = $\frac{p^2}{2m}$ $\therefore \frac{K_G}{K_B} = \frac{m}{M} < 1$ $(\because M >> m)$. Thus kinetic energy of gun is lesser than that of bullet i.e., kinetic energy of bullet and gun will not be equal. Initial kinetic energy of the system is zero as both are at rest. Final kinetic energy of the system is greater than zero. So, here kinetic energy of the system is not constant but increases. If PE is assumed to be constant then Mechanical energy = (kinetic energy + potential energy) will also increase. However, energy is always conserved. Here chemical energy of gun powder is converted into KE.

Example : Block-Bullet System :



(a) When bullet remains embedded in the block

Conserving momentum of bullet and block

mv + 0 = (M + m) V

Velocity of block V = $\frac{mv}{M+m}$ …..(i)

By conservation of mechanical energy

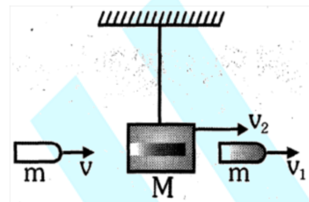
$\frac{1}{2}(M + m)V^2 = (M + m)gh \Rightarrow V = \sqrt{2gh}$ …..(ii)

From eq^n. (i) and eq^n. (ii) $\frac{mv}{M+m} = \sqrt{2gh}$

Speed of bullet v = $\frac{(M+m)\sqrt{2gh}}{m}$, Maximum height gained by block $h = \dfrac{V^2}{2g} = \dfrac{m^2v^2}{2g(M+m)^2}$

$\because h = L - L\cos\theta \ \therefore \cos\theta = 1 - \dfrac{h}{L} \Rightarrow \theta = \cos^{-1}\left(1 - \dfrac{h}{L}\right)$

(b) If bullet emerges out of the block



Conserving momentum $mv + 0 = mv_1 + Mv_2$

$M(v - v_1) = Mv_2$ .....(i)

Conserving energy $\dfrac{1}{2}Mv_2^2 = Mgh \Rightarrow v_2 = \sqrt{2gh}$ .....(ii)

From eq^n. (i) & eq^n. (ii) $m(v - v_1) = M\sqrt{2gh} \Rightarrow h = \dfrac{m^2(v-v_1)^2}{2gM^2}$

Example : Explosion of a Bomb at rest

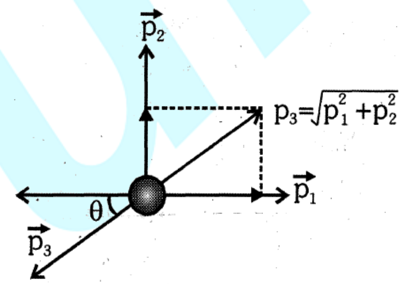
Conserving momentum

$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \vec{0} \Rightarrow \vec{p}_3 = -(\vec{p}_1 + \vec{p}_2) \Rightarrow \vec{p}_3 = \sqrt{p_1^2 + p_2^2}$ as $\vec{p}_1 \perp \vec{p}_2$

Angle made by $\vec{p}_3$ with $\vec{p}_1 = \pi + \theta$

Angle made by $\vec{p}_3$ with $\vec{p}_2 = \dfrac{\pi}{2} + \theta$

where $\theta = \tan^{-1}\left(\dfrac{p_2}{p_1}\right)$.



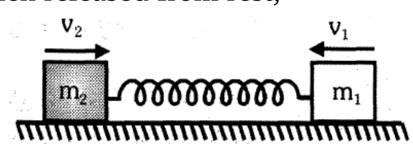
Energy released in explosion $= K_f - K_i = \dfrac{p_1^2}{2m_1} + \dfrac{p_2^2}{2m_2} + \dfrac{p_3^2}{2m_3} - 0 = \dfrac{p_1^2}{2m_1} + \dfrac{p_2^2}{2m_2} + \dfrac{p_3^2}{2m_3}$

Example : Motion of Two Masses Connected to a Spring

Consider two blocks, resting on a frictionless surface and connected by a massless spring as shown in figure. If the spring is stretched (or compressed) and then released from rest,

Then $F_{ext}= 0$ so $\vec{p}_s = \vec{p}_1 + \vec{p}_2 =$ constant

However, initially both the blocks were at rest so, $\vec{p}_1 + \vec{p}_2 = \vec{0}$



It is clear that :

- $\vec{p}_2 = \vec{p}_1$ i.e., at any instant the two blocks will have momentum equal in magnitude but opposite in direction (though they have different values of momentum at different positions).
- As momentum $\vec{p} = m\vec{v}$, $m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{0} \Rightarrow \vec{v}_2 = -\left(\dfrac{m_1}{m_2}\right)\vec{v}_1$

  The two blocks always move in opposite directions with lighter block moving faster.
- Kinetic energy $KE = \dfrac{p^2}{2m}$ and $|\vec{p}_1| = |\vec{p}_2|$, $\dfrac{KE_1}{KE_2} = \dfrac{m_2}{m_1}$ or the kinetic energy of two blocks will not be equal but in the inverse ratio of their masses and so lighter block will have greater kinetic energy.
- Initially kinetic energy of the blocks is zero (as both are at rest) but after some time kinetic energy of the blocks is not zero (as both are in motion). So, kinetic energy is not

constant but changes. Here during the motion of the blocks KE is converted into elastic potential energy of the spring and viceversa but total mechanical energy of the system remain constant.

Kinetic energy + Potential energy = Mechanical Energy = Constant

GOLDEN KEY POINTS

- For an isolated system, initial momentum of the system is equal to the final momentum of the system. If the system consists of n bodies having momenta $\vec{p}_1, \vec{p}_2, \vec{p}_3, ...\vec{p}_n$ then $\vec{p}_1, \vec{p}_2, \vec{p}_3 + ... + \vec{p}_n$ = constant
- As linear momentum depends on frame of reference, observers in different frames would find different values of linear momenta of a given system but each would agree that his own value of linear momentum does not change with time. But the system should be isolated and closed, i.e. law of conservation of linear momentum is independent bf frame of reference though linear momentum depends on the frame of reference.
- Conservation of linear momentum is equivalent to Newton's III law of motion for a system of two particles.

  In the absence of external force from law of conservation of linear momentum,

  $\Rightarrow \vec{p}_1 + \vec{p}_2 = \text{constant}$ i.e. $m_1\vec{v}_1 + m_2\vec{v}_2$ = constant

  Differentiating the above expression with respect to time $m_1\dfrac{d\vec{v}_1}{dt} + m_2\dfrac{d\vec{v}_2}{dt} = \vec{0}$ [as m is constant]

  $\Rightarrow m_1\vec{a}_1 + m_2\vec{a}_2 = \vec{0}$ $[\because \dfrac{d\vec{v}_1}{dt} = \vec{a}] \Rightarrow \vec{F}_1 + \vec{F}_2 = \vec{0}$ $[\because \vec{F} = m\vec{a}] \Rightarrow \vec{F}_1 = -\vec{F}_2$

  i.e., for every action there is equal and opposite reaction which is Newton's III law of motion.
- This law is universal, i.e.' it applies to macroscopic as well as microscopic systems.
- Sum of mass moments in centroidal frame (i.e. centre of mass frame) become zero. It implies $\sum m_n\vec{r}_n = \vec{0}$ **or** $m_1\vec{r}_1 + m_2\vec{r}_2 + ... + m_n\vec{r}_n = \vec{0}$.
- Total linear momentum of the system in centroidal frame is zero. It implies $\sum m_n\vec{v}_n = \vec{0}$ **or** $m_1\vec{v}_1 + m_2\vec{v}_2 + ... + m_n\vec{v}_n = \vec{0}$.

