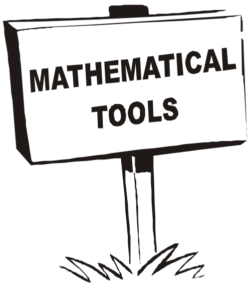
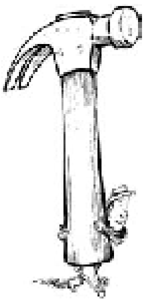
CHAPTER-1
MATHEMATICAL TOOLS

Mathematics is the language of physics. It becomes easier to describe, understand and apply the physical principles, if one has a good knowledge of mathematics.

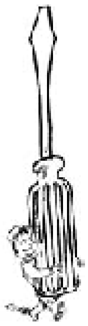


⑩Tools are required to do physical work easily and mathematical tools are required to solve numerical problems easily.

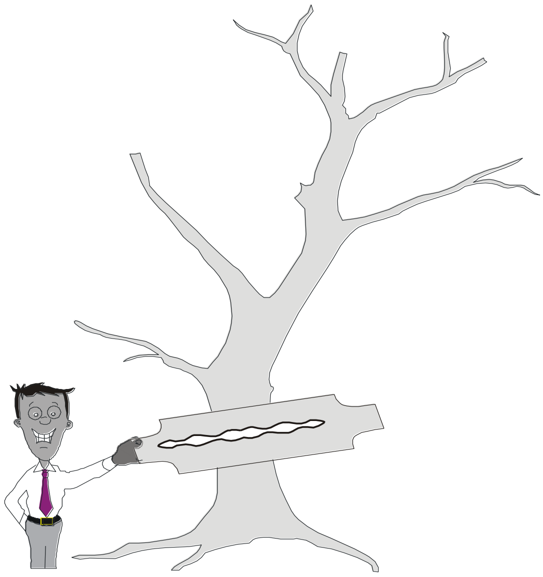
MATHEMATICAL TOOLS

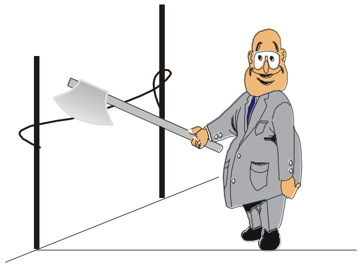

Differentiation


Integration


Vectors

To solve the problems of physics Newton made significant contributions to Mathematics by inventing differentiation and integration.


Cutting a tree with a blade


Cutting a string with an axe

APPROPRIATE CHOICE OF TOOL IS VERY IMPORTANT

1. FUNCTION

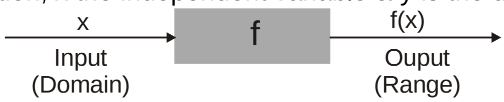
Function is a rule of relationship between two variables in which one is assumed to be dependent and the other independent variable, for example :

e.g. The temperatures at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). Here elevation above sea level is the independent & temperature is the dependent variable

e.g. The interest paid on a cash investment depends on the length of time the investment is held. Here time is the independent and interest is the dependent variable.

In each of the above example, value of one variable quantity (dependent variable) , which we might call y, depends on the value of another variable quantity (independent variable), which we might call x. Since the value of y is completely determined by the value of x, we say that y is a function of x and represent it mathematically as $y = f(x)$.

Here f represents the function, x the independent variable & y is the dependent variable.



All possible values of independent variables (x) are called **domain** of function.

All possible values of dependent variable (y) are called **range** of function.

Think of a function f as a kind of machine that produces an output value f(x) in its range whenever we feed it an input value x from its domain (figure).

When we study circles, we usually call the area A and the radius r. Since area depends on radius, we say that A is a function of r, $A = f(r)$. The equation $A = \pi r^2$ is a rule that tells how to calculate a unique (single) output value of A for each possible input value of the radius r.

$A = f(r) = \pi r^2$. (Here the rule of relationship which describes the function may be described as square & multiply by π).

If $r = 1$ $A = \pi$; if $r = 2$ $A = 4\pi$; if $r = 3$ $A = 9\pi$

The set of all possible input values for the radius is called the domain of the function. The set of all output values of the area is the range of the function.

We usually denote functions in one of the two ways :

1. By giving a formula such as $y = x^2$ that uses a dependent variable y to denote the value of the function.
2. By giving a formula such as $f(x) = x^2$ that defines a function symbol f to name the function.

   Strictly speaking, we should call the function f and not f(x),
   y = sin x. Here the function is sine, x is the independent variable.

Example 1. The volume V of a ball (solid sphere) of radius r is given by the function $V(r) = (4/3)\pi(r)^3$. The volume of a ball of radius 3m is ?

Solution : $V(3) = 4/3\pi(3)^3 = 36\pi\ m^3$.

Example 2. Suppose that the function F is defined for all real numbers r by the formula $F(r) = 2(r - 1) + 3$. Evaluate F at the input values 0, 2, x + 2, and F(2).

Solution : In each case we substitute the given input value for r into the formula for F :

$F(0) = 2(0 - 1) + 3 = -2 + 3 = 1$;

$F(2) = 2(2 - 1) + 3 = 2 + 3 = 5$

$F(x + 2) = 2(x + 2 - 1) + 3 = 2x + 5$;

$F(F(2)) = F(5) = 2(5 - 1) + 3 = 11$.

Example 3. A function $f(x)$ is defined as $f(x) = x^2 + 3$, Find $f(0)$, $f(1)$, $f(x^2)$, $f(x+1)$ and $f(f(1))$.

Solution : $f(0) = 0^2 + 3 = 3$; $f(1) = 1^2 + 3 = 4$; $f(x^2) = (x^2)^2+3 = x^4+3$

$f(x+1) = (x + 1)^2 + 3 = x^2 + 2x + 4$; $f(f(1)) = f(4) = 4^2+3 = 19$

Example 4. If function F is defined for all real numbers x by the formula $F(x) = x^2$. Evaluate F at the input values 0, 2, x + 2 and F(2)

Solution : $F(0) = 0$; $F(2) = 2^2 = 4$; $F(x+2) = (x+2)^2$

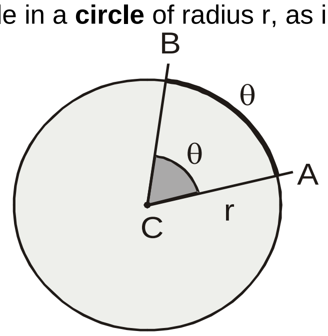
$F(F(2)) = F(4) = 4^2 = 16$

2. TRIGONOMETRY

2.1 MEASUREMENT OF ANGLE AND RELATIONSHIP BETWEEN DEGREES AND RADIAN

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called radians because of the way they simplify later calculations.

Let ACB be a central angle in a **circle** of radius r, as in figure.



Then the angle ACB or θ is defined in radius as $\theta = \dfrac{\text{Arc length}}{\text{Radius}} \Rightarrow \theta = \dfrac{\overset{\frown}{AB}}{r}$

If r = 1 then θ = AB

The **radian measure** for a circle of unit radius of angle ACB is defined to be the length of the circular arc AB. Since the circumference of the circle is 2π and one complete revolution of a circle is 360^o, the relation between radians and degrees is given by : π radians = 180^o

Angle Conversion formulas

1 degree = $\dfrac{\pi}{180}$ (≈ 0.02) radian

1 radian $\dfrac{\pi}{180} \approx 57$ degrees

Degrees to radians : multiply by $\dfrac{\pi}{180}$
Radians to degrees : multiply by $\dfrac{180}{\pi}$

Example 5. (i) Convert 45^o to radians. (ii) Convert $\dfrac{\pi}{6}$ rad to degrees.

Solution : (i) $45 \bullet \dfrac{\pi}{180} = \dfrac{\pi}{4}$ rad (ii) $\dfrac{\pi}{6} \bullet \dfrac{180}{\pi} = 30^o$

Example 6. Convert 30^o to radians.

Solution : $30^o \times \dfrac{\pi}{180} = \dfrac{\pi}{6}$ rad

Example 7. Convert $\dfrac{\pi}{3}$ rad to degrees.

Solution : $\dfrac{\pi}{3} \times \dfrac{180}{\pi} = 60^o$

Standard values

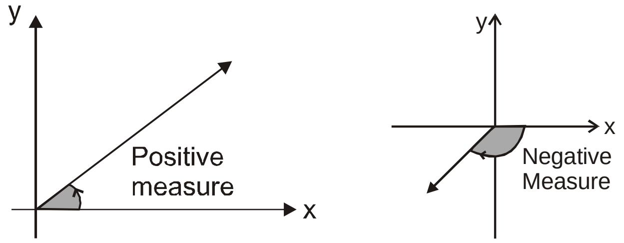
(1) $30° = \dfrac{\pi}{6}$ rad (2) $45^o = \dfrac{\pi}{4}$ rad (3) 60^o = rad

(4) $90^o = \dfrac{\pi}{2}$ rad (5) $120^o = \dfrac{2\pi}{3}$ rad (6) 135^o = rad

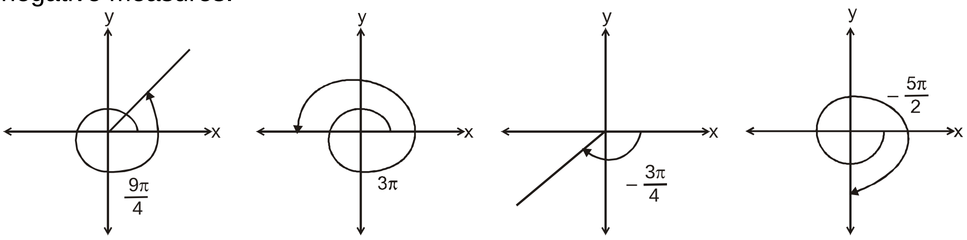
(7) $150^o = \dfrac{5\pi}{6}$ rad (8) $180^o = \pi$ rad (9) $360^o = 2\pi$ rad

(Check these values yourself to see that they satisfy the conversion formulae)

2.2. MEASUREMENT OF POSITIVE AND NEGATIVE ANGLES

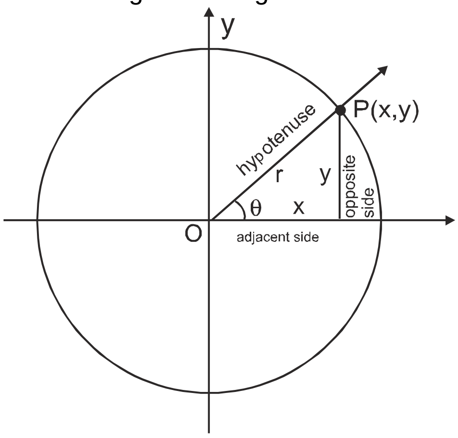


An angle in the xy-plane is said to be in standard position if its vertex lies at the origin and its initial ray lies along the positive x-axis (Fig.). Angles measured counterclockwise from the positive x-axis are assigned positive measures; angles measured clockwise are assigned negative measures.



2.3 SIX BASIC TRIGONOMETRIC FUNCTIONS

The trigonometric function of a general angle θ are defined in terms of x, y, and r.



Sine : $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$ Cosecant : $\text{cosec}\theta = \frac{\text{hyp}}{\text{opp}} = \frac{r}{y}$

Cosine : $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$ Secant : $\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{r}{x}$

Tangent : $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$ Cotangent : $\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{x}{y}$

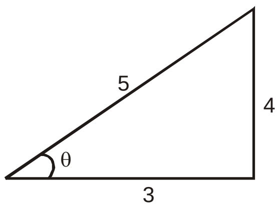
VALUES OF TRIGONOMETRIC FUNCTIONS

If the circle in (Fig. above) has radius r = 1, the equations defining sinθ and cos θ become

$\cos\theta = x$, $\sin\theta = y$

We can then calculate the values of the cosine and sine directly from the coordinates of P.

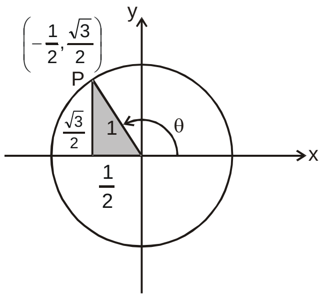
Example 8. Find the six trigonometric ratios from given figure



Solution : $\sin\theta = \frac{opp}{hyp} = \frac{4}{5}$; $\cos\theta = \frac{adj}{hyp} = \frac{3}{5}$; $\tan\theta = \frac{opp}{adj} = \frac{4}{3}$; $\text{cosec }\theta = \frac{hyp}{opp} = \frac{5}{4}$;

$\sec\theta = \frac{hyp}{adj} = \frac{5}{3}$; $\cot\theta = \frac{adj}{opp} = \frac{3}{4}$

Example 9. Find the sine and cosine of angle θ shown in the unit circle if coordinate of point p are as shown.



