# Binomial Theorem

## 1. Binomial Expressions

An algebraic expression containing two terms is called a **binomial expression**.

For example , $2x + 3$, $x^2 – x/3$, $x + a$ etc. are Binomial Expressions.

## 2. Binomial Theorem

The rule by which any power of a binomial can be expanded is called the **Binomial Theorem**.

## 3. Binomial Theorem for Positive Integral Index

If x and a are two real numbers and n is a positive integer then

$(x + a)^n = {}^nC_0\, x^n a^0 + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots\ldots + {}^nC_r x^{n-r} a^r + \ldots\ldots + {}^nC_n x^0 a^n$.

Where ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ${}^nC_3$,......${}^nC_r$..... are called **binomial coefficients** which can be denoted by $C_0, C_1, C_2, C_3$, .......$C_r$ ......

**3.1 General Term** : In the expansion of $(x+a)^n$, $(r+1)^{th}$ term is called the **general term** which can be represented by $T_{r+1}$.

$$T_{r+1} = {}^nC_r\, x^{n-r} a^r$$

$= {}^nC_r(\text{ first term})^{n-r}$ (second term)$^r$.

**3.2 Characteristics of the expansion of $(x + a)^n$**

Observing to the expansion of $(x + a)^n$, $n \in N$, we find that-

**(i)** The total number of terms in the expansion = (n + 1) i.e. one more than the index n.

**(ii)** In every successive term of the expansion the power of x (first term) decreases by 1and the power of (second term) increases by 1. Thus in every term of the expansion, the sum of the powers of x and a is equal to n (index).

**(iii)** The binomial coefficients of the terms which are at equidistant from the beginning and from the end are always equal i.e.

$${}^nC_r = {}^nC_{n-r}$$

Thus ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$,

${}^nC_2 = {}^nC_{n-2}$ etc.

**(iv)** ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

**3.3 Some deduction of Binomial Theorem :**

**(i)** Expansion of $(x-a)^n$.

$(x – a)^n = {}^nC_0\, x^n a^0 – {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 – {}^nC_3 x^{n-3} a^3 + \ldots + (-1)^r\, {}^nC_r x^{n-r} a^r + \ldots + (-1)^n\, {}^nC_n\, x^o a^n$

This expansion can be obtained by putting (–a) in place of a in the expansion of $(x+a)^n$.

**General term = $(r + 1)^{th}$ term**

$$T_{r+1} = {}^nC_r(-1)^r . x^{n-r} a^r$$

**(ii)** By putting x = 1 and a = x in the expansion of $(x + a)^n$, we get the following result $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + \ldots\ldots + {}^nC_n x^n$

which is the standard form of binomial expansion.

**General term = $(r + 1)^{th}$ term**

$$T_{r+1} = {}^nC_r\, x^r$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!} . x^r$$

**(iii)** By putting (–x) in place of x in the expansion of $(1+ x)^n$

$(1–x)^n = {}^nC_0 – {}^nC_1 x + {}^nC_2 x^2 – {}^nC_3 x^3 + \ldots + (-1)^r\, {}^nC_r x^r + \ldots + {}^nC_n x^n$.

**General term = $(r + 1)^{th}$ term**

$$T_{r+1} = (-1)^r . {}^nC_r\, x^r$$

$$= (-1)^r . \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!} . x^r$$

## 4. Number of Terms in the Expansion of $(x + y + z)^n$

$(x + y + z)^n$ can be expanded as-

$(x + y + z)^n = \{(x + y) + z\}^n$

$= (x + y)^n + {}^nC_1(x + y)^{n-1}.z + {}^nC_2(x + y)^{n-2} z^2 + \ldots\ldots + {}^nC_n z^n$.

= (n + 1) terms + n terms + (n–1) terms + .........+ 1 term

$\therefore$ Total number of terms $= (n+1) + n + (n-1) + \ldots + 1$

$$= \frac{(n+1)(n+2)}{2}$$

## 5. Middle Term in the Expansion of $(x + a)^n$

**(a)** If n is even, then the number of terms in the expansion i.e. (n+1) is odd, therefore, there will be only one middle term which is $\left(\frac{n+2}{2}\right)^{th}$ term.

i.e. $\left(\frac{n}{2}+1\right)^{th}$ term.

so middle term = $\left(\frac{n}{2}+1\right)^{th}$ term.