Illustrations

Illustration 10.

Two particles of masses 1 kg and 0.5 kg are moving in the same direction with speeds of 2 m/s and 6 m/s, respectively, on a smooth horizontal surface. Find the speed of the centre of mass of the system.

Solution:

Velocity of centre of mass of the system $\vec{v}_{cm} = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$. Since the two particles are moving in same direction, $m_1\vec{v}_1$ and $m_2\vec{v}_2$ are parallel.

$$\Rightarrow |m_1\vec{v}_1 + m_2\vec{v}_2| = m_1v_1 + m_2v_2.$$

Therefore, $v_{cm} = \dfrac{|m_1\vec{v}_1 + m_2\vec{v}_2|}{m_1 + m_2} = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \dfrac{(1)(2) + \left(\frac{1}{2}\right)(6)}{\left(1 + \frac{1}{2}\right)} = 3.33$ m/s.

Illustration 11.

Tow particles of masses 2 kg and 4 kg are approaching towards each other with accelerations of m/s^2 and 2 m/s^2 respectively, on a smooth horizontal surface. Find the acceleration of centre of mass of the system.

Solution:

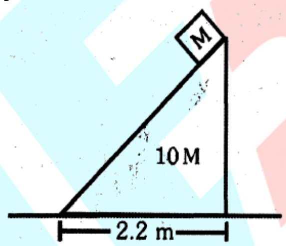
The acceleration of centre of mass of the system $\vec{a}_{cm} = \dfrac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2} \Rightarrow a_{cm} = \dfrac{|m_1\vec{a}_1 + m_2\vec{a}_2|}{m_1 + m_2}$

Since $\vec{a}_1$ and $\vec{a}_2$ are anti-parallel, so $a_{cm} = \dfrac{|m_1a_1 - m_2a_2|}{m_1 + m_2} = \dfrac{|(2)(1) - (4)(2)|}{2+4} = 1$ m/

Since $m_2a_2 > m_1a_1$ so the direction of acceleration of centre of mass is along in the direction of a_2.

Illustration 12.

A block of mass M is placed on the top of a bigger block of mass 10 M as shown in figure. All the surfaces are frictionless. The system is released from rest.



Find the distance moved by the bigger block at the instant when the smaller block reaches the ground.

Solution:

If the bigger block moves toward right. by a distance (x) then the smaller block will move toward left by a distance (2.2 – x).

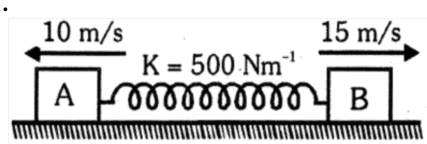
Now considering both the blocks together as a system, horizontal position of CM remains same.

As the sum of mass moments about centre of mass is zero i.e. $\sum m_i x_{i/cm} = 0$.

$M(2.2 - x) = 10\,Mx \Rightarrow x = 0.2$ m.

Illustration 13.

Two blocks A and B are joined together with a compressed spring. When the system is released, the two blocks appear to be moving with unequal speeds in opposite directions as shown in figure. Select the correct statement :



(A) The centre of mass of the system will remain stationary.

(B) Mass of block A is equal to that of block B.

(C) The centre of mass of the system will move towards right.

(D) It is an impossible physical situation.

Ans. **(A)**

Solution:

As net force on the system = 0 (after being released)
So centre of mass of the system remains stationary.

Illustration 14.

A man of mass 80 kg stands on a plank of mass 40 kg. The plank is lying on a smooth horizontal floor. Initially both are at rest. The man starts walking on the plank towards north and stops after moving a distance of 6 m on the plank. Then

(A) the centre of mass of plank-man system remains stationary.
(B) the plank will slide to the north by a distance of 4 m
(C) the plank will slide to the south by a distance of 4 m
(D) the plank will slide to the south by a distance of 12 m

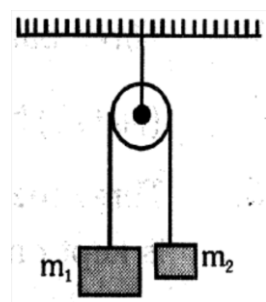
Ans. **(A,C)**

Solution:

Since net force is zero so centre of mass remains stationary
Let x be the displacement of the plank.
Since CM of the system remains stationary
so 80 (6 – x) = 40 x ⇒ 12 – 2x = x ⇒ x ,= 4 m.

Illustration 15.

Two bodies of masses m_1 and m_2 ($<m_1$) are connected to the ends of a massless cord and allowed to move as shown in figure. Toe pulley is massless and frictionless. Calculate the acceleration of the centre of mass.



Solution:

If $\vec{a}$ is the acceleration of m_1, then $-\vec{a}$ is the acceleration of m_2 then

$$\vec{a}_{cm} = \frac{m_1\vec{a} + m_2(-\vec{a})}{m_1 + m_2} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\vec{a}$$

But $\vec{a} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\vec{g}$ so $\vec{a}_{cm} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 \vec{g}$.

Illustration 16.

In a gravity free room a man of mass m_1 is standing at a height h above the floor. He throws a ball of mass m_2 vertically downward with a speed u. Find the distance of the man from the floor when the ball reaches the ground.

Solution:

Time taken by ball to reach the ground $t = \frac{h}{u}$

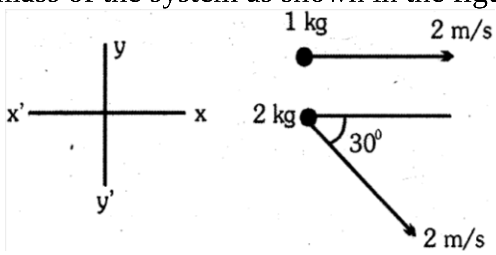
By conservation of linear momentum, speed of man $v = \left(\frac{m_2 u}{m_1}\right)$

Therefore, the man will move upward by a distance $= vt = \left(\frac{h}{u}\right)\left(\frac{m_2 u}{m_1}\right) = \frac{m_2}{m_1}h$

Total distance of the man from the floor = h + $\frac{m_2}{m_1}$h = $\left(1+\frac{m_2}{m_1}\right)$h.