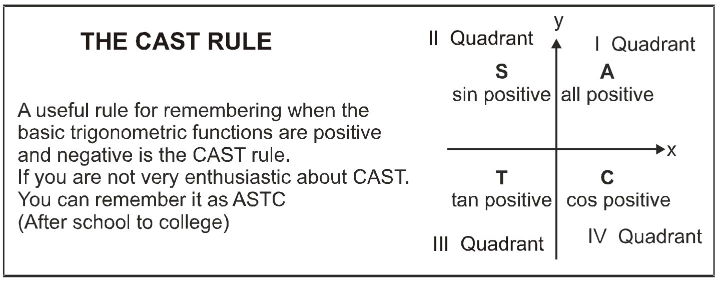
Solution : $\cos\theta$ = x-coordinate of P = $-\frac{1}{2}$; $\sin\theta$ = y-coordinate of P = $\frac{\sqrt{3}}{2}$.

2.4 RULES FOR FINDING TRIGONOMETRIC RATIO OF ANGLES GREATER THAN 90°

Step 1 → Identify the quadrant in which angle lies.

Step 2 →

(a) If angle = $(n\pi \pm \theta)$ where n is an integer. Then trigonometric function of $(n\pi \pm \theta)$ = same trigonometric function of θ and sign will be decided by CAST Rule.



(b) If angle = $\left[(2n+1)\frac{\pi}{2} \pm \theta\right]$ where n is an integer. Then trigonometric function of $\left[(2n+1)\frac{\pi}{2} \pm \theta\right]$

= complimentary trignometric function of θ and sign will be decided by CAST Rule.

Degree	0	30	37	45	53	60	90	120	135	180
Radians	0	$\pi/6$	$37\pi/180$	$\pi/4$	$53\pi/180$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$\sin\theta$	0	$1/2$	$3/5$	$1/\sqrt{2}$	$4/5$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	0
$\cos\theta$	1	$\sqrt{3}/2$	$4/5$	$1/\sqrt{2}$	$3/5$	$1/2$	0	$-1/2$	$-1/\sqrt{2}$	-1
$\tan\theta$	0	$1/\sqrt{3}$	$3/4$	1	$4/3$	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	0

Values of sin θ, cos θ and tan θ for some standard angles.

Example 10. Evaluate sin 120°

Solution : $\sin 120° = \sin(90° + 30°) = \cos 30° = \frac{\sqrt{3}}{2}$

Aliter $\sin 120° = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$

Example 11. Evaluate cos 135°

Solution : $\cos 135° = \cos (90° + 45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$

Example 12. Evaluate cos 210°

Solution : $\cos 210° = \cos (180° + 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$

Example 13. Evaluate tan 210°

Solution : $\tan 210° = \tan (180° + 30°) = \tan 30° = \frac{1}{\sqrt{3}}$

2.5 GENERAL TRIGONOMETRIC FORMULAS :

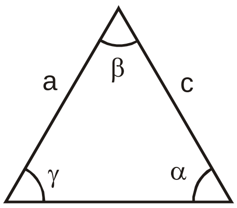
1.
$\cos^2\theta + \sin^2\theta = 1$
$1 + \tan^2\theta = \sec^2\theta.$
$1 + \cot^2\theta = \text{cosec}^2\theta.$

2.
$\cos(A + B) = \cos A \cos B - \sin A \sin B$
$\sin(A + B) = \sin A \cos B + \cos A \sin B$
$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

3. $\sin 2\theta = 2 \sin\theta \cos\theta$; $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

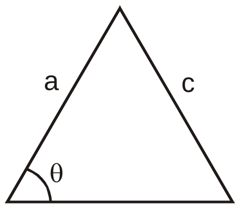
$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$; $\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

4. sine rule for triangles



$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$

5. cosine rule for triangles



$c^2 = a^2 + b^2 - 2ab\cos\theta$

3. DIFFERENTIATION

3.1 FINITE DIFFERENCE

The finite difference between two values of a physical quantity is represented by Δ notation.

For example :

Difference in two values of y is written as Δy as given in the table below.

y_2	100	100	100
y_1	50	99	99.5
$\Delta y = y_2 - y_1$	50	1	0.5

INFINITELY SMALL DIFFERENCE :

The infinitely small difference means very-very small difference. And this difference is represented by ‘d’ notation instead of ‘Δ’.

For example infinitely small difference in the values of y is written as ‘dy’

if $y_2 = 100$ and $y_1 = 99.99999999........$

then dy = 0.000000..................00001

3.2 DEFINITION OF DIFFERENTIATION

Another name for differentiation is derivative. Suppose y is a function of x or y = f(x).

Differentiation of y with respect to x is denoted by symbol $f'(x)$ where $f'(x) = \frac{dy}{dx}$

dx is very small change in x and dy is corresponding very small change in y.

NOTATION : There are many ways to denote the derivative of a function y = f(x). Besides $f'(x)$, the most common notations are these :

y'	"y prime" or "y dash"	Nice and brief but does not name the independent variable.
$\frac{dy}{dx}$	"dy by dx"	Names the variables and uses d for derivative.
$\frac{df}{dx}$	"df by dx"	Emphasizes the function's name.
$\frac{d}{dx}f(x)$	"d by dx of f"	Emphasizes the idea that differentiation is an operation performed on f.
D_xf	"dx of f"	A common operator notation.
$\dot{y}$	"y dot"	One of Newton's notations, now common for time derivatives i.e. $\frac{dy}{dt}$.
$f'(x)$	f dash x	Most common notation, it names the independent variable and Emphasize the function's name.

3.3 SLOPE OF A LINE

It is the tan of angle made by a line with the positive direction of x-axis, measured in anticlockwise direction.

Slope = $\tan\theta$ (In 1st quadrant $\tan\theta$ is +ve & 2nd quadrant $\tan\theta$ is –ve)

In Figure - 1 slope is positive

$\theta < 90°$ (1st quadrant)

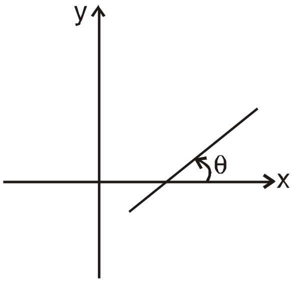


Figure - 1

In Figure - 2 slope is negative

$\theta > 90°$ (2nd quadrant)

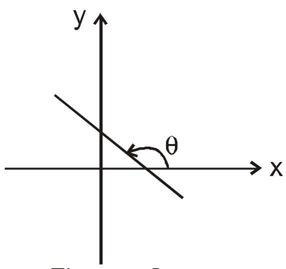


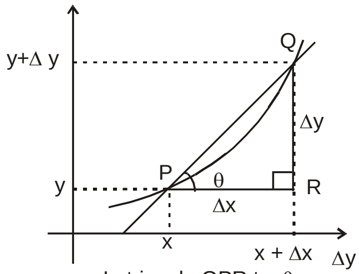
Figure - 2

3.4 AVERAGE RATES OF CHANGE :

Given an arbitrary function y = f(x) we calculate the average rate of change of y with respect to x over the interval (x, x + Δx) by dividing the change in value of y, i.e. Δy = f(x + Δx) – f(x), by length of interval Δx over which the change occurred.

The average rate of change of y with respect to x over the interval [x, x + Δx]

$$= \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$



In triangle QPR $\tan\theta = \frac{\Delta y}{\Delta x}$

Geometrically, $\frac{\Delta y}{\Delta x} = \frac{QR}{PR}$ = tan θ = Slope of the line PQ

therefore we can say that average rate of change of y with respect to x is equal to slope of the line joining P & Q.

3.5 THE DERIVATIVE OF A FUNCTION

We know that, average rate of change of y w.r.t. x is $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$. If the limit of this ratio exists as Δx → 0, then it is called the derivative of given function f(x) and is denoted as

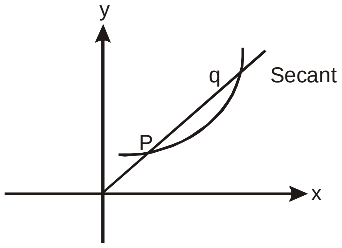
$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

3.6 GEOMETRICAL MEANING OF DIFFERENTIATION

The geometrical meaning of differentiation is very much useful in the analysis of graphs in physics. To understand the geometrical meaning of derivatives we should have knowledge of secant and tangent to a curve.

Secant and tangent to a curve

Secant : A secant to a curve is a straight line, which intersects the curve at any two points.



Tangent : A tangent is a straight line, which touches the curve at a particular point. Tangent is a limiting case of secant which intersects the curve at two overlapping points.

In the figure-1 shown, if value of Δx is gradually reduced then the point Q will move nearer to the point P. If the process is continuously repeated (Figure - 2) value of Δx will be infinitely small and secant PQ to the given curve will become a tangent at point P.

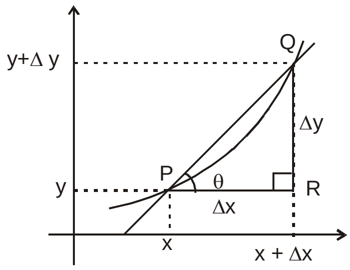


Figure - 1

Therefore $\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \frac{dy}{dx} = \tan\theta$

we can say that differentiation of y with respect to x, i.e., $\left(\frac{dy}{dx}\right)$ is equal to slope of the tangent at point P (x, y) or $\tan\theta = \frac{dy}{dx}$ (From fig. 1, the average rate of change of y from x to x + Δx is identical with the slope of secant PQ.)

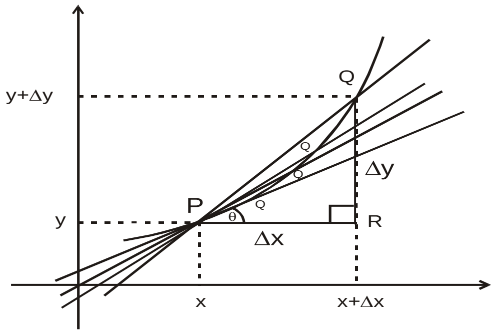


Figure - 2

3.7 RULES FOR DIFFERENTIATION

RULE NO. 1 : DERIVATIVE OF A CONSTANT

The first rule of differentiation is that the derivative of every constant function is zero.

If c is constant, then $\frac{d}{dx}c = 0$.

Example 14. $\frac{d}{dx}(8) = 0$, $\frac{d}{dx}\left(-\frac{1}{2}\right) = 0$, $\frac{d}{dx}\left(\sqrt{3}\right) = 0$

RULE NO. 2 : POWER RULE

If n is a real number, then $\frac{d}{dx}x^n = nx^{n-1}$.

To apply the power Rule, we subtract 1 from the original exponent (n) and multiply the result by n.

Example 15.

f	x	x^2	x^3	x^4	....
f'	1	2x	$3x^2$	$4x^3$	....

Example 16. (i) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$

(ii) $\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$.

Example 17. (a) $\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

| |

Function defined for $x \ge 0$ derivative defined only for $x > 0$

(b) $\frac{d}{dx}(x^{1/5})$ $= \frac{1}{5}x^{-4/5}$

| |

Function defined for $x \ge 0$ derivative not defined at $x = 0$

RULE NO. 3 : THE CONSTANT MULTIPLE RULE

If u is a differentiable function of x, and c is a constant, then $\frac{d}{dx}(cu) = c\frac{du}{dx}$

In particular, if n is a positive integer, then $\frac{d}{dx}(cx^n) = cn\,x^{n-1}$

Example 18. The derivative formula $\frac{d}{dx}(3x^2) = 3(2x) = 6x$ says that if we rescale the graph of $y = x^2$ by multiplying each y–coordinate by 3, then we multiply the slope at each point by 3.

Example 19. A useful special case. The derivative of the negative of a differentiable function is the negative of the function's derivative. Rule 3 with c = –1 gives.

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1\cdot u) = -1\cdot\frac{d}{dx}(u) = -\frac{d}{dx}(u)$$

RULE NO. 4 : THE SUM RULE

The derivative of the sum of two differentiable functions is the sum of their derivatives.

If u and v are differentiable functions of x, then their sum u + v is differentiable at every point where u and v are both differentiable functions is their derivatives.

$$\frac{d}{dx}(u-v) = \frac{d}{dx}[u+(-1)v] = \frac{du}{dx}+(-1)\frac{dv}{dx} = \frac{du}{dx}-\frac{dv}{dx}$$

The Sum Rule also extends to sums of more than two functions, as long as there are only finitely many functions in the sum. If $u_1, u_2, ...u_n$ are differentiable at x, then so is $u_1 + u_2 + ...+ u_n$, and $\frac{d}{dx}(u_1+u_2+.....+u_n) = \frac{du_1}{dx}+\frac{du_2}{dx}+.......+\frac{du_n}{dx}$.

