ARITHMETIC PROGRESSION

DEFINITION

When the terms of a sequence or series are arranged under a definite rule then they are said to be in a Progression. Progression can be classified into 5 parts as -

(i) Arithmetic Progression (A.P.)
(ii) Geometric Progression (G.P.)
(iii) Arithmetic Geometric Progression (A.G.P.)
(iv) Harmonic Progression (H.P.)
(v) Miscellaneous Progression

ARITHMETIC PROGRESSION (A.P.)

Arithmetic Progression is defined as a series in which difference between any two consecutive terms is constant throughout the series. This constant difference is called common difference.

If 'a' is the first term and 'd' is the common difference, then an AP can be written as

$a + (a + d) + (a + 2d) + (a + 3d) + ...$

Note: If a,b,c, are in AP $\Leftrightarrow 2b = a + c$

General Term of an AP

General term (n^{th} term) of an AP is given by

$T_n = a + (n-1)\,d$

Note :

(i) General term is also denoted by ℓ (last term)

(ii) n (No. of terms) always belongs to set of natural numbers.

(iii) Common difference can be zero, + ve or – ve.

$d = 0 \Rightarrow$ then all terms of AP are same

Eg. 2, 2, 2, 2, ……… d = 0

$d = +ve \Rightarrow$ increasing AP

Eg. $\frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, ………$ $d = +\frac{1}{2}$

$d = -ve \Rightarrow$ decreasing A.P.

Eg. 57, 52, 47, 42, 37, …….. $d = -\frac{1}{2}$

r^{th} term from end of an A.P.

If number of terms in an A.P. is n then

T_r from end $= T_n - (r - 1)d = (n - r + 1)^{th}$ from beginning

or we can use last term of series as first term and use 'd' with opposite sign of given A.P.

Eg. : Find 26^{th} term from last of an AP 7, 15, 23……., 767 consits 96 terms.

Sol. Method : I

r^{th} term from end is given by

$= T_n - (r - 1)\,d$

or $= (n - r + 1)^{th}$ term from beginning

where n is total no. of terms.

m = 96, n = 26

$\therefore$ T_{26} from last $= T_{(96-26+1)}$ from beginning

$= T_{71}$ from beginning

$= a + 70d$

$= 7 + 70\,(8) = 7 + 560 = 567$

Method : II

$d = 15 - 7 = 8$

$\therefore$ from last, a = 767 and d = –8

$\therefore$ $T_{26} = a + 25d = 767 + 25\,(-8)$

$= 767 - 200$

$= 567.$

Sum of n terms of an A.P.

The sum of first n terms of an A.P. is given by

$S_n = \frac{n}{2}[2a + (n-1)\,d]$ or $S_n = \frac{n}{2}[a + T_n]$

Note :

(i) If sum of n terms S_n is given then general term $T_n = S_n - S_{n-1}$ where S_{n-1} is sum of $(n - 1)$ terms of A.P.

(ii) n^{th} term of an AP is linear in 'n'

Eg. : $a_n = 2 - n$, $a_n = 5n + 2$........

Also we can find common difference 'd' from a_n or T_n : d = coefficient of n

For $a_n = 2 - n$

$\therefore$ d = –1 Ans.

Verification : by putting n = 1, 2, 3, 4,.........

we get AP : 1, 0, –1, –2,........

$\therefore$ d = 0 – 1 = –1