BEGINNER'S BOX - 2

1. The velocity of centre of mass of the system as shown in the figure is :-



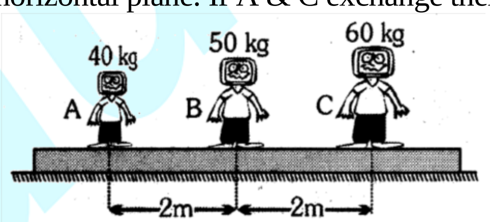
(A) $\left(\frac{2-2\sqrt{3}}{3}\right)\hat{i}-\frac{1}{3}\hat{j}$ (B) $\left(\frac{2+2\sqrt{3}}{3}\right)\hat{i}-\frac{2}{3}\hat{j}$ (C) $4\hat{i}$ (D) None of these

2. Two bodies of masses 10 kg and 2 kg are moving with velocities $(2\hat{i}-7\hat{j}+3\hat{k})$ m/s and $(-10\hat{i}+35\hat{j}-3\hat{k})$ m/s respectively. Find the velocity of their centre of mass.

3. Two blocks of masses 5 kg and 2 kg placed on a frictionless surface are connected by a spring. An external kick gives a velocity of 14 m/s to the heavier block in the direction of the lighter one. Calculate the velocity gained by the centre of mass.

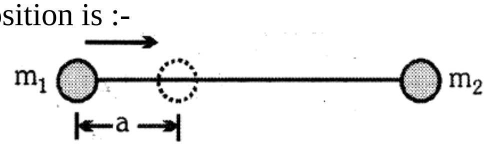
4. Three particles of masses 1 kg, 2 kg and 3 kg are subjected to forces $(3\hat{i}-2\hat{j}+2\hat{k})$ N, $(-\hat{i}+2\hat{j}-\hat{k})$ and $(\hat{i}+\hat{j}+\hat{k})$ N respectively. Find the magnitude of the acceleration of the CM of the system.

5. Three men A, B & C of masses 40 kg, 50 kg and 60 kg are standing on a plank of mass 90 kg, which is kept on a smooth horizontal plane. If A & C exchange their positions then mass B will shift



(A) 1/3 m towards left (B) 1/3 m towards right
(C) will not move w.r.t. ground (D) 5/3 m towards left

6. Consider a system having two masses m_1 and m_2 in which first mass is pushed towards the centre of mass by a distance a. The distance by which the second mass should be moved to keep the centre of mass at same position is :-

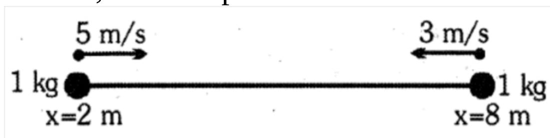


(A) $\frac{m_1}{m_2}a$ (B) $\frac{m}{(m_1+m_2)}a$ (C) $\frac{m_2}{m_1}a$ (D) $\left(\frac{m_2}{m_1+m_2}a\right)$

7. Two particles A and B initially at rest, move towards each other under their mutual force of attraction. At the instant when the speed of A is v and the speed of B is 2v, the speed of the centre of mass of the system is:-

(A) 3v (B) v (C) 1.5v (D) zero

8. The figure shows the positions and velocities of two particles. If the· particles move under the mutual attraction of each other, then the position of centre of mass at t = 1 s is :-



(A) x = 5 m (B) x = 6 m (C) x = 3 m (D) x = 2 m

9. An isolated particle of mass m is moving in a horizontal plane (x – y), along the x-axis, at a certain height above the ground. It suddenly explodes into two fragments of masses $\frac{m}{4}$ and $\frac{3m}{4}$. An instant later, the smaller fragment is at y = +15 cm. The larger fragment at this instant is at :-

(A) y = –5 cm (B) y = +20 cm (C) y = +5 cm (D) y = –20 cm

10. In which of the following cases, the centre of mass of a rod may be at its centre?

(1) The linear mass density decreases continuously from left to right.
(2) The linear mass density increases continuously from left to right.
(3) The linear mass density decreases from left to right upto the centre and then increases.
(4) The linear mass density increases from left to right upto the centre and then decreases.

(A)1, 2 (B) 3, 4 (C) 2, 4 (D) 1, 4

4. COLLISION

Impact or collision is the interaction between two bodies during very small duration in which they exert relatively large forces on each other. Interaction forces during an impact are created either due to direct contact or strong repulsive force fields or some. connecting links. These forces are so large as compared to other external forces acting on either of the bodies that the effects of later can be neglected. The duration of the interaction is short enough to permit us only to consider the states of motion just before and after the event and not during the impact. Duration of an impact ranges from 10^{-23} s for impacts between elementary particles to millions of years for impacts between galaxies. The impacts we observe in our everyday life such as that between two balls last from 10^{-3} s to few seconds.

For example, when an α–particle passes by the nucleus of a gold atom in Rutherford's experiment, it gets deflected in a very short time. Deflection means a change in the direction of motion- a change in velocity.

In this process, the particles do not touch each other.

Let us take another example, when a rubber ball strikes a floor, it remains in contact with the floor for very short time in which it changes its velocity. This is an example of collision where physical contact takes place between the colliding bodies.

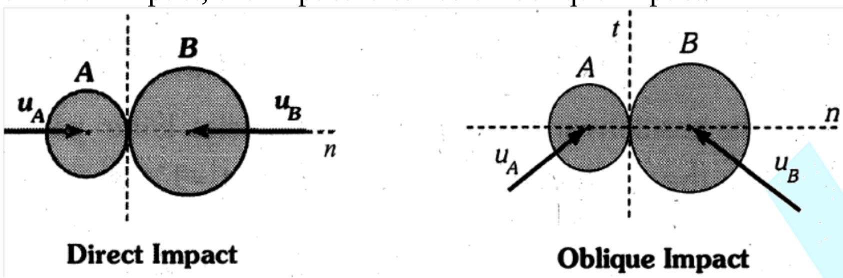
As a result of collision, the momentum and kinetic energy of the interacting bodies change.

Forces involved in a collision are action-reaction forces, i.e., the internal forces of the system.

The total momentum remains conserved in any type of collision.

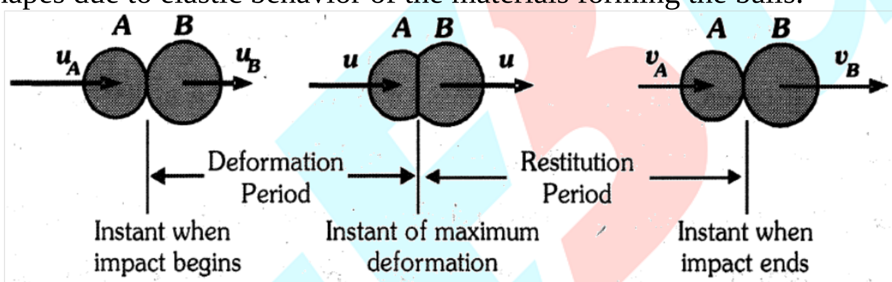
Head-on (Direct) and Oblique collision (impact)

If velocity vectors of the colliding bodies are directed along the line of impact, the impact is called a direct or head-on impact; and if velocity vectors of both or of any one of the bodies are not along the line of impact, the impact is called an oblique impact.