Example 20. **(a)** $y = x^4 + 12x$ **(b)** $y = x^3 + \frac{4}{3}x^2 - 5x + 1$

(a)

$$\frac{dy}{dx} = \frac{d}{dx}(x^4)+\frac{d}{dx}(12x)$$

$$= 4x^3 + 12$$

(b)

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left(\frac{4}{3}x^2\right)-\frac{d}{dx}(5x)+\frac{d}{dx}(1)$$

$$= 3x^2 + \frac{4}{3}.2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5.$$

Notice that we can differentiate any polynomial term by term, the way we differentiated the polynomials in above example.

RULE NO. 5 : THE PRODUCT RULE

If u and v are differentiable at x, then so is their product uv, and $\frac{d}{dx}(uv) = u\frac{dv}{dx}+v\frac{du}{dx}$.

The derivative of the product uv is u times the derivative of v plus v times the derivative of u.
In prime notation $(uv)' = uv' + vu'$.
While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is not the product of their derivatives. For instance,

$$\frac{d}{dx}(x.x) = \frac{d}{dx}(x^2) = 2x, \quad \text{while } \frac{d}{dx}(x).\frac{d}{dx}(x) = 1.1 = 1.$$

Example 21. Find the derivatives of $y = \frac{4}{3}(x^2 + 1)(x^3 + 3)$.

Solution : From the product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$\frac{d}{dx}[(x^2+1)(x^3+3)] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$

$= 3x^4 + 3x^2 + 2x^4 + 6x = 5x^4 + 3x^2 + 6x.$

Example can be done as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial. We now check : $y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$

$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$

This is in agreement with our first calculation. There are times, however, when the product Rule must be used. In the following examples. We have only numerical values to work with.

Example 22. Let y = uv be the product of the functions u and v. Find $y'(2)$ if $u(2) = 3$, $u'(2) = -4$, $v(2) = 1$, and $v'(2) = 2$.

Solution : From the Product Rule, in the form $y' = (uv)' = uv' + vu'$,

we have $y'(2) = u(2)\,v'(2) + v(2)\,u'(2) = (3)(2) + (1)(-4) = 6 - 4 = 2.$

RULE NO. 6 : THE QUOTIENT RULE

If u and v are differentiable at x, and $v(x) \neq 0$, then the quotient u/v is differentiable at x,

and $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives.

Example 23. Find the derivative of $y = \frac{t^2-1}{t^2+1}$

Solution : We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$\frac{dy}{dt} = \frac{(t^2+1).2t - (t^2-1).2t}{(t^2+1)^2} \Rightarrow \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2} = \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}.$

RULE NO. 7 : DERIVATIVE OF SINE FUNCTION

$\frac{d}{dx}(\sin x) = \cos x$

Example 24. (a) $y = x^2 - \sin x$: $\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x)$ Difference Rule

$= 2x - \cos x$

(b) $y = x^2 \sin x$: $\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + 2x \sin x$ Product Rule

$= x^2 \cos x + 2x \sin x$

(c) $y = \frac{\sin x}{x}$: $\frac{dy}{dx} = \frac{x.\frac{d}{dx}(\sin x) - \sin x.1}{x^2}$ Quotient Rule

$= \frac{x\cos x - \sin x}{x^2}.$

RULE NO. 8 : DERIVATIVE OF COSINE FUNCTION

$$\frac{d}{dx}(\cos x) = -\sin x$$

Example 25. (a) $y = 5x + \cos x$

$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x)$ Sum Rule

$= 5 - \sin x$

(b) $y = \sin x \cos x$

$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x(\sin x) \frac{d}{dx}$ Product Rule

$= \sin x\,(-\sin x) + \cos x\,(\cos x) = \cos^2 x - \sin^2 x$

RULE NO. 9 : DERIVATIVES OF OTHER TRIGONOMETRIC FUNCTIONS

Because sin x and cos x are differentiable functions of x, the related functions

$\tan x = \frac{\sin x}{\cos x}$; $\sec x = \frac{1}{\cos x}$

$\cot x = \frac{\cos x}{\sin x}$; $\operatorname{cosec} x = \frac{1}{\sin x}$

are differentiable at every value of x at which they are defined. There derivatives. Calculated from the Quotient Rule, are given by the following formulas.

$\frac{d}{dx}(\tan x) = \sec^2 x$; $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$; $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Example 26. Find dy / dx if y = tan x.

Solution : $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$

$= \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

Example 27. (a) $\frac{d}{dx}(3x + \cot x) = 3 + \frac{d}{dx}(\cot x) = 3 - \operatorname{cosec}^2 x$

(b) $\frac{d}{dx}\left(\frac{2}{\sin x}\right) = \frac{d}{dx}(2\operatorname{cosec} x) = 2\frac{d}{dx}(\operatorname{cosec} x) = 2(-\operatorname{cosec} x \cot x) = -2 \operatorname{cosec} x \cot x$

RULE NO. 10 : DERIVATIVE OF LOGARITHM AND EXPONENTIAL FUNCTIONS

$$\frac{d}{dx}(\log_e x) = \frac{1}{x} \Rightarrow \frac{d}{dx}(e^x) = e^x$$

Example 28. $y = e^x . \log_e(x)$

$\frac{dy}{dx} = \frac{d}{dx}(e^x) . \log(x) +. \frac{d}{dx}[\log_e(x)]\, e^x \Rightarrow \frac{dy}{dx} = e^x . \log_e(x) + \frac{e^x}{x}$

RULE NO. 11 : CHAIN RULE OR "OUTSIDE INSIDE" RULE

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

It sometimes helps to think about the Chain Rule the following way. If y = f(g(x)),

$$\frac{dy}{dx} = f'[g(x)].g'(x) .$$

In words : To find dy/dx , differentiate the "outside" function f and leave the "inside" g(x) alone ; then multiply by the derivative of the inside.

We now know how to differentiate sin x and $x^2 - 4$, but how do we differentiate a composite like sin $(x^2 - 4)$? The answer is, with the Chain Rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The Chain Rule is probably the most widely used differentiation rule in mathematics. This section describes the rule and how to use it. We begin with examples.

Example 29. The function y = 6x – 10 = 2(3x – 5) is the composite of the functions y = 2u and u = 3x – 5. How are the derivatives of these three functions related ?

Solution : We have $\frac{dy}{dx} = 6, \ \frac{dy}{du} = 2, \ \frac{du}{dx} = 3$.

Since 6 = 2.3, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Is it an accident that $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$?

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. For y = f(u) and u = g(x) , if y changes twice as fast as u and u changes three times as fast as x, then we expect y to change six times as fast as x.

Example 30. We sometimes have to use the Chain Rule two or more times to find a derivative. Here is an example. Find the derivative of g(t) = tan (5 – sin 2t)

Solution : $g'(t) = \frac{d}{dt}$ (tan(5– sin 2t)

$= \sec^2 (5 - \sin 2t). \frac{d}{dt} (5 - \sin 2t)$

Derivative of
tan u with
u = 5 – sin 2t

Derivative of
5 – sin u
with u = 2t

$= \sec^2 (5 - \sin 2t) . (0 - (\cos 2t). \frac{d}{dt} (2t)$

$= \sec^2 (5 - \sin 2t) . (- \cos 2t) . 2$

$= -2(\cos 2t) \sec^2 (5 - \sin 2t)$

Example 31. (a) $\frac{d}{dx} (1 - x^2)^{1/4} = \frac{1}{4} (1 - x^2)^{-3/4} (-2x)$ u = $1 - x^2$ and n = 1/4

Function defined
on [–1 , 1]

$= \dfrac{-x}{2(1-x^2)^{3/4}}$

derivative defined only on (–1 ,1)

(b) $\dfrac{d}{dx}$ sin 2x = cos 2x $\dfrac{d}{dx}$ (2x) = cos 2x . 2 = 2 cos 2x

(c) $\dfrac{d}{dt}$ (A sin (ωt + φ)

= A cos (ωt + φ) $\dfrac{d}{dt}$ (ωt + φ) = A cos (ωt + φ). ω. = A ω cos (ωt + φ)

RULE NO. 12 : POWER CHAIN RULE

If u(x) is a differentiable function and where n is a Real number, then u^n is differentiable and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}, \ \forall n \in R$$

Example 32. (a) $\dfrac{d}{dx}\sin^5 x = 5\sin^4 x \dfrac{d}{dx}(\sin x) = 5\sin^4 x \cos x$

(b) $\dfrac{d}{dx}(2x+1)^{-3} = -3(2x+1)^{-4}\dfrac{d}{dx}(2x+1) = -3(2x+1)^{-4}(2) = -6(2x+1)^{-4}$

(c) $\dfrac{d}{dx}\left(\dfrac{1}{3x-2}\right) = \dfrac{d}{dx}(3x-2)^{-1} = -1(3x-2)^{-2}(3x-2)\dfrac{d}{dx} = -1(3x-2)^{-2}(3) = -\dfrac{3}{(3x-2)^2}$

In part (c) we could also have found the derivative with the Quotient Rule.

Example 33. Find the value of $\dfrac{d}{dx}(Ax+B)^n$

Solution : Here u = Ax + B, = $\dfrac{du}{dx}$ A

$\therefore \dfrac{d}{dx}(Ax+B)^n = n(Ax+B)^{n-1} . A$

RULE NO. 13 : RADIAN VS. DEGREES

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°) .$$

3.8 DOUBLE DIFFERENTIATION

If f is differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the second derivative of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of y = f (x) as

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$. Another notation is $f''(x) = D_2 f(x) = D^2 f(x)$

INTERPRETATION OF DOUBLE DERIVATIVE

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is acceleration, which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we known that its first derivative represents the velocity $v(t)$ of the object as a function of time : $v(t) = s'(t) = \frac{ds}{dt}$

The instantaneous rate of change of velocity with respect to time is called the acceleration $a(t)$ of the object. Thus, the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function : $a(t) = v'(t) = s''(t)$ or in Leibniz notation, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Example 34. If $f(x) = x \cos x$, find $f''(x)$.

Solution : Using the Product Rule, we have $f'(x) = x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x)$

$= -x \sin x + \cos x$

To find $f''(x)$ we differentiate $f'(x)$: $f''(x) = \frac{d}{dx}(-x \sin x + \cos x)$

$= -x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(-x) + \frac{d}{dx}(\cos x) = -x \cos x - \sin x - \sin x = -x \cos x - 2 \sin x$

Example 35. The position of a particle is given by the equation $s = f(t) = t^3 - 6t^2 + 9t$ where t is measured in seconds and s in meters. Find the acceleration at time t. What is the acceleration after 4s ?

Solution : The velocity function is the derivative of the position function : $s = f(t) = t^3 - 6t^2 + 9t$

$\Rightarrow \quad v(t) = \frac{ds}{dt} = 3t^2 - 12t + 9$

The acceleration is the derivative of the velocity function : $a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t - 12$

$\Rightarrow \quad a(4) = 6(4) - 12 = 12 \text{ m/s}^2$

3.9 APPLICATION OF DERIVATIVES

3.9.1 DIFFERENTIATION AS A RATE OF CHANGE

$\frac{dy}{dx}$ is rate of change of 'y' with respect to 'x' :

For examples :

(i) $v = \frac{dx}{dt}$ this means velocity 'v' is rate of change of displacement 'x' with respect to time 't'

(ii) $a = \frac{dv}{dt}$ this means acceleration 'a' is rate of change of velocity 'v' with respect to time 't' .

(iii) $F = \frac{dp}{dt}$ this means force 'F' is rate of change of momentum 'p' with respect to time 't' .

(iv) $\tau = \frac{dL}{dt}$ this means torque 'τ' is rate of change of angular momentum 'L' with respect to time 't'

(v) Power = $\frac{dW}{dt}$ this means power 'P' is rate of change of work 'W' with respect to time 't'

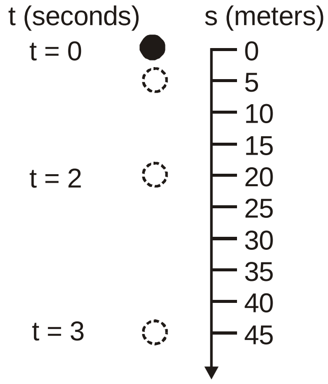
(vi) $I = \frac{dq}{dt}$ this means current 'I' is rate of flow of charge 'q' with respect to time 't'

Example 36. The area A of a circle is related to its diameter by the equation $A = \frac{\pi}{4}D^2$. How fast is the area changing with respect to the diameter when the diameter is 10 m ?