**(b)** If n is odd, then the number of terms in the expansion i.e. (n +1) is even, therefore there will be two middle terms which are

$$= \left(\frac{n+1}{2}\right)^{th} \text{ and } \left(\frac{n+3}{2}\right)^{th} \text{ term.}$$

**Note :** (i) When there are two middle terms in the expansion then their Binomial coefficients are equal.

(ii) Binomial coefficient of middle term is the greatest Binomial coefficient.

## 6. To Determine a Particular Term in the Expansion

In the expansion of $\left(x^{\alpha} \pm \frac{1}{x^{\beta}}\right)^n$, if $x^m$ occurs in $T_{r+1}$, then r is given by

$$n\alpha - r(\alpha + \beta) = m$$

$$\Rightarrow r = \frac{n\alpha - m}{\alpha + \beta}$$

Thus in above expansion if constant term i.e. the term which is independent of x, occurs in $T_{r+1}$

then r is determined by

$$n\alpha - r(\alpha + \beta) = 0$$

$$\Rightarrow r = \frac{n\alpha}{\alpha + \beta}$$

## 7. To Find a Term the end in the Expansion of $(x + A)^N$

It can be easily seen that in the expansion of $(x+a)^n$.

$(r+1)^{th}$ term from end = $(n-r+1)^{th}$ term from beginning.

i.e. $T_{r+1}(E) = T_{n-r+1}(B)$

$\therefore$ $T_r(E) = T_{n-r+2}(B)$

## 8. Binomial Coefficients & Their Properties

In the expansion of $(1+x)^n$; i.e.$(1+x)^n = {}^nC_0 + {}^nC_1x + \ldots\ldots + {}^nC_r x^r + \ldots + {}^nC_nx^n$

The coefficients ${}^nC_0$, ${}^nC_1$ ,${}^nC_n$ of various powers of x, are called binomial coefficients and they are written as

$$C_0, C_1, C_2, \ldots\ldots, C_n$$

Hence

$$(1+x)^n = C_0+C_1x + C_2x^2+\ldots..+ C_rx^r + \ldots\ldots+ C_nx^n \quad \ldots(1)$$

Where $C_0 = 1$, $C_1 = n$, $C_2 = \frac{n(n-1)}{2!}$

$C_r = \frac{n(n-1)\ldots\ldots(n-r+1)}{r!}$, $C_n = 1$

Now, we shall obtain some important expressions involving binomial coefficients-

**(a) Sum of Coefficient :** putting x = 1 in (1), we get

$$C_0+ C_1 + C_2 + \ldots\ldots+ C_n = 2^n \quad \ldots(2)$$

**(b) Sum of coefficients with alternate signs :** putting x = –1 in(1)

We get

$$C_0 - C_1 + C_2 - C_3 + \ldots\ldots = 0 \quad \ldots(3)$$

**(c) Sum of coefficients of even and odd terms:** from (3), we have

$$C_0+ C_2+ C_4 + \ldots\ldots. = C_1+ C_3+ C_5 + \ldots\ldots.. \quad \ldots(4)$$

i.e. sum of coefficients of even and odd terms are equal.

from (2) and (4)

$$\Rightarrow C_0+ C_2 + \ldots\ldots.. = C_1+ C_3 + \ldots\ldots. = 2^{n-1}$$

**(d) Sum of products of coefficients :** Replacing x by 1/x in (1)

We get

$$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots.+ \frac{C_n}{x^n} + \ldots. \quad \ldots(5)$$

Multiplying (1) by (5), we get

$$\frac{(1+x)^{2n}}{x^n} = (C_0+ C_1x + C_2x^2 +\ldots.)$$

$$\left(C_0+ \frac{C_1}{x} + \frac{C_2}{x} + \ldots.\right)$$

Now, comparing coefficients of $x^r$ on both the sides, we get

$$C_0C_r + C_1C_{r+1} + \ldots\ldots.+ C_{n-r}C_n = {}^{2n}C_{n-r}$$

$$= \frac{2n!}{(n+1)!(n-r)!} \quad \ldots(6)$$

**(e) Sum of squares of coefficients :**

putting r = 0 in (6), we get

$$C_0^2 + C_1^2 + C_2^2 + \ldots\ldots+ C_n^2 = \frac{2n!}{n!n!}$$

**(f)** putting r = 1 in (6), we get

$$C_0C_1 + C_1C_2 + C_2C_3 + \ldots\ldots.+ C_{n-1}C_n = {}^{2n}C_{n-1}$$

$$= \frac{2n!}{(n+1)!(n-1)!} \quad \ldots(7)$$

**(g)** putting r = 2 in (6) , we get

$$C_0C_2 + C_1C_3 + C_2C_4 +\ldots.+ C_{n-2}C_n = {}^{2n}C_{n-2}$$

$$= \frac{2n!}{(n+2)!(n-2)!} \quad \ldots(8)$$

**(h)** Differentiating both sides of (1) w.r.t. x, we get

$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nC_n x^{n-1}$

Now putting x = 1 and x = – 1 respectively

$C_1 + 2C_2 + 3C_3 + \ldots\ldots + nC_n = n.2^{n-1}$ .....(9)