4.1 Head-on (Direct) Impact

To understand what happens in a head-on impact let us consider two balls A and B of masses m_A and m_B moving with velocities u_A and u_B in the same direction as shown. Velocity u_A is larger than u_B so the ball A hits the ball B. During the impact, both the bodies push each other and first they get deformed till the deformation reaches a maximum value and then they try to regain their original shapes due to elastic behavior of the materials forming the balls.

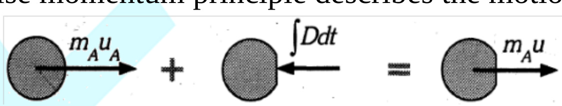


The time interval during which deformation takes place is called the **deformation period** and the time interval in which the bodies try to regain their original shapes is called the **restitution period**. Due to push applied by the balls on each other during period of deformation speed of ball A decreases and that of ball B increases and at the end of the deformation period, when the deformation is maximum both the balls move with the same velocity say it is u.

Thereafter, the balls will either move together with this velocity or follow the period of restitution. During the period of restitution due to push applied by the balls on each other, speed of the ball A decreases further and that of ball B increases further till they separate from each other. Let us denote the velocities of the balls A and B after the impact by v_A and v_B respectively.

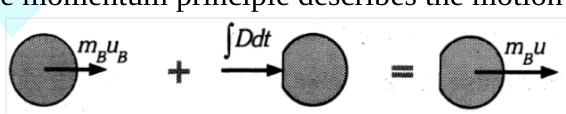
Equation of Impulse and Momentum during impact

Impulse momentum principle describes the motion of ball A during deformation period.



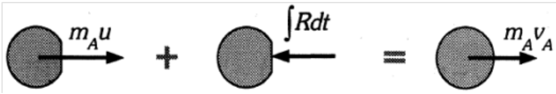
$$m_Au_A - \int Ddt = m_Au \quad \text{...(i)}$$

Impulse momentum principle describes the motion of ball B during deformation period.



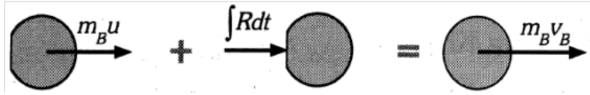
$$m_Bu_B + \int Ddt = m_Bu \quad \text{...(ii)}$$

Impulse momentum principle describes the motion of ball A during restitution period.



$$m_A u - \int Rdt = m_A v_A \qquad \text{...(iii)}$$

Impulse momentum principle describes the motion of ball B during restitution period.



$$m_B u + \int Rdt = m_B v_B \qquad \text{...(iv)}$$

Conservation of Momentum during impact

From equations, (i) and (ii) we have $\quad m_A u_A + m_B u_B = (m_A + m_B)\, u \quad$ ….(v)

From equations, (iii) and (iv) we have $\quad (m_A + m_B)\, u = m_A v_A + m_B v_B \quad$ ….(vi)

From equations, (v) and (vi) we obtain the following equation.

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \qquad \text{….(vii)}$$

The above equation elucidates the principle of conservation of momentum.

Coefficient of Restitution

Usually the force D applied by the bodies A and B on each other during the period of deformation differs from the force R applied by the bodies on each other during the period of restitution. Therefore, it is hot necessary that the magnitude of impulse $\int Ddt$ due to deformation equals to that of impulse $\int Rdt$ due to restitution.

The ratio of magnitudes of impulse of restitution to that of deformation is called the coefficient of restitution and is denoted by e.

$$\boxed{e = \frac{\text{impulse of recovery}}{\text{impulse of deformation}} = \frac{\int Rdt}{\int Ddt}}$$

$$\boxed{e = \frac{\text{velocity of separation along line of impact}}{\text{velocity of approach along line of impact}} = \frac{|\vec{v}_B - \vec{v}_A|}{|\vec{u}_A - \vec{u}_B|}}$$

From equations (i), (ii), (iii) and (iv), we have

coefficient of restitution depends on various factors as elastic properties of materials forming the bodies, velocities of the contact points before impact, state of rotation of the bodies and temperature of the bodies. In general, its value ranges from zero to one but in collisions where additional kinetic energy is generated, its value may exceed one.

Depending on the values of coefficient of restitution, two particular cases are of special interest.

Perfectly Plastic or Inelastic Impact — For these impacts e = 0, and bodies undergoing impact stick to each other after the impact.

Perfectly Elastic Impact — For these impacts e = 1.

Strategy to solve problems of head-on impact :

Write the momentum conservation equation $\quad m_A v_A + m_B v_B = m_A u_A + m_B u_B \quad$ …(A)

Write the equation involving coefficient of restitution $\quad v_B - v_A = e(u_A - u_B) \quad$ …(B)

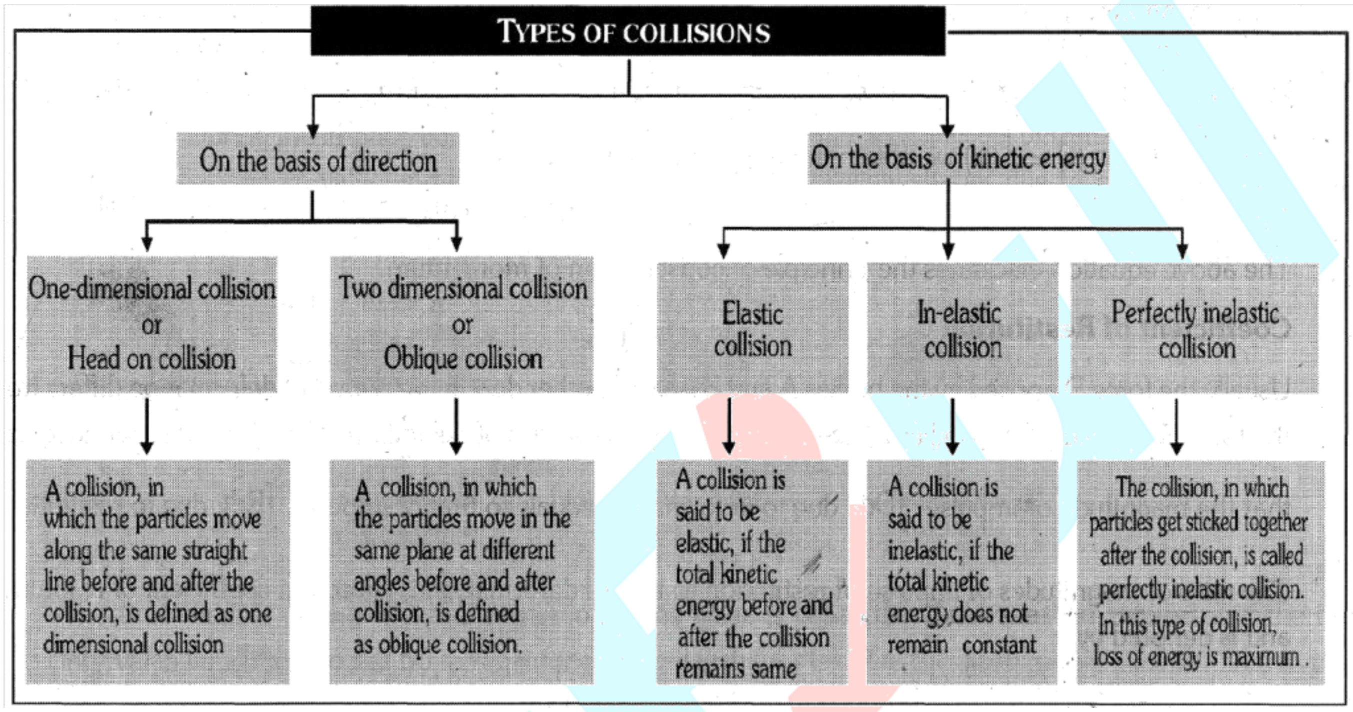
4.2 Types of collisions according to the conservation law of kinetic energy :