Solution : The (instantaneous) rate of change of the area with respect to the diameter is $\frac{dA}{dD} = \frac{\pi}{4}2D = \frac{\pi D}{2}$.

When D = 10 m, the area is changing at rate $(\pi/2)\ 10 = 5\pi\ m^2/m$. This means that a small change ΔD m in the diameter would result in a change of about $5\pi\Delta D m^2$ in the area of the circle.

Example 37. Experimental and theoretical investigations revealed that the distance a body released from rest falls in time t is proportional to the square of the amount of time it has fallen. We express this by saying that



A ball bearing falling from rest

$$s = \frac{1}{2}gt^2,$$

where s is distance and g is the acceleration due to Earth's gravity. This equation holds in a vacuum, where there is no air resistance, but it closely models the fall of dense, heavy objects in air. Figure shows the free fall of a heavy ball bearing released from rest at time t = 0 sec.

(a) How many meters does the ball fall in the first 2 sec?

(b) What is its velocity, speed, and acceleration then?

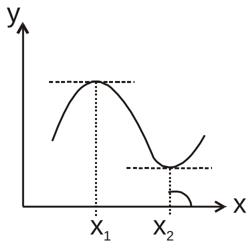
Solution : (a) The free–fall equation is $s = 4.9\ t^2$. During the first 2 sec. the ball falls $s(2) = 4.9(2)^2 = 19.6$ m,

(b) At any time t, velocity is derivative of displacement : $v(t) = s'(t) = \frac{d}{dt}(4.9t^2) = 9.8\ t$.

At t = 2, the velocity is v(2) = 19.6 m/sec in the downward (increasing s) direction. The speed at t = 2 is speed = |v(2)| = 19.6 m/sec. $a = \frac{d^2s}{dt^2} = 9.8\ m/s^2$

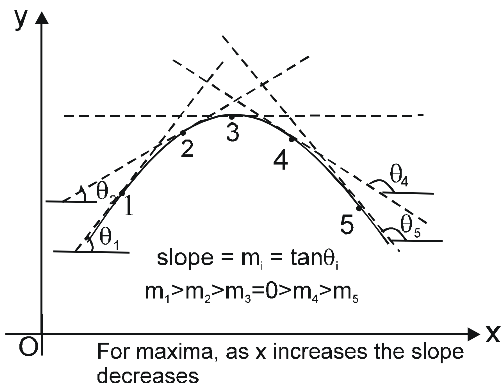
3.9.2 MAXIMA AND MINIMA

Suppose a quantity y depends on another quantity x in a manner shown in the figure. It becomes maximum at x_1 and minimum at x_2. At these points the tangent to the curve is parallel to the x-axis and hence its slope is tan $\theta = 0$. Thus, at a maximum or a minimum, slope = $\frac{dy}{dx} = 0$.



MAXIMA

Just before the maximum the slope is positive, at the maximum it is zero and just after the maximum it is negative. Thus, $\frac{dy}{dx}$ decreases at a maximum and hence the rate of change of $\frac{dy}{dx}$ is negative at a maximum i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$ at maximum.

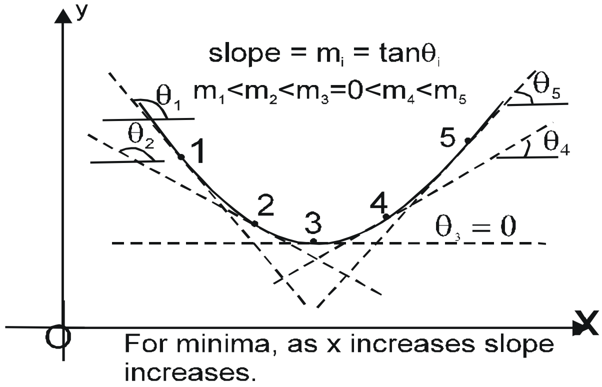


The quantity $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the rate of change of the slope. It is written as $\frac{d^2y}{dx^2}$.

Conditions for maxima are: (a) $\frac{dy}{dx} = 0$ (b) $\frac{d^2y}{dx^2} < 0$

MINIMA

Similarly, at a minimum the slope changes from negative to positive. Hence with the increases of x. the slope is increasing that means the rate of change of slope with respect to x is positive hence $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$.



Conditions for minima are : (a) $\frac{dy}{dx} = 0$ (b) $\frac{d^2y}{dx^2} > 0$

Quite often it is known from the physical situation whether the quantity is a maximum or a minimum. The test on $\frac{d^2y}{dx^2}$ may then be omitted.

Example 38. Particle's position as a function of time is given as $x = 5t^2 - 9t + 3$. Find out the maximum value of position co-ordinate? Also, plot the graph.

Solution : $x = 5t^2 - 9t + 3$

$\frac{dx}{dt} = 10t - 9 = 0$ $\therefore$ $t = 9/10 = 0.9$

Check, whether maxima or minima exists. $\frac{d^2x}{dt^2} = 10 > 0$

$\therefore$ there exists a minima at t = 0.9

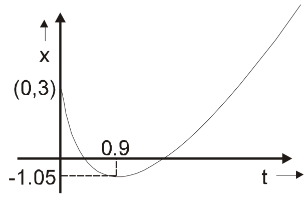
Now, Check for the limiting values.

When t = 0 ; x = 3

t = ∞ ; x = ∞

So, the maximum position co-ordinate does not exist.

Graph :



Putting t = 0.9 in the equation $x = 5(0.9)^2 - 9(0.9) + 3 = -1.05$

NOTE : If the coefficient of t^2 is positive, the curve will open upside.

SOLVED EXAMPLES ON APPLICATION OF DERIVATIVE

Example 39. Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents ? If so, where ?

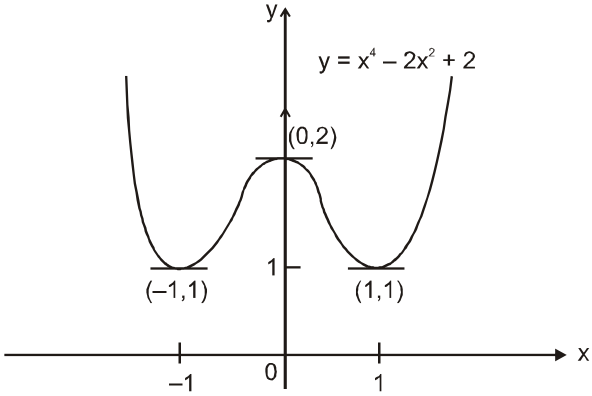
Solution : The horizontal tangents, if any, occur where the slope dy/dx is zero. To find these points. We

1. Calculate dy/dx : $\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x$

2. Solve the equation : $\frac{dy}{dx} = 0$ for x : $4x^3 - 4x = 0$

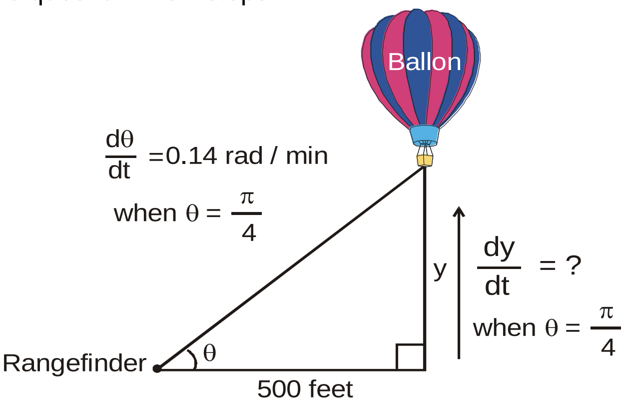
$4x(x^2 - 1) = 0$; $x = 0, 1, -1$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at x = 0, 1 and –1. The corresponding points on the curve are (0, 2) (1, 1) and (–1, 1). See figure.



Example 40. A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at the moment ?

Solution : We answer the question in six steps.



Step 1 : Draw a picture and name the variables and constants (Figure). The variables in the picture are

θ = the angle the range finder makes with the ground (radians).

y = the height of the balloon (feet).

We let t represent time and assume θ and y to be differentiable functions of t.

The one constant in the picture is the distance from the range finder to the lift-off point (500 ft.) There is no need to give it a special symbol s.

Step 2 : Write down the additional numerical information. $\frac{d\theta}{dt}$ = 0.14 rad/min when $\theta = \pi / 4$.

Step 3 : Write down what we are asked to find. We want dy/dt when $\theta = \pi/4$.

Step 4 : Write an equation that relates the variables y and θ. = tan θ, $\frac{y}{500}$ or $y = 500 \tan \theta$.

Step 5 : Differentiate with respect to t using the Chain Rule. The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 500 \sec^2 \theta \ \frac{d\theta}{dt}$$

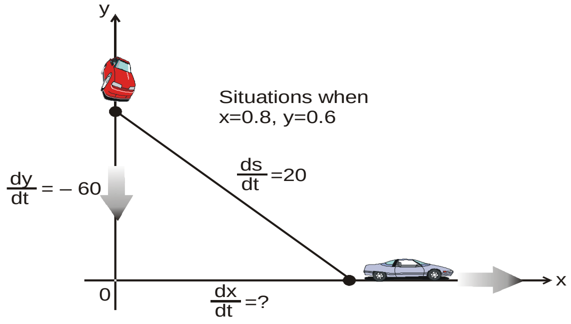
Step 6 : Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find dy/dt.

$$\frac{dy}{dt} = 500\left(\sqrt{2}\right)^2 (0.14) = (1000)\ (0.14) = 140 \quad \left(\sec\frac{\pi}{4} = \sqrt{2}\right)$$

At the moment in question, the balloon is rising at the rate of 140 ft./min.

Example 41. A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the Cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution : We carry out the steps of the basic strategy.



☞ **Step 1 :** Picture and variables. We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the northbound highway (Figure). We let t represent time and set x = position of car at time t.
y = position of cruiser at time t, s = distance between car and cruiser at time t.
We assume x, y and s to be differentiable functions of t.

$x = 0.8$ mi, $y = 0.6$ mi, $\frac{dy}{dt} = -60$ mph, $\frac{ds}{dt} = 20$ mph

(dy/dt is negative because y is decreasing.)

☞ **Step 2 :** To find : $\frac{dx}{dt}$

☞ **Step 3 :** How the variables are related : $s^2 = x^2 + y^2$ Pythagorean theorem

(The equation $s = \sqrt{x^2 + y^2}$ would also work.)

☞ **Step 4 :** Differentiate with respect to t. $2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ Chain Rule

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

☞ **Step 5 :** Evaluate, with x = 0.8, y = 0.6, dy/dt = – 60, ds/dt = 20, and solve for dx/dt.

$$20 = \underbrace{\frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}}_{1}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right) \Rightarrow 20 = 0.8\frac{dx}{dt} - 36 \Rightarrow \frac{dx}{dt} = \frac{20+36}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph.

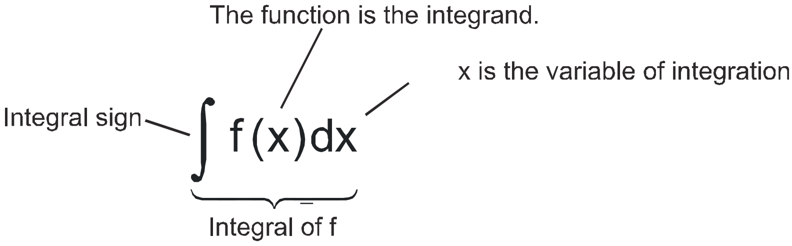
4. INTEGRATION

In mathematics, for each mathematical operation, there has been defined an inverse operation.
For example - Inverse operation of addition is subtruction, inverse operation of multiplication is division and inverse operation of square is square root. Similarly there is a inverse operation for differentiation which is known as integration

4.1 ANTIDERIVATIVES OR INDEFINITE INTEGRALS

Definitions :

A function F(x) is an antiderivative of a function f(x) if F´(x) = f(x) for all x in the domain of f.
The set of all antiderivatives of f is the indefinite integral of f with respect to x, denoted by



The symbol $\int$ is an integral sign. The function f is the integrand of the integral and x is the variable of integration.

For example f(x) = x^3 then f′(x) = $3x^2$

So the integral of $3x^2$ is x^3

Similarly if f(x) = x^3 + 4 then f′(x) = $3x^2$

So the integral of $3x^2$ is x^3 + 4

there for general integral of $3x^2$ is x^3 + c where c is a constant

One antiderivative F of a function f, the other antiderivatives of f differ from F by a constant. We indicate this in integral notation in the following way :

$\int f(x)dx = F(x) + C.$ .....(i)

The constant C is the constant of integration or arbitrary constant, Equation (1) is read, "The indefinite integral of f with respect to x is F(x) + C." When we find F(x) + C, we say that we have integrated f and evaluated the integral.

Example 42. Evaluate $\int$ 2x dx.