and $C_1 - 2C_2 + 3C_3 - \ldots\ldots = 0$ ...(10)

**(i)** adding (2) and (9)

$C_0 + 2C_1 + 3C_2 + \ldots + {}^{(n+1)}C_n = 2^{n-1}(n+2)$ ...(11)

**(j)** Integrating (1) w.r.t. x between the limits 0 to 1, we get,

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots\ldots + \frac{C_nX^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1} \quad \ldots(12)$$

Integrating (1) w.r.t. x between the limits –1 to 0, we get

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^0$$

$$= \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots\ldots + \frac{C_nx^{n+1}}{n+1}\right]_{-1}^0$$

$$\Rightarrow C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \ldots + \frac{(-1)^n.C_n}{n+1} = \frac{1}{(n+1)} \quad \ldots(13)$$

## 9. Greatest Term in the Expansion of $(X + A)^N$

**(a) The term in the expansion of $(x+a)^n$ of greatest coefficient**

$$= \begin{cases} T_{\frac{n+2}{2}}, \text{ when } n \text{ is even} \\ T_{\frac{n+1}{2}}, T_{\frac{n+3}{2}}, \text{ when } n \text{ is odd} \end{cases}$$

**(b) The greatest term**

$$= \begin{cases} T_p \ \& \ T_{p+1} \text{ when } \frac{(n+1)a}{x+a} = p \in Z \\ T_{q+1} \text{ when } \frac{(n+1)a}{x+a} \notin Z \text{ and } q < \frac{(n+1)a}{x+a} < q+1 \end{cases}$$

## 10. Binomial Theorem For Any Index

When n is a negative integer or a fraction then the expansion of a binomial is possible only when

(i) Its first term is 1, and

(ii) Its second term is numerically less than 1.

Thus when $n \notin N$ and $|x| < 1$, then it states

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3$$

$$+ \ldots + \frac{n(n-1)(n-r+1)}{r!}x^r + \ldots\infty$$

### 10.1 General Term :

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!} . x^r$$

**Note** :

**(i)** In this expansion the coefficient of different terms can not be expressed as ${}^nC_0, {}^nC_1, {}^nC_2\ldots$ because **n** is not a positive integer.

**(ii)** In this case there are infinite terms in the expansion.

### 10.2 Some Important Expansions :

If $|x| < 1$ and $n \in Q$ but $n \notin N$, then

**(a)** $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots\ldots(n-r+1)}{r!}x^r + \ldots\ldots$

**(b)** $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots + \frac{n(n-1)\ldots\ldots(n-r+1)}{r!}(-x)^r + \ldots$

**(c)** $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots + \frac{n(n+1)\ldots(n+r-1)}{r!}x^r + \ldots$

**(d)** $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots + \frac{n(n+1)\ldots(n+r-1)}{r!}(-x)^r + \ldots$

By putting n = 1, 2, 3 in the above results (c) and (d), we get the following results-

**(e)** $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots + x^r + \ldots\ldots$

**General term** $T_{r+1} = x^r$

**(f)** $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots(-x)^r + \ldots\ldots$

**General term** $T_{r+1} = (-x)^r$

**(g)** $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots + (r+1)x^r + \ldots\ldots$

**General term** $T_{r+1} = (r+1)x^r$

**(h)** $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots + (r+1)(-x)^r + \ldots\ldots$

**General term** $T_{r+1} = (r+1)(-x)^r$.

**(i)** $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots + \frac{(r+1)(r+2)}{2!}x^r + \ldots\ldots$

**General term** $= \frac{(r+1)(r+2)}{2!}x^r$

**(j)** $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots\ldots + \frac{(r+1)(r+2)}{2!}(-x)^r + \ldots\ldots$

**General term** $= \frac{(r+1)(r+2)}{2!}(-x)^r$.