(a) **Elastic collision :** $KE_{\text{before collision}} = KE_{\text{after collision}}$

(b) **Inelastic collision :** Kinetic energy is not conserved.
Some energy is lost in collision; Therefore $KE_{\text{before collision}} > KE_{\text{after collision}}$

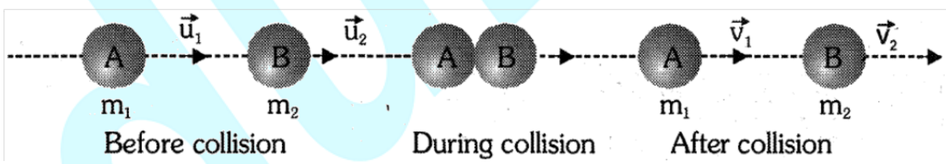
(c) **Perfect inelastic collision :** Both the bodies stick together after collision.

momentum remains conserved in all types of collisions.



Head on Elastic collision

The head on elastic collision is one in which the colliding bodies move along the same straight line path before and after the collision.



Assuming initial direction of motion to be positive and $u_1 > u_2$ (so that collision may take place) and applying law of conservation of linear momentum

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \Rightarrow m_1(u_1 - v_1) = m_2(v_2 - u_2) \quad \text{....(i)}$$

For elastic collision, kinetic energy before collision must be equal to kinetic energy after collision, i.e:,

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \Rightarrow m_1(u_1^2 - v_1^2) = m_2(v_2^2 - u_2^2) \quad \text{....(ii)}$$

Dividing equation (ii) by (i) $\quad u_1 + v_1 = v_2 + u_2 \Rightarrow (u_1 - u_2) = (v_2 - v_1) \quad$ ....(iii)

In 1–D elastic collision 'velocity of approach' before collision is equal to the 'velocity of separation after collision, no matter what the masses of the colliding particles are.

This law is called **Newton's law for elastic collision.**

If we multiply equation (iii) by m_2 and subtract it from (i)

$$(m_1 - m_2)\vec{u}_1 + 2m_2\vec{u}_2 = (m_1 + m_2)\vec{v}_1 \Rightarrow \vec{v}_1 = \frac{m_1 - m_2}{m_1 + m_2}\vec{u}_1 + \frac{2m_2}{m_1 + m_2}\vec{u}_2 \quad \text{....(iv)}$$

Similarly, multiplying equation (iii) by m_1 and adding it to equation (i)

$$2m_1\vec{u}_1 + (m_2 - m_1)\vec{u}_2 = (m_2 + m_1)\vec{v}_2 \Rightarrow \vec{v}_2 = \frac{m_2 - m_1}{m_1 + m_2}\vec{u}_1 + \frac{2m_2\vec{u}_1}{m_1 + m_2} \quad \text{....(v)}$$

Note : If masses are different and collision is inelastic then by momentum conservation

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_2\vec{v}_2 \quad \text{....(i)}$$

By definition of coefficient of restitution

$$\vec{v}_2 - \vec{v}_1 = e(\vec{u}_1 - \vec{u}_2) \quad \text{....(ii)}$$

$$\vec{v}_1 = \left(\frac{m_1 - em_2}{m_1 + m_2}\right)\vec{u}_1 + \left(\frac{(1+e)m_2}{m_1 + m_2}\right)\vec{u}_2 \qquad \text{Loss in KE} = \frac{1}{2}\frac{m_1 m_2}{(m_1 + m_2)}(1 - e^2)(\vec{u}_1 - \vec{u}_2)^2$$

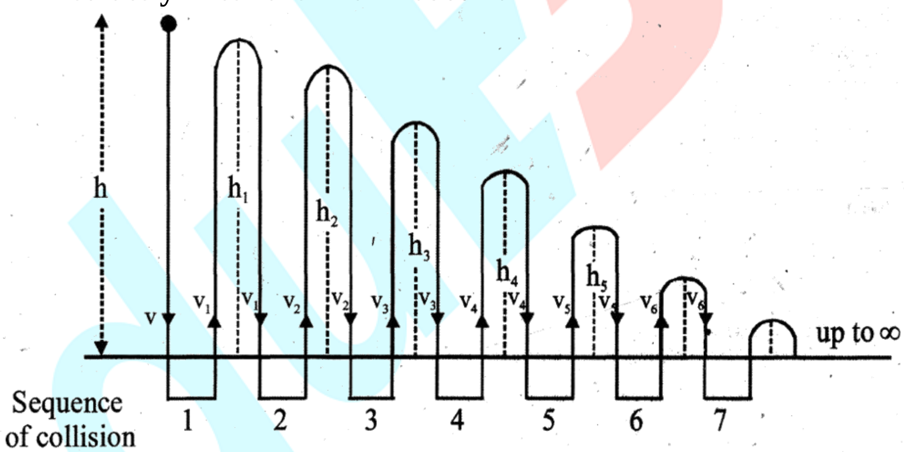
$$\vec{v}_2 = \left(\frac{m_2 - em_2}{m_1 + m_2}\right)\vec{u}_1 + \left(\frac{(1+e)m_2}{m_1 + m_2}\right)\vec{u}_2$$

4.3 Bouncing of Ball

Let a ball fall from a height (h) and let it touch the ground with a velocity v taking time (t) to reach the ground.

Let v_1, v_2, v_3..... be the velocities immediately after first, second, third..... collisions with the ground.