Solution :

an antiderivative of 2x

$\int 2x\,dx = x^2 + C$

the arbitrary constant

The formula x^2 + C generates all the antiderivatives of the function 2x. The function $x^2 + 1$, $x^2 - \pi$, and $x^2 + \sqrt{2}$ are all antiderivatives of the function 2x, as you can check by differentiation. Many of the indefinite integrals needed in scientific work are found by reversing derivative formulas.

4.2 INTEGRAL FORMULAS

Indefinite Integral	**Reversed derivative formula**
1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$, n rational	$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$
$\int dx = \int 1dx = x + C$ (special case)	$\frac{d}{dx}(x) = 1$
2. $\int \sin(Ax+B)dx = \frac{-\cos(Ax+B)}{A} + C$	$\frac{d}{dx}\left(-\frac{\cos kx}{k}\right) = \sin kx$
3. $\int \cos kx dx = \frac{\sin kx}{k} + C$	$\frac{d}{dx}\left(\frac{\sin kx}{k}\right) = \cos kx$
4. $\int \sec^2 x dx = \tan x + C$	$\frac{d}{dx}\tan x = \sec^2 x$
5. $\int \operatorname{cosec}^2 x dx = -\cot x + C$	$\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x$
6. $\int \sec x \tan x\, dx = \sec x + C$	$\frac{d}{dx}\sec x = \sec x \tan x$

7. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + C$ $\qquad \frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$

8. $\int \frac{1}{(ax+b)} = \frac{1}{a}\ell n(ax+b) + C$

Example 43. Examples based on above formulas :

(a) $\int x^5 dx = \frac{x^6}{6} + C$ — Formula 1 with n = 5

(b) $\int \frac{1}{\sqrt{x}} dx = \int x^{-1/2} dx = 2x^{1/2} + C = 2\sqrt{x} + C$ — Formula 1 with n = –1/2

(c) $\int \sin 2x\, dx = \frac{-\cos 2x}{2} + C$ — Formula 2 with k = 2

(d) $\int \cos\frac{x}{2} dx = \int \cos\frac{1}{2}x dx = \frac{\sin(1/2)x}{1/2} + C = 2\sin\frac{x}{2} + C$ — Formula 3 with k = 1/2

Example 44. Right : $\int$ x cosx dx = x sin x + cos x + C

Reason : The derivative of the right-hand side is the integrand:

Check : $\frac{d}{dx}$ (x sin x + cos x + C) = x cos x + sin x – sin x + 0 = x cos x.

Wrong : x cosx dx = x sin x + C

Reason : $\int$ The derivative of the right-hand side is not the integrand:

Check : $\frac{d}{dx}$ (x sin x + C) = x cos x + sin x + 0 x cos x.

4.3 RULES FOR INTEGRATION

RULE NO. 1 : CONSTANT MULTIPLE RULE

A function is an antiderivative of a constant multiple kf of a function f if and only if it is k times an antiderivative of f.

$\int kf(x)dx = k\int f(x)dx$; where k is a constant

Example 45. Rewriting the constant of integration $\int$ 5 sec x tan x d x = 5 $\int$ sec x tan x d x

Rule 1

= 5 (sec x + C) — Formula 6

= 5 sec x + 5C — First form

= 5 sec x + C´ — Shorter form, where C’ is 5C

= 5 sec x + C — Usual form–no prime. Since 5 times an arbitrary constant is an arbitrary constant, we rename C’.

What about all the different forms in example? Each one gives all the antiderivatives of f (x) = 5 sec x tan x. so each answer is correct. But the least complicated of the three, and the usual choice, is

$\int$ 5 sec x tan x d x = 5 sec x + C.

Just as the Sum and Difference Rule for differentiation enables us to differentiate expressions term by term, the Sum and Difference Rule for integration enables us to integrate expressions term by term. When we do so, we combine the individual constants of integration into a single arbitrary constant at the end.

RULE NO. 2 : SUM AND DIFFERENCE RULE

A function is an antiderivative of a sum or difference $f \pm g$ if and only if it is the sum or difference of an antiderivative of f an antiderivative of g.

$$\int [f(x) \pm g(x)]\,dx = \int f(x)dx \pm \int g(x)dx$$

Example 46. Term–by–term integration. Evaluate : $\int (x^2 - 2x + 5)\,dx$.

Solution : If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$(x^2 - 2x + 5)dx = \underbrace{\frac{x^3}{3} - x^2 + 5x}_{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}$$

If we do not recognize the antiderivative right away, we can generate it term by term with the sum and difference Rule:

$$\int (x^2 - 2x + 5)dx = \int x^2dx - \int 2xdx + \int 5dx = \frac{x^3}{3} + C_1 - x^2 + C_2 + 5x + C_3.$$

This formula is more complicated than it needs to be. If we combine C_1, C_2 and C_3 into a single constant $C = C_1 + C_2 + C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and still gives all the antiderivatives there are. For this reason we recommend that you go right to the final form even if you elect to integrate term by term. Write

$$\int (x^2 - 2x + 5)dx = \int x^2dx - \int 2xdx + \int 5dx = \frac{x^3}{3} - x^2 + 5x + C.$$

Find the simplest antiderivative you can for each part add the constant at the end.

Example 47. We can sometimes use trigonometric identities to transform integrals we do not know how to evaluate into integrals we do know how to evaluate. The integral formulas for $\sin^2 x$ and $\cos^2 x$ arise frequently in applications.

(a) $\int \sin^2 x dx = \int \frac{1 - \cos 2x}{2} dx \quad \sin^2 x = \frac{1 - \cos 2x}{2}$

$$= \frac{1}{2}\int (1 - \cos 2x)dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$

$= \frac{x}{2} - \left(\frac{1}{2}\right)\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$

(b) $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}dx$ $\cos^2 x = \frac{1+\cos 2x}{2}$

$= \frac{x}{2} + \frac{\sin 2x}{4} + C$ As in part (a), but with a sign change

Example 48. Find a body velocity from its acceleration and initial velocity. The acceleration of gravity near the surface of the earth is 9.8 m/sec^2. This means that the velocity v of a body falling freely in a vacuum changes at the rate of $\frac{dv}{dt}$ = 9.8 m/sec^2. If the body is dropped from rest, what will its velocity be t seconds after it is released?

Solution : In mathematical terms, we want to solve the initial value problem that consists of

The differential condition : $\frac{dv}{dt} = 9.8$

The initial condition : v = 0 when t = 0 (abbreviated as v (0) = 0)

We first solve the differential equation by integrating both sides with respect to t:

$\frac{dv}{dt} = 9.8$ The differential equation

$\int \frac{dv}{dt}dt = \int 9.8dt$ Integrate with respect to t.

$v + C_1 = 9.8t + C_2$ Integrals evaluated

$v = 9.8t + C.$ Constants combined as one

This last equation tells us that the body's velocity t seconds into the fall is 9.8t + C m/sec.

For value of C : What value? We find out from the initial condition :

$v = 9.8t + C$

$0 = 9.8(0) + C$ v(0) = 0

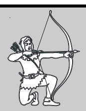
$C = 0.$

Conclusion : The body's velocity t seconds into the fall is

$v = 9.8t + 0 = 9.8t$ m/sec.

The indefinite integral F(x) + C of the function f(x) gives the general solution y = F(x) + C of the differential equation dy/dx = f(x). The general solution gives all the solutions of the equation (there are infinitely many, one for each value of C). We solve the differential equation by finding its general Solution : We then solve the initial value problem by finding the particular solution that satisfies the initial condition $y(x_o) = y_o$ (y has the value y_o when $x = x_o$).

RULE NO. 3 : RULE OF SUBSTITUTION



$\int f(g(x)) \cdot g'(x)\,dx = \int f(u)du$ 1. Substitute u = g(x), du = g'(x) dx.

$= F(u) + C$ 2. Evaluate by finding an antiderivative F (u) of f(u). (Any one will do.)

$= F(g(x)) + C$ 3. Replace u by g(x).

Example 49. Evaluate $\int (x+2)^5 dx$.

We can put the integral in the form $\int u^n du$

by substituting $u = x + 2$, $du = d(x + 2) = \frac{d}{dx}(x + 2).\ dx = 1.dx = dx$.

Then $\int (x+2)^5 dx = \int u^5 du$ u = x + 2, du = dx

$= \frac{u^6}{6} + C$ Integrate, using rule no. 3 with n = 5.

$= \frac{(x+2)^6}{6} + C$. Replace u by x + 2.

Example 50. Evaluate $\int \sqrt{1+y^2} \bullet 2y\ dy = \int u^{1/2} du$. Let $u = 1 + y^2$, $du = 2y\ dy$.

$= \frac{u^{(1/2)+1}}{(1/2)+1}$ Integrate, using rule no. 3 with n = 1/2.

$= \frac{2}{3}u^{3/2} + C$ Simpler form

$= \frac{2}{3}(1+y^2)^{3/2} + C$ Replace u by $1 + y^2$.

Example 51. Evaluate $\int \cos\ (7\theta + 5)\ d\theta = \int \cos u \cdot \frac{1}{7} du$ Let $u = 7\theta + 5$, $du = 7d\theta$, $(1/7)\ du = d\theta$.

$= \frac{1}{7}\int \cos u\, du$ With (1/7) out front, the integral is now in standard form.

$= \frac{1}{7}\sin u + C$ Integrate with respect to u.

$= \frac{1}{7}\sin\ (7\theta + 5) + C$ Replace u by $7\theta + 5$.

Example 52. Evaluate $\int x^2 \sin(x)^3 dx = \int \sin(x)^3 \bullet x^2\ dx$

$= \int \sin u \bullet \frac{1}{3} du$ Let $u = x^3$, $du = 3x^2\ dx$, $(1/3)\ du = x^2 dx$.

$= \frac{1}{3}\int \sin u\, du$

$= \frac{1}{3}(-\cos u) + C$ Integrate with respect to u.

$= -\frac{1}{3}\cos(x^3) + C$ Replace u by x^3.

Example 53. $\int \frac{1}{\cos^2 2\theta} d\theta = \int \sec^2\ 2\theta\ d\theta$ $\sec 2\theta = \frac{1}{\cos 2\theta}$

$= \int \sec^2 u \bullet \frac{1}{2} du$ Let $u = 2\theta$, $du = 2d\theta$, $d\theta = (1/2)du$.

$= \frac{1}{2}\int \sec^2 u\ du$

$= \frac{1}{2}\tan u + C$ Integrate, using eq. (4).

$= \frac{1}{2}\tan 2\theta + C$ Replace u by 2θ.

Check: $\frac{d}{d\theta}\left(\frac{1}{2}\tan 2\theta + C\right) = \frac{1}{2} \bullet \frac{d}{d\theta}(\tan 2\theta) + 0 = \frac{1}{2} \bullet \left(\sec^2 2\theta \cdot \frac{d}{d\theta}2\theta\right)$ Chain Rule

$= \frac{1}{2} \bullet \sec^2 2\theta \bullet 2 = \frac{1}{\cos^2 2\theta}$.

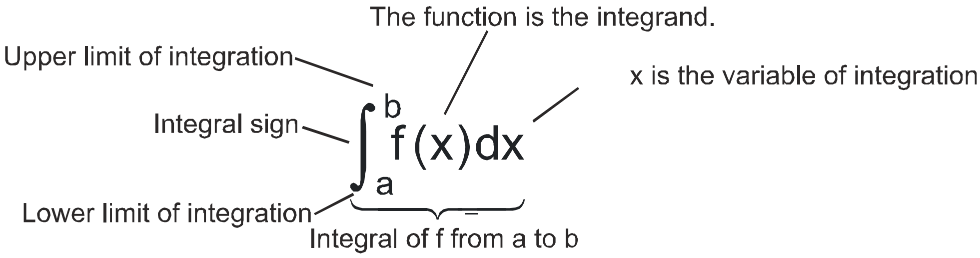
Example 54. $\int \sin^4 t \cos t\, dt = \int u^4 du$ Let u = sin t, du = cos t dt.

$= \frac{u^5}{5} + C$ Integrate with respect to u.

$= \frac{\sin^5 t}{5} + C$ Replace u.

The success of the substitution method depends on finding a substitution that will change an integral we cannot evaluate directly into one that we can. If the first substitution fails, we can try to simplify the integrand further with an additional substitution or two.