- **Velocity immediately After the 'n'th Rebound**



By Newton's formula $(\vec{v}_2 - \vec{v}_2) = (\vec{u}_1 - \vec{u}_2)$

$v = \sqrt{2gh}$ here $\vec{v}_2 = 0$, $\vec{u}_2 = 0$ (surface at rest)

$v_1 = ev$ (opposite direction)

$v = u_1$

$v_1 = ev$ ……(1), $v_2 = ev_1$ ……(2), $v_2 = e(ev) \Rightarrow v_2 = e^2v$

Similarly, $v_3 = e^3v$, $v_4 = e^4v$……. $v_n = v^nv$

$V_n = e^n\sqrt{2gh}$

- **Height Attained by the Ball After the 'n'th Rebound**

$v_1 = ev \Rightarrow \sqrt{2gh_1} = e\sqrt{2gh} \Rightarrow h_1 = e^2h,$

$v_2 = e^2h \Rightarrow \sqrt{2gh_2} = e^2\sqrt{2gh}, \Rightarrow h_2 = e^4h.$

Similarly $h_n = e^{2n}h$

- **Time Taken in n^th Rebound**

$$h_1 = e^2h,\ \frac{1}{2}gt_1^2 = e^2\frac{1}{2}gt^2 \Rightarrow t_1^2 = e^2t^2, t_1 = et$$

$$t_1 = e\sqrt{\frac{2h}{g}}, h_2 = e^4h,\ \frac{1}{2}gt_2^2 = e^4\left(\frac{1}{2}gt^2\right) \Rightarrow t_2^2 = e^4t^2, t_2 = e^2t, t_2 = e^2\sqrt{\frac{2h}{g}}$$

Similarly $t_n = e^n\sqrt{\frac{2h}{g}}$ $\quad t_n = e^nt$

Total time taken in bouncing. (i.e., total time elapsed before the ball stops)

$$T = t + 2t_1 + 2t_2 + \ldots\ldots\ldots$$
$$= t + 2et + 2e^2t + 2e^3t + \ldots\ldots..$$
$$= t + 2t\,(e + e^2 + e^3 + \ldots\ldots.)$$
$$= t + 2t\left(\frac{e}{1-e}\right) = t\left(\frac{1+e}{1-e}\right) = \sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)$$
$$T = \sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)$$

- **Distance Covered by The Ball Before it Stops**

$$s = 2h_1 + 2h_2 + \ldots\ldots.. + \infty = h + 2e^2h + 2e^4h + 2e^6h + \ldots\ldots..$$
$$= h + 2e^2h\,(1 + e^2 + e^4 + e^6 + \ldots\ldots.)$$
$$= h + 2e^2h\left(\frac{1}{1-e^2}\right) = h\left[1 + \frac{2e^2}{1-e^2}\right], \qquad s = h\left(\frac{1+e^2}{1-e^2}\right)$$

- **Average Speed**

$$v_{av.} = \frac{\text{Total distance}}{\text{Total Time}} = \frac{h\left(\frac{1+e^2}{1-e^2}\right)}{\sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)}$$

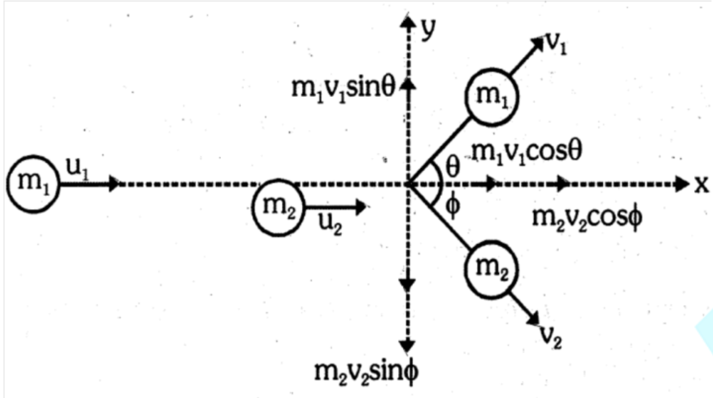
$$v_{av.} = \sqrt{\frac{2h}{2}}\left[\frac{1+e^2}{(1+e)^2}\right]$$

- **Average Velocity**

$$v_{av.} = \frac{\text{Total distance}}{\text{Total Time}} = \frac{h}{\sqrt{\frac{2h}{g}}\left(\frac{1+e}{1-e}\right)}$$

$$v_{av.} = \sqrt{\frac{2h}{2}}\left(\frac{1-e}{1+e}\right)$$

4.4 Oblique collision



By COLM along x-axis $m_1u_1 + m_2u_2 = m_1v_1\cos\theta + m_2v_2\cos\phi$

By COLM along y-axis $0 + 0 = m_1v_1\sin\theta - m_2v_2\sin\phi$

If collision is elastic then,

By conservation of kinetic energy $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$

GOLDEN KEY POINTS

- **If the two bodies are of equal masses :** $m_1 = m_2 = m$ the yield $v_1 = u_2$ and $v_2 = v_1$
  Thus, if two bodies of equal masses undergo elastic collision in one dimension, then the bodies exchange their velocities after the collision.
- **If the two bodies are of equal masses and second body is at rest**.
  $m_1 = m_2$ and initial velocity of second body $u_2 = 0$, $v_1 = 0$, $v_2 = u_1$
  When body A collides against body B of equal mass at rest, then body A comes to rest and body B moves on with the velocity of body A. In this case transfer of energy is hundred percent
  e.g., Billiard's Ball, Nuclear moderation.
- **If the mass of one body is negligible as compared to the other.**
  If $m_1 >> m_2$ and $u_2 = 0$ then $v_1 = u_1$, $v_2 = 2u_1$
  When a heavy body A collides against a light body Bat rest, then body A should keep on moving with same velocity whereas body B moves with velocity double that of A.
  If $m_2 >> m_1$ and $u_2 = 0$ then $v_2 = 0$, $v_1 = -u_1$
  When a light body A collides against a heavy body B at rest, the body A starts moving with same speed just in opposite direction while the body B practically remains at rest.
- Linear momentum remains conserved in all types of collisions .
- Total energy remains conserved in all types of collisions .
- Only conservative forces work in elastic collisions.
- In inelastic collisions all the forces are not conservative.

Illustrations

Illustration 17.

Two balls each of mass 5 kg moving in opposite directions with equal speeds 5 m/s collide head on with each other. Find out the final velocities of the balls if the collision is perfectly elastic.

Solution:

Here $m_1 = m_2 = 5$ kg, $u_1 = 5$ m/s, $u_2 = -5$ m/s

In such a condition velocities get interchanged so $v_2 = u_1 = 5$ m/s and $v_1 = u_2 = -5$ m/s

Illustration 18.

A 0.1 kg ball makes an elastic head on collision with a ball of unknown mass which is initially at rest. If the 0.1 kg ball rebounds with one third of its original speed, what is the mass of other ball?

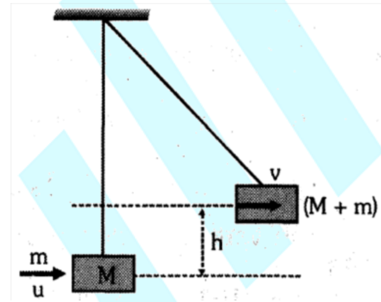
Solution:

Here $m_1 = 0.1$ kg, $m_2 = ?$, $u_2 = 0$, $u_1 = u$, $v_1 = -u/3$

Using, $v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \frac{2m_2u_2}{m_1 + m_2} \Rightarrow -\frac{u}{3} = \left(\frac{0.1 - m_2}{0.1 + m_2}\right)u \Rightarrow m_2 = 0.2$ kg.

Illustration 19.

A simple pendulum of length 1m has a wooden bob of mass 1kg. It is struck by a bullet of mass 10^{-2} kg moving with a speed of 2×10^2 m/s. The bullet gets embedded within the bob. Obtain the height to which the bob rises before swinging back.



Solution:

Applying principle of conservation of linear momentum

$mu = (M + m)\, v \Rightarrow 10^{-2} \times (2 \times 10^2) = (1 + .01)\, v \Rightarrow v = \frac{2}{1.01}$ m/s

Initial KE of the block with bullet in it, is fully converted into PE as it rises through a height h, given by

$$\frac{1}{2}(M + m)v^2 = (M + m)gh \Rightarrow v^2 = 2gh \Rightarrow h = \frac{v^2}{2g} = \left(\frac{2}{1.01}\right)^2 \times \frac{1}{2 \times .98} = 0.2$$

Illustration 20.

A body falling on the ground from a height of 10m, rebounds to a height 2.5 m calculate the : (i) percentage loss in K.E. (ii) ratio of the velocities of the body just before and after the collision.

Solution:

Let v_1 and v_2 be the velocities of the body just before and just after the collision.

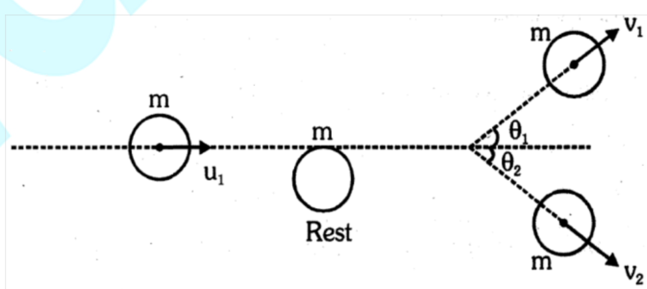
$KE_1 = \frac{1}{2}mv_1^2 = mgh_1$….(i) and $KE_2 = \frac{1}{2}mv_2^2 = mgh_2$…..(ii) $\Rightarrow \frac{v_1^2}{v_2^2} = \frac{h_1}{h_2} = \frac{10}{2.5} = 4 \Rightarrow \frac{v_1}{v_2} = 2.$

Percentage loss in KE $= \frac{mg(h_1 - h_2)}{mgh_1} \times 100 = \frac{10 - 2.5}{10} \times 100 = 75\%.$

Illustration 21.

A body collides obliquely with another identical stationary body elastically. Prove that they will move perpendicular to each other after collision.

Solution:



Conservation of linear momentum in x-direction gives

$mu_1 = mv_1\cos\theta_1 + mv_2\cos\theta_2 \Rightarrow u_1 = v_1\cos\theta_1 + v_2\cos\theta_2$ …..(i)

Conservation of linear momentum in y-direction gives

$0 = mv_1\sin\theta_1 + mv_2\sin\theta_2 \Rightarrow 0 = v_1\sin\theta_1 + v_2\sin\theta_2$ …..(ii)

Conservation of kinetic energy yields,

$$\frac{1}{2}mu_1^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 \quad \Rightarrow \quad u_1^2 = v_1^2 + v_2^2 \qquad .....(iii)$$

Squaring and adding equations (i) and (ii)

$\Rightarrow u_1^2 + 0 = v_1^2\cos^2\theta_1 + v_2^2\cos^2\theta_2 + 2v_1v_2\cos\theta_1\cos^2\theta_2 + v_1^2\sin^2\theta_1 + v_2^2\sin^2\theta_2 - 2v_1v_2\sin\theta_1\sin\theta_2$

$\Rightarrow u_1^2 = v_1^2(\cos^2\theta_1 + \sin^2\theta_1) + v_2^2(\cos^2\theta_2 + \sin^2\theta_2) + 2v_1v_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$

$\Rightarrow u_1^2 = v_1^2 + v_2^2 + 2v_1v_2\cos(\theta_1 + \theta_2) \quad \{\because u_1^2 = v_1^2 + v_2^2\}$

$\Rightarrow \cos(\theta_1 + \theta_2) = 0 \Rightarrow \theta_1 + \theta_2 = 90^\circ.$

Illustration 22.

A ball of mass m hits a floor with a speed v making an angle of incidence $\theta = 45°$ with the normal to the floor. If the coefficient of restitution is $e = \frac{1}{\sqrt{2}}$, find the speed of the reflected ball and the angle of reflection. **AIPMT (Mains) 2005]**

Solution:

Since the floor exerts a force on the ball along the normal during the collision so horizontal component of velocity remains same and only the vertical component changes.

Therefore, $\quad v'\sin\theta' = v\sin\theta = \frac{v}{\sqrt{2}}$

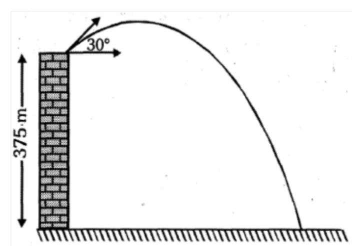
and $\quad v'\cos\theta' = ev\cos\theta = \frac{1}{\sqrt{2}}v \times \frac{1}{\sqrt{2}} = \frac{v}{2}.$

$\Rightarrow \quad v'^2 = \frac{v^2}{2} + \frac{v^2}{4} = \frac{3}{4}v^2 \Rightarrow v' = \frac{\sqrt{3}}{2}v$

and $\quad \tan\theta' = \sqrt{2}$ Þ $\theta' = \tan^{-1}\sqrt{2}$.

Illustration 23.

A particle mass 1 kg is projected from a tower of height 375m with an initial velocity of 100 m/s at an angle 30° to the horizontal. Find its kinetic energy in joules just after the collision with ground if the collision is inelastic with $e = \frac{1}{2}$ (take $g = 10\ m/s^2$)



Solution:

$v_y^2 = u_y^2 + 2gh \Rightarrow v_y = \sqrt{(50)^2 + 2\times10\times375} = 100\ m/s$

Horizontal velocity just after collision = $50\sqrt{3}$ m/s

Vertical velocity just after collision = $100 \times \frac{1}{2} = 50$ m/s

Kinetic energy just after the collision = $\frac{1}{2} \times 1 \times \left[\left(50\sqrt{3}\right)^2 + (50)^2\right] = 5000$ J.

Illustration 24.

A body moving towards a body of finite mass at rest, collides with it. It is impossible that

(A) both bodies come tcn·est

(B) both bodies move after collision

(C) the moving body stops and body at rest starts moving

(D) the stationary body remains~it~fionary and the moving body rebounds

Ans. **(A,D)**

Solution:

For (A) : Momentum cannot be destroyed by internal forces.

For (D) : If mass of stationary body is infinite then the moving body rebounds.

Illustration 25.

A ball of mass 2 kg moving with a speed of 5 m/s collides directly with another ball of mass 3 kg moving in the same direction with a speed of 4 m/s. The coefficient of restitution is 2/3. Find the velocities after collision.

Solution.

Denoting the first ball by A and the second ball by B, velocities immediately before and after the impact are shown in the figure.