4.3 DEFINITE INTEGRATION OR INTEGRATION WITH LIMITS



$$\int_a^b f(x)dx = [g(x)]_a^b = g(b) - g(a)$$

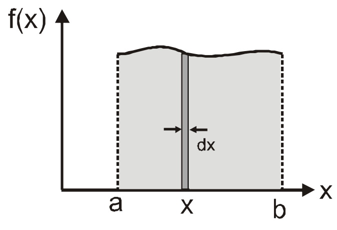
where g(x) is the antiderivative of f(x) i.e. $g'(x) = f(x)$

Example 55. $\int_{-1}^{4} 3dx = 3\int_{-1}^{4} dx = 3[x]_{-1}^{4} = 3[4 - (-1)] = (3)(5) = 15$

$\int_0^{\pi/2} \sin x\, dx = [-\cos x]_0^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos(0) = -0 + 1 = 1$

4.4 APPLICATION OF DEFINITE INTEGRAL : CALCULATION OF AREA OF A CURVE

From graph shown in figure if we divide whole area in infinitely small strips of dx width. We take a strip at x position of dx width. Small area of this strip dA = f(x) dx



So, the total area between the curve and x-axis

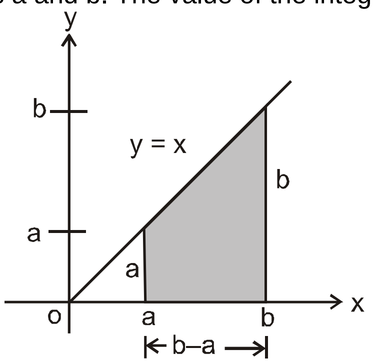
= sum of area of all strips = $\int_a^b f(x)dx$

Let $f(x) \geq 0$ be continuous on [a, b]. The area of the region between the graph of f and the x-axis is

A = $\int_a^b f(x)dx$

Example 56. Using an area to evaluate a definite integral $\int_a^b x dx \quad 0 < a < b$.

Solution : We sketch the region under the curve y = x, $a \leq x \leq b$ (figure) and see that it is a trapezoid with height (b – a) and bases a and b. The value of the integral is the area of this trapezoid :



The region in Example

$\int_a^b x\,dx = (b - a) \bullet \frac{a+b}{2} = \frac{b^2}{2} - \frac{a^2}{2}$.

Notice that $x^2/2$ is an antiderivative of x, further evidence of a connection between antiderivatives and summation.

5. VECTOR

In physics we deal with two type of physical quantity one is scalar and other is vector. Each scalar quantities has magnitude.

Magnitude of a physical quantity means product of numerical value and unit of that physical quantity.

For example mass = 4 kg

Magnitude of mass = 4 kg and unit of mass = kg

Example of scalar quantities : mass, speed, distance etc.

Scalar quantities can be added, subtracted and multiplied by simple laws of algebra.

5.1 DEFINITION OF VECTOR

If a physical quantity in addition to magnitude -

(a) has a specified direction.

(b) It should obey commutative law of additions $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

(c) obeys the law of parallelogram of addition, then and then only it is said to be a vector. If any of the above conditions is not satisfied the physical quantity cannot be a vector.

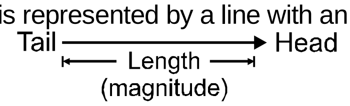
If a physical quantity is a vector it has a direction, but the converse may or may not be true, i.e. if a physical quantity has a direction, it may or may not a be vector. example : pressure, surface tension or current etc. have directions but are not vectors because they do not obey parallelogram law of addition.

The magnitude of a vector ($\vec{A}$) is the absolute value of a vector and is indicated by $|\vec{A}|$ or A.

Example of vector quantity : Displacement, velocity, acceleration, force etc.

Representation of vector :

Geometrically, the vector is represented by a line with an arrow indicating the direction of vector as

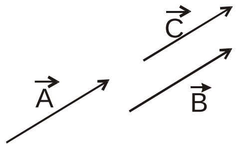


Mathematically, vector is represented by $\vec{A}$

Sometimes it is represented by bold letter **A**.

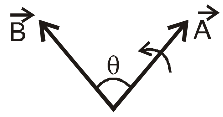
IMPORTANT POINTS :

☞ If a vector is displaced parallel to itself it does not change (see Figure)



$\vec{A} = \vec{B} = \vec{C}$
Transition of a vector parallel to itself

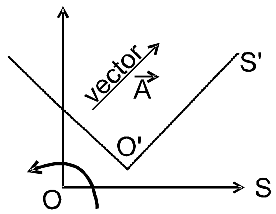
☞ If a vector is rotated through an angle other than multiple



$\vec{A} \neq \vec{B}$ Rotation of a vector

of 2π (or 360º) it changes (see Figure).

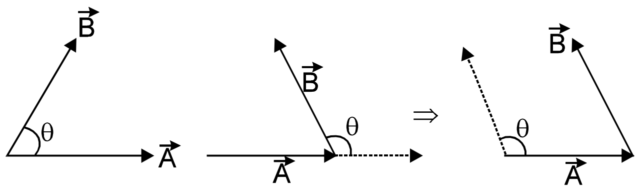
☞ If the frame of reference is translated or rotated the vector does not change (though its components may change). (see Figure).



moving frame

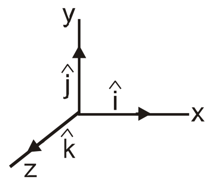
☞ Two vectors are called equal if their magnitudes and directions are same, and they represent values of same physical quantity.

☞ Angle between two vectors means smaller of the two angles between the vectors when they are placed tail to tail by displacing either of the vectors parallel to itself (i.e., $0 \leq \theta \leq \pi$).



5.2 UNIT VECTOR

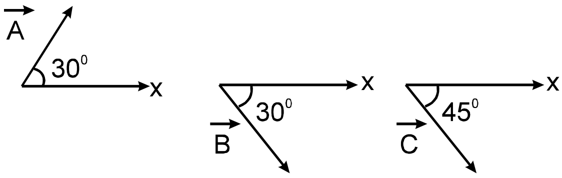
Unit vector is a vector which has a unit magnitude and points in a particular direction. Any vector ($\vec{A}$) can be written as the product of unit vector ($\hat{A}$) in that direction and magnitude of the given vector.



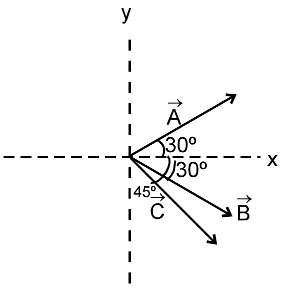
$\vec{A} = A\hat{A}$ or $\hat{A} = \dfrac{\vec{A}}{A}$

A unit vector has no dimensions and unit. Unit vectors along the positive x-, y- and z-axes of a rectangular coordinate system are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively such that $|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$.

Example 57. Three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ are shown in the figure. Find angle between (i) $\vec{A}$ and $\vec{B}$, (ii) $\vec{B}$ and $\vec{C}$, (iii) $\vec{A}$ and $\vec{C}$.



Solution : To find the angle between two vectors we connect the tails of the two vectors. We can shift $\vec{B}$ such that tails of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are connected as shown in figure.



Now we can easily observe that angle between $\vec{A}$ and $\vec{B}$ is 60º, $\vec{B}$ and $\vec{C}$ is 15º and between $\vec{A}$ and $\vec{C}$ is 75º.

Example 58. A unit vector along East is defined as $\hat{i}$. A force of 10^5 dynes acts west wards. Represent the force in terms of $\hat{i}$.

Solution : $\vec{F} = -10^5 \hat{i}$ dynes

5.3 MULTIPLICATION OF A VECTOR BY A SCALAR

Multiplying a vector $\vec{A}$ with a positive number λ gives a vector $\vec{B}\,(= \lambda \vec{A})$ whose magnitude is changed by the factor λ but the direction is the same as that of $\vec{A}$. Multiplying a vector $\vec{A}$ by a negative number λ gives a vector $\vec{B}$ whose direction is opposite to the direction of $\vec{A}$ and whose magnitude is $-\lambda$ times.

Example 59. A physical quantity (m = 3kg) is multiplied by a vector $\vec{a}$ such that $\vec{F} = m\vec{a}$. Find the magnitude and direction of $\vec{F}$ if

(i) $\vec{a} = 3 m/s^2$ East wards

(ii) $\vec{a} = -4 m/s^2$ North wards

Solution : (i) $\vec{F} = m\vec{a} = 3 \times 3\ ms^{-2}$ East wards = 9 N East wards

(ii) $\vec{F} = m\vec{a} = 3 \times (-4)$ N North wards = – 12N North wards = 12 N South wards

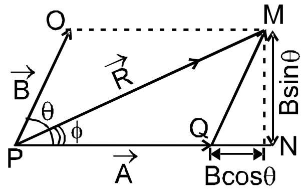
5.4 ADDITION OF VECTORS

Addition of vectors is done by parallelogram law or the triangle law :

(a) Parallelogram law of addition of vectors : If two vectors $\vec{A}$ and $\vec{B}$ are represented by two adjacent sides of a parallelogram both pointing outwards (and their tails coinciding) as shown. Then the diagonal drawn through the intersection of the two vectors represents the resultant (i.e., vector sum of $\vec{A}$ and $\vec{B}$).

$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

The direction of resultant vector from $\vec{R}$ is $\vec{A}$ given by

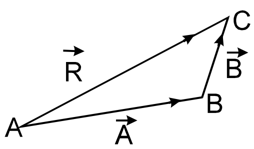


$$\tan\phi = \frac{MN}{PN} = \frac{MN}{PQ + QN} = \frac{B\sin\theta}{A + B\cos\theta}$$

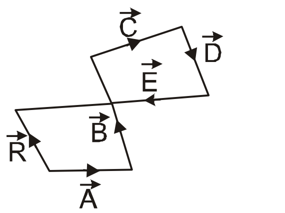
$$\phi = \tan^{-1}\left(\frac{B\sin\theta}{A + B\cos\theta}\right)$$

(b) Triangle law of addition of vectors: To add two vectors $\vec{A}$ and $\vec{B}$ shift any of the two vectors parallel to itself until the tail of $\vec{B}$ is at the head of $\vec{A}$. The sum $\vec{A} + \vec{B}$ is a vector $\vec{R}$ drawn from the tail of $\vec{A}$ to the head of $\vec{B}$, i.e., $\vec{A} + \vec{B} = \vec{R}$. As the figure formed is a triangle, this method is called 'triangle method' of addition of vectors.

If the 'triangle method' is extended to add any number of vectors in one operation as shown. Then the figure formed is a polygon and hence the name Polygon Law of addition of vectors is given to such type of addition.



$\vec{A} + \vec{B} = \vec{R}$



$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$

IMPORTANT POINTS :

☞ To a vector only a vector of same type can be added that represents the same physical quantity and the resultant is a vector of the same type.

☞ As $R = [A^2 + B^2 + 2AB\cos\theta]^{1/2}$ so R will be maximum when, $\cos\theta$ = max = 1, i.e., $\theta = 0^o$, i.e. vectors are like or parallel and $R_{max} = A + B$.

☞ The resultant will be minimum if, $\cos\theta$ = min = -1, i.e., $\theta = 180^o$, i.e., vectors are antiparallel and $R_{min} = A \sim B$.

☞ If the vectors A and B are orthogonal, i.e., $\theta = 90^o$, $R = \sqrt{A^2 + B^2}$

☞ As previously mentioned that the resultant of two vectors can have any value from (A ~ B) to (A + B) depending on the angle between them and the magnitude of resultant decreases as θ increases 0^o to 180^o

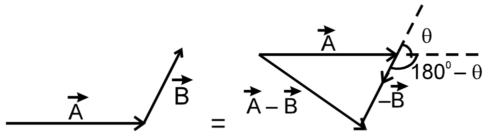
☞ Minimum number of unequal coplanar vectors whose sum can be zero is three.

☞ The resultant of three non-coplanar vectors can never be zero, or minimum number of non coplanar vectors whose sum can be zero is four.

☞ Subtraction of a vector from a vector is the addition of negative vector, i.e.,

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

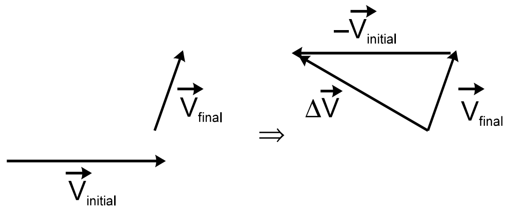
(a) From figure it is clear $\vec{A} - \vec{B}$ that is equal to addition of $\vec{A}$ with reverse of $\vec{B}$



$|\vec{A} - \vec{B}| = [(A)^2 + (B^2) + 2AB\cos(180^o - \theta)]^{1/2}$

$|\vec{A} - \vec{B}| = \sqrt{A^2 + B^2 - 2AB\cos\theta}$

(b) Change in a vector physical quantity means subtraction of initial vector from the final vector.



Example 60. Find the resultant of two forces each having magnitude F_0, and angle between them is θ.

Solution : $F^2_{Resultant} = F_0^2 + F_0^2 + 2F_0^2 \cos\theta$

$= 2F_0^2 (1 + \cos\theta) = 2F_0^2 (1 + 2\cos^2\frac{\theta}{2} - 1) = 2F_0^2 \times 2\cos^2\frac{\theta}{2}$

$F_{resultant} = 2F_0 \cos\frac{\theta}{2}$

Example 61. Two non zero vectors $\vec{A}$ and $\vec{B}$ are such that $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$. Find angle between $\vec{A}$ and $\vec{B}$?

Solution : $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}| \Rightarrow A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 - 2AB\cos\theta$

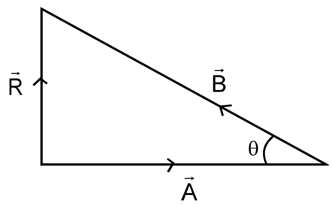
$\Rightarrow 4AB\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

Example 62. The resultant of two velocity vectors $\vec{A}$ and $\vec{B}$ is perpendicular to $\vec{A}$. Magnitude of Resultant is $\vec{R}$ equal to half magnitude of $\vec{B}$. Find the angle between $\vec{A}$ and $\vec{B}$?

Solution : Since $\vec{R}$ is perpendicular to $\vec{A}$. Figure shows the three vectors $\vec{A}$, $\vec{B}$ and $\vec{R}$ .angle between $\vec{A}$ and $\vec{B}$ is $\pi - \theta$

$\sin\theta = \frac{R}{B} = \frac{B}{2B} = \frac{1}{2} \Rightarrow \theta = 30^o$

$\Rightarrow$ angle between A and B is 150º.



Example 63. If the sum of two unit vectors is also a unit vector. Find the magnitude of their difference?

Solution : Let $\hat{A}$ and $\hat{B}$ are the given unit vectors and $\hat{R}$ is their resultant then $|\hat{R}| = |\hat{A} + \hat{B}|$

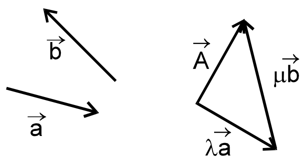
$1 = \sqrt{(\hat{A})^2 + (\hat{B})^2 + 2|\hat{A}||\hat{B}|\cos\theta}$

$1 = 1 + 1 + 2\cos\theta \Rightarrow \cos\theta = -\frac{1}{2}$

$|\vec{A} - \vec{B}| = \sqrt{(\hat{A})^2 + (\hat{B})^2 - 2|\hat{A}||\hat{B}|\cos\theta} = \sqrt{1 + 1 - 2\times1\times1(-\frac{1}{2})} = \sqrt{3}$

5.5 RESOLUTION OF VECTORS

If $\vec{a}$ and $\vec{b}$ be any two nonzero vectors in a plane with different directions and $\vec{A}$ be another vector in the same plane. $\vec{A}$ can be expressed as a sum of two vectors - one obtained by multiplying $\vec{a}$ by a real number and the other obtained by multiplying $\vec{b}$ by another real number.

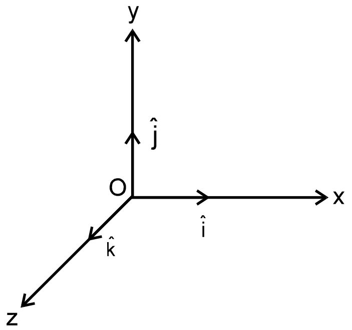


$\vec{A} = \lambda\vec{a} + \mu\vec{b}$ **(**where λ and μ are real numbers)

We say that $\vec{A}$ has been resolved into two component vectors namely $\lambda\vec{a}$ and $\mu\vec{b}$

$\lambda\vec{a}$ and $\mu\vec{b}$ along $\vec{a}$ and $\vec{b}$ respectively. Hence one can resolve a given vector into two component vectors along a set of two vectors – all the three lie in the same plane.

Resolution along rectangular component :
It is convenient to resolve a general vector along axes of a rectangular coordinate system using vectors of unit magnitude, which we call as unit vectors. $\hat{i}, \hat{j}, \hat{k}$ are unit vector along x, y and z-axis as shown in figure below:



Resolution in two Dimension

Consider a vector $\vec{A}$ that lies in xy plane as shown in figure,

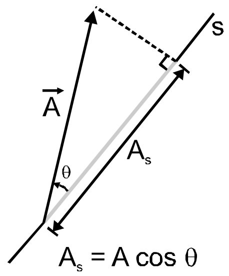
$\vec{A} = \vec{A}_1 + \vec{A}_2$

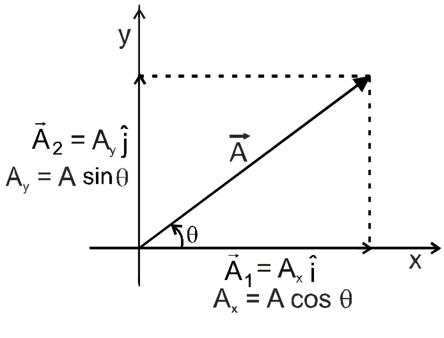
$\vec{A}_1 = A_x\hat{i}$, $\vec{A}_2 = A_y\hat{j}$

$\Rightarrow \vec{A} = A_x\hat{i} + A_y\hat{j}$

The quantities A_x and A_y are called x- and y-components of the vector.

A_x is itself not a vector but $A_x\hat{i}$ is a vector and so is $A_y\hat{j}$





$A_x = A\cos\theta$ and $A_y = A\sin\theta$

Its clear from above equation that a component of a vector can be positive, negative or zero depending on the value of θ. A vector $\vec{A}$ can be specified in a plane by two ways :

(a) its magnitude A and the direction θ it makes with the x-axis; or

(b) its components A_x and A_y. $A = \sqrt{A_x^2 + A_y^2}$, $\theta = \tan^{-1}\dfrac{A_y}{A_x}$

Note : If $A = A_x \Rightarrow A_y = 0$ and if $A = A_y \Rightarrow A_x = 0$ i.e., components of a vector perpendicular to itself is always zero.

The rectangular components of each vector and those of the sum $\vec{C} = \vec{A} + \vec{B}$ are shown in figure. We saw that

$\vec{C} = \vec{A} + \vec{B}$ is equivalent to both $C_x = A_x + B_x$ and $C_y = A_y + B_y$

Resolution in three dimensions. A vector $\vec{A}$ in components along x-, y- and z-axis can be written as :

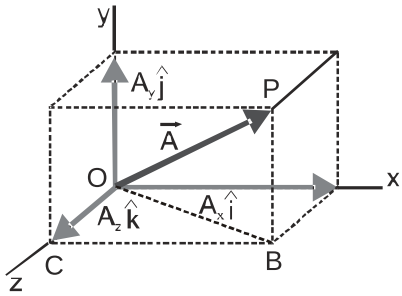
$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP} = \overrightarrow{OC} + \overrightarrow{CB} + \overrightarrow{BP}$

$\Rightarrow \vec{A} = \vec{A}_Z + \vec{A}_x + \vec{A}_y = \vec{A}_x + \vec{A}_y + \vec{A}_Z = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

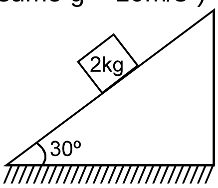
$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

$A_x = A\cos\alpha$, $A_y = A\cos\beta$, $A_z = A\cos\gamma$

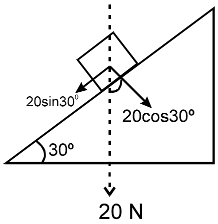
where $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are termed as **Direction Cosines** of a given vector $\vec{A}$.$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$



Example 64. A mass of 2 kg lies on an inclined plane as shown in figure. Resolve its weight along and perpendicular to the plane.(Assume g = 10m/s^2)



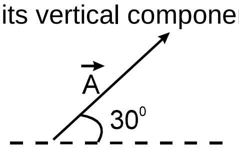
Solution : Component along the plane



$= 20 \sin 30 = 10\sqrt{3}$ N.

Component perpendicular to the plane = 20 cos 30 = 10N

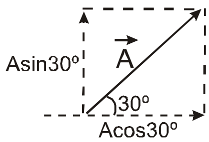
Example 65. A vector makes an angle of 30º with the horizontal. If horizontal component of the vector is 250. Find magnitude of vector and its vertical component ?



Solution : Let vector is $\vec{A}$

$A_x = A\cos30^0 = 250 = \frac{A\sqrt{3}}{2} \Rightarrow A = \frac{500}{\sqrt{3}}$

$A_y = A\sin30^0 = \frac{500}{\sqrt{3}} \times \frac{1}{2} = \frac{250}{\sqrt{3}}$



Example 66. $\vec{A} = \hat{i} + 2\hat{j} - 3\hat{k}$, when a vector $\vec{B}$ is added to $\vec{A}$, we get a unit vector along x-axis. Find the value of $\vec{B}$? Also find its magnitude

Solution : $\vec{A} + \vec{B} = \hat{i}$

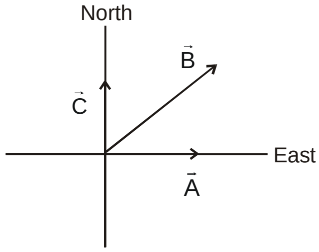
$\vec{B} = \hat{i} - \vec{A} = \hat{i} - (\hat{i} + 2\hat{j} - 3\hat{k}) = -2\hat{j} + 3\hat{k} \Rightarrow |\vec{B}| = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$

Example 67. In the above question find a unit vector along $\vec{B}$?

Solution : $\hat{B} = \frac{\vec{B}}{B} = \frac{-2\hat{j} + 3\hat{k}}{\sqrt{13}}$

Example 68. Vector $\vec{A}$, $\vec{B}$ and $\vec{C}$ have magnitude 5, $5\sqrt{2}$ and 5 respectively, direction of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are towards east, North-East and North respectively. If $\hat{i}$ and $\hat{j}$ are unit vectors along East and North respectively. Express the sum $\vec{A} + \vec{B} + \vec{C}$ in terms of $\hat{i}$, $\hat{j}$. Also find magnitude and direction of the resultant.

Solution : $\vec{A} = 5\hat{i}$ $\qquad \vec{C} = 5\hat{j}$



$\vec{B} = 5\sqrt{2}\cos 45\,\hat{i} + 5\sqrt{2}\sin 45\,\hat{j} = 5\hat{i} + 5\hat{j}$

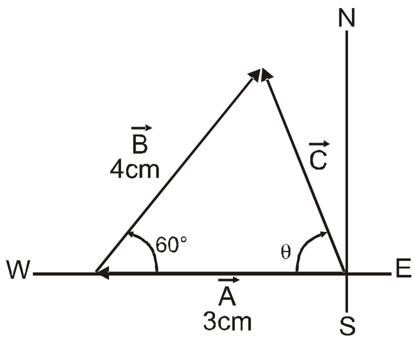
$\vec{A} + \vec{B} + \vec{C} = 5\hat{i} + 5\hat{i} + 5\hat{j} + 5\hat{j} = 10\hat{i} + 10\hat{j}$

$|\vec{A} + \vec{B} + \vec{C}| = \sqrt{(10)^2 + (10)^2} = 10\sqrt{2}$

$\tan\theta = \frac{10}{10} = 1 \quad \Rightarrow \quad \theta = 45^o$ from East

Example 69. You walk 3 Km west and then 4 Km headed 60° north of east. Find your resultant displacement
(a) graphically and (b) using vector components.

Solution : Picture the Problem : The triangle formed by the three vectors is not a right triangle, so the magnitudes of the vectors are not related by the Pythagoras theorem. We find the resultant graphically by drawing each of the displacements to scale and measuring the resultant displacement.



(a) If we draw the first displacement vector 3 cm long and the second one 4 cm long, we find the resultant vector to be about 3.5 cm long. Thus the magnitude of the resultant displacement is 3.5 Km. The angle θ made between the resultant displacement and the west direction can then be measured with a protractor. It is about 75°.

(b) 1. Let $\vec{A}$ be the first displacement and choose the x-axis to be in the easterly direction. Compute A_x and A_y, $A_x = -3$, $A_y = 0$

2. Similarly, compute the components of the second displacement $\vec{B}$, $B_x = 4\cos 60° = 2$, $B_y = 4\sin 60° = 2\sqrt{3}$

3. The components of the resultant displacement $\vec{C} = \vec{A} + \vec{B}$ are found by addition,

$\vec{C} = (-3+2)\hat{i} + (2\sqrt{3})\hat{j} = -\hat{i} + 2\sqrt{3}\,\hat{j}$

4. The Pythagorean theorem gives the magnitude of $\vec{C}$.

$C = \sqrt{1^2 + \left(2\sqrt{3}\right)^2} = \sqrt{13} = 3.6$

5. The ratio of C_y to C_x gives the tangent of the angle θ between $\vec{C}$ and the x axis.

$\tan\theta = \frac{2\sqrt{3}}{-1} \Rightarrow \theta = -74°$

Remark : Since the displacement (which is a vector) was asked for, the answer must include either the magnitude and direction, or both components. in (b) we could have stopped at step

3 because the x and y components completely define the displacement vector. We converted to the magnitude and direction to compare with the answer to part (a). Note that in step 5 of (b), a calculator gives the angle as –74°. But the calculator can't distinguish whether the x or y components is negative. We noted on the figure that the resultant displacement makes an angle of about 75° with the negative x axis and an angle of about 105° with the positive x axis. This agrees with the results in (a) within the accuracy of our measurement.