By CLOM : $2(5) + 3(4) = 2v_1 + 3v_2 \Rightarrow 2v_1 + 3v_2 = 22$ .....(i)

By definition of e : $e = \frac{v_2 - v_1}{u_1 - u_2 3} \Rightarrow \frac{2}{3} = \frac{v_2 - v_1}{5 - 4} \Rightarrow 3v_2 - 3v_1 = 2$ .....(ii)

by solving equations (i) and (ii), we have $v_1 = 4$ m/s and $v_2 = 4.67$ m/s

Illustration 26.

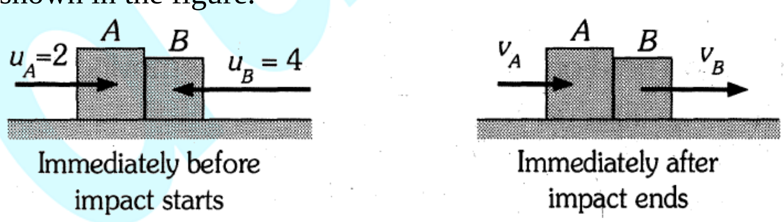
A block of mass 5 kg moves from left to right with a velocity of 2 m/s and collides with another block of mass 3 kg moving along the same line in the opposite direction with velocity 4 m/s.

(a) If the collision is perfectly elastic, determine the velocities of both the blocks after their collision.

(b) If coefficient of restitution is 0.6, determine the velocities of both the blocks after their collision.

Solution.

Denoting the first block by A and the second block by B, velocities immediately before and after the impact are shown in the figure.



Applying principle of conservation of momentum,

$m_Bv_B + m_Av_A = m_Au_A + m_Bu_B$ we have $3v_B + 5v_B = 5 \times 2 + 3 \times (-4) \Rightarrow 3v_B + 5v_A = -2$ .....(i)

Applying equation of coefficient of restitution,

$v_B - v_A = e(u_A - u_B)$ we have $v_B - v_A = e(2 -(-4)) \Rightarrow v_B - v_A = 6e$ .....(ii)

(a) For perfectly elastic impact e =1. Using this value in equation (ii), we have

$v_B - v_A = 6$ .....(iia)

Solving equations (i) and (iia), we obtain $\Rightarrow v_A = -2.5$ m/s and $v_B = 3.5$ m/s

(b) For value e = 0.6, equation 2 is modified as $\Rightarrow v_B - v_A = 3.6$ ......(iib)

Solvin equations (i) and (iib), we obtain $\Rightarrow v_A = 1.6$ m/s and $v_B = 2.0$ m/s

Block A reverses back with speed 1.6 m/s and B too moves in a direction opposite to its original direction with speed 2.0 m/s.

BEGINNER'S BOX - 3

1. A body of mass 2 kg makes an elastic collision with another body at rest and continues to move in the original direction with one fourth of its original speed. Find the mass of the second body.

2. A particle of mass m moving with a velocity v makes a head on elastic collision with another particle of same mass initially at rest. Find the velocity of the first particle after the collision.

3. A particle of mass m moving with velocity v strikes a stationary particle of mass 2m and sticks to it. Find the speed of the system.

4. Two putty balls of equal masses moving in mutually perpendicular directions with equal speed, stick together after collision. If the balls were initially moving with a velocity of $45\sqrt{2}$ m/s each, find the velocity of the combined mass after collision.

5. A body of 2 kg mass having velocity 3 m/s collides with a body of 1 kg mass moving with a velocity of 4 m/s in the opposite direction. After collision both bodies stick together and move with a common velocity. Find the velocity in m/s.

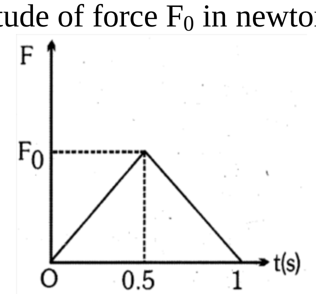
6. A ball of mass 1 kg is dropped from 20 m height. Find (i) velocity of ball after second collision (ii) maxirnuin height attained by the ball after second collision (iii) average speed for whole interval (If e = 0.5) (g = 10 m/s^2)

7. A ball is thrown vertically upward from ground with speed 40 m/s. It collides with ground after returning. Find the total distance travelled and time taken during its bouncing. (e=0.5) (g=10 m/s^2)

8. A particle falls from a height 'h' upon a fixed horizontal plane and rebounds. If e = 0.2 is the coefficient of restitution. Find the total distance travelled before rebounding has stopped.

9. Two balls of equal masses undergo a head-on collision with speeds 6 m/s moving in opposite direction. If the coefficient of restitution is $\frac{1}{3}$, find the speed of each ball after impact in m/s.

10. A body of mass 1 kg moving with velocity 1 m/s makes an elastic one dimensional collision with an identical stationary body. They are in contact for a brief period 1 s. Their force of interaction increases from zero to F_0 linearly in 0.5 s and decreases linearly to zero in a further 0.5 s as shown in figure. Find the magnitude of force F_0 in newtons.



11. An object A of mass 1 kg is projected vertically upward with a speed of 20 m/s. At the same moment another object B of mass 3 kg, which is initially above the object A, is dropped from a height h = 20 m. The two point like objects collide and stick to each other. Find the kinetic energy of the combined mass just after the collision.

12. A particle of mass 2 kg moving with a velocity $5\hat{i}$ m/s collides head-on with another particle of mass 3 kg moving with a velocity $2\hat{i}$ m/s. After the collision the first particle has speed of 1.6 m/s in negative x direction.
Find the :
(a) velocity of the centre of mass after the .collision
(b) velocity of the second particle after the collision
(c) coefficient of restitution.

ANSWERS

BEGINNER'S BOX - 1

1. 1.1 m, 1.3 m **2.** $\left(\frac{a}{2}, \frac{7}{10}a\right)$ **3.** $\left(\frac{14}{5}\hat{i}+\frac{19}{5}\hat{j}\right)$m, (2.8, 3.8)

4. (a) at O; (b) III quadrant; (c) on OY' axis; (d) at O; (e) iv quadrant; (f) at O

5. $x = \frac{-cb^2}{(a^2-b^2)}$

BEGINNER'S BOX - 2

1. (B) **2.** $2\hat{k}$ m/s **3.** 10 m/s **4.** $\frac{\sqrt{14}}{6}$ m/s^2

5. (B) **6.** (A) **7.** (D) **8.** (B)

9. (A) **10.** (B)

BEGINNER'S BOX - 3

1. 1.2 kg **2.** 0 **3.** $\frac{v}{3}$ **4.** 45 m/s

5. $\frac{2}{3}$ m/s, towards initial direction of velocity of 2 kg mass

6. (i) 5 m/s; (ii) $\frac{5}{4}$ m; (iii) $\frac{50}{9}$ m/s **7.** 213.33 m, 16 s

8. $\frac{13}{12}$h **9.** 2 m/s **10.** 2 N **11.** 50 J

12. (a) $0.8\hat{i}$ m/s; (b) $2.4\hat{i}$ m/s; (c) $\frac{4}{7}$