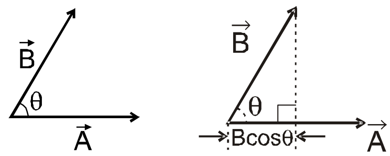
5.6 MULTIPLICATION OF VECTORS

5.6.1 THE SCALAR PRODUCT

The scalar product or dot product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A}\cdot\vec{B}$ (read $\vec{A}$ dot $\vec{B}$) is defined as the product of their magnitude with cosine of angle between them. Thus, $\vec{A}\cdot\vec{B} = AB\cos\theta$ {here θ is the angle between the vectors}

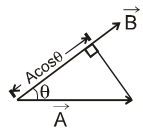
PROPERTIES :

- It is always a scalar which is positive if angle between the vectors is acute (i.e. < 90º) and negative if angle between them is obtuse (i.e. $90^\circ < \theta \le 180^\circ$)
- It is commutative, i.e., $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
- It is distributive, i.e. $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$
- As by definition $\vec{A}\cdot\vec{B} = AB\cos\theta$. The angle between the vectors $\theta = \cos^{-1}\left[\frac{\vec{A}\cdot\vec{B}}{AB}\right]$
- $\vec{A}\cdot\vec{B} = A(B\cos\theta) = B(A\cos\theta)$
  Geometrically, $B\cos\theta$ is the projection of $\vec{B}$ onto $\vec{A}$ and $A\cos\theta$ is the projection of $\vec{A}$ onto $\vec{B}$ as shown. So $\vec{A}\cdot\vec{B}$ is the product of the magnitude of $\vec{A}$ and the component of $\vec{B}$ along $\vec{A}$ and vice versa.



Component of $\vec{B}$ along $\vec{A} = B\cos\theta = \frac{\vec{A}\cdot\vec{B}}{A} = \hat{A}\cdot\vec{B}$

Component of $\vec{A}$ along $\vec{B} = A\cos\theta = \frac{\vec{A}\cdot\vec{B}}{B} = \vec{A}\cdot\hat{B}$



- Scalar product of two vectors will be maximum when $\cos\theta$ = max = 1, i.e., $\theta = 0^\circ$, i.e., vectors are parallel $\Rightarrow (\vec{A}\cdot\vec{B})_{max} = AB$
- If the scalar product of two nonzero vectors vanishes then the vectors are perpendicular.
- The scalar product of a vector by itself is termed as self dot product and is given by
  $(\vec{A})^2 = \vec{A}\cdot\vec{A} = AA\cos\theta = AA\cos0^\circ = A^2 \quad\Rightarrow\quad A = \sqrt{\vec{A}\cdot\vec{A}}$
- In case of unit vector $\hat{n}$,
  $\hat{n}\cdot\hat{n} = 1 \times 1 \times \cos 0^\circ = 1 \quad\Rightarrow\quad \hat{n}\cdot\hat{n} = \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$
- In case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$; $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
  $\vec{A}\cdot\vec{B} = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)\cdot(\hat{i}B_x + \hat{j}B_y + \hat{k}B_z) = [A_xB_x + A_yB_y + A_zB_z]$

Example 70. If the Vectors $\vec{P} = a\hat{i} + a\hat{j} + 3\hat{k}$ and $\vec{Q} = a\hat{i} - 2\hat{j} - \hat{k}$ are perpendicular to each other. Find the value of a?

Solution : If vectors $\vec{P}$ and $\vec{Q}$ are perpendicular

$\Rightarrow \vec{P} \cdot \vec{Q} = 0 \quad \Rightarrow (a\hat{i} + a\hat{j} + 3\hat{k}).(a\hat{i} - 2\hat{j} - \hat{k}) = 0 \quad \Rightarrow a^2 - 2a - 3 = 0$

$\Rightarrow a^2 - 3a + a - 3 = 0 \quad \Rightarrow a(a-3) + 1(a-3) = 0 \quad \Rightarrow a = -1, 3$

Example 71. Find the component of $3\hat{i} + 4\hat{j}$ along $\hat{i} + \hat{j}$?

Solution : Component of $\vec{A}$ along $\vec{B}$ is given by $\frac{\vec{A}\cdot\vec{B}}{B}$ hence required component

$= \frac{(3\hat{i}+4\hat{j})\cdot(\hat{i}+\hat{j})}{\sqrt{2}} = \frac{7}{\sqrt{2}}$

Example 72. Find angle between $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = 12\hat{i} + 5\hat{j}$?

Solution : We have $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{(3\hat{i}+4\hat{j})\cdot(12\hat{i}+5\hat{j})}{\sqrt{3^2+4^2}\sqrt{12^2+5^2}}$

$\cos\theta = \frac{36+20}{5\times13} = \frac{56}{65} \qquad \theta = \cos^{-1}\frac{56}{65}$

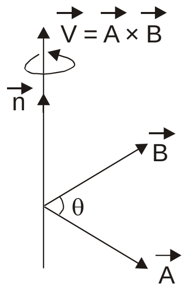
5.6.2 VECTOR PRODUCT

The vector product or cross product of any two vectors $\vec{A}$ and $\vec{B}$, denoted as $\vec{A}\times\vec{B}$ (read $\vec{A}$ cross $\vec{B}$) is defined as : $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$.

Here θ is the angle between the vectors and the direction $\hat{n}$ is given by the right-hand-thumb rule.

Right-Hand-Thumb Rule:

To find the direction of $\hat{n}$, draw the two vectors $\vec{A}$ and $\vec{B}$ with both the tails coinciding. Now place your stretched right palm perpendicular to the plane of $\vec{A}$ and $\vec{B}$ in such a way that the fingers are along the vector $\vec{A}$ and when the fingers are closed they go towards $\vec{B}$.



The direction of the thumb gives the direction of $\hat{n}$.

PROPERTIES :

☞ Vector product of two vectors is always a vector perpendicular to the plane containing the two vectors i.e. orthogonal to both the vectors $\vec{A}$ and $\vec{B}$, though the vectors $\vec{A}$ and $\vec{B}$ may or may not be orthogonal.

☞ Vector product of two vectors is not commutative i.e. $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$.

But $|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{A}| = AB\sin\theta$

☞ The vector product is distributive when the order of the vectors is strictly maintained i.e.

$A\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$.

☞ The magnitude of vector product of two vectors will be maximum when $\sin\theta$ = max = 1, i.e, $\theta = 90^o$. $|\vec{A}\times\vec{B}|_{max} = AB$ i.e., magnitude of vector product is maximum if the vectors are orthogonal.

☞ The magnitude of vector product of two non–zero vectors will be minimum when $|\sin\theta|$ = minimum = 0, i.e., $\theta = 0^\circ$ or 180° and $|\vec{A}\times\vec{B}|_{min} = 0$ i.e., if the vector product of two non–zero vectors vanishes, the vectors are collinear.

Note : When $\theta = 0^\circ$ then vectors may be called as like vector or parallel vectors and when $\theta = 180^\circ$ then vectors may be called as unlike vectors or antiparallel vectors.

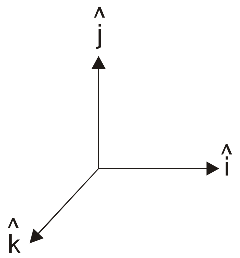
☞ The self cross product i.e. product of a vector by itself vanishes i.e. is a null vector.

Note : Null vector or zero vector : A vector of zero magnitude is called zero vector. The direction of a zero vector is in determinate (unspecified). $\vec{A}\times\vec{A}$ = AA sin 0º $\hat{n}$ = $\vec{0}$.

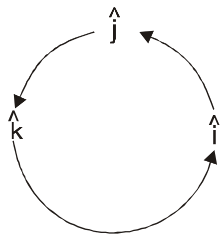
In case of unit vector $\hat{n}$, $\hat{n} \times \hat{n} = \vec{0} \Rightarrow \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$

☞ In case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ in accordance with right-hand-thumb-rule,

$\hat{i}\times\hat{j} = \hat{k}$ $\quad \hat{j}\times\hat{k} = \hat{i}$ $\quad \hat{k}\times\hat{i} = \hat{j}$



(A)



(B)

☞ In terms of components,

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$\vec{A}\times\vec{B} = \hat{i}(A_yB_z - A_zB_y) + \hat{j}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$$

☞ The magnitude of area of the parallelogram formed by the adjacent sides of vectors $\vec{A}$ and $\vec{B}$ equal to $|\vec{A}\times\vec{B}|$

Example 73. $\vec{A}$ is Eastwards and $\vec{B}$ is downwards. Find the direction of $\vec{A} \times \vec{B}$?

Solution : Applying right hand thumb rule we find that $\vec{A} \times \vec{B}$ is along North.

Example 74. If $\vec{A} \cdot \vec{B} = |\vec{A} \times \vec{B}|$, find angle between $\vec{A}$ and $\vec{B}$

Solution : $\vec{A} \cdot \vec{B} = |\vec{A}\times\vec{B}|$ AB cos θ = AB sin θ $\quad$ tan θ = 1 $\quad \Rightarrow \theta = 45^\circ$

Example 75. Two vectors $\vec{A}$ and $\vec{B}$ are inclined to each other at an angle θ. Find a unit vector which is perpendicular to both $\vec{A}$ and $\vec{B}$

Solution : $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n} \quad \Rightarrow \quad \hat{n} = \dfrac{\vec{A}\times\vec{B}}{AB\sin\theta}$ here $\hat{n}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

Example 76 Find $\vec{A} \times \vec{B}$ if $\vec{A} = \hat{i} - 2\hat{j} + 4\hat{k}$ and $\vec{B} = 2\hat{i} - \hat{j} + 2\hat{k}$.

Solution : $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 4 \\ 2 & -1 & 2 \end{vmatrix} = \hat{i}\ (-4-(-4)) - \hat{j}\ (2-12) + \hat{k}\ (-1-(-6)) = 10\hat{j} + 5\hat{k}$

Problem 1. Find the value of

(a) $\sin(-\theta)$ (b) $\cos(-\theta)$ (c) $\tan(-\theta)$ (d) $\cos(\frac{\pi}{2}-\theta)$

(e) $\sin(\frac{\pi}{2}+\theta)$ (f) $\cos(\frac{\pi}{2}+\theta)$ (g) $\sin(\pi-\theta)$ (h) $\cos(\pi-\theta)$

(i) $\sin(\frac{3\pi}{2}-\theta)$ (j) $\cos(\frac{3\pi}{2}-\theta)$ (k) $\sin(\frac{3\pi}{2}+\theta)$ (l) $\cos(\frac{3\pi}{2}+\theta)$

(m) $\tan(\frac{\pi}{2}-\theta)$ (n) $\cot(\frac{\pi}{2}-\theta)$

Answers :
(a) $-\sin\theta$ (b) $\cos\theta$ (c) $-\tan\theta$ (d) $\sin\theta$
(e) $\cos\theta$ (f) $-\sin\theta$ (g) $\sin\theta$ (h) $-\cos\theta$
(i) $-\cos\theta$ (j) $-\sin\theta$ (k) $-\cos\theta$ (l) $\sin\theta$
(m) $\cot\theta$ (n) $\tan\theta$

Problem 2. (i) For what value of m the vector $\vec{A} = 2\hat{i} + 3\hat{j} - 6\hat{k}$ is perpendicular to $\vec{B} = 3\hat{i} - m\hat{j} + 6\hat{k}$

(ii) Find the components of vector $\vec{A} = 2\hat{i} + 3\hat{j}$ along the direction of $\hat{i} + \hat{j}$?

Answers : (i) m = –10 (ii) $\frac{5}{\sqrt{2}}$.

Problem 3. (i) $\vec{A}$ is North–East and $\vec{B}$ is down wards, find the direction of $\vec{A} \times \vec{B}$.

(ii) Find $\vec{B} \times \vec{A}$ if $\vec{A} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and $\vec{B} = \hat{i} - \hat{j} + \hat{k}$.

Answers : (i) North - West. (ii) $-4\hat{i} - 3\hat{j} + \hat{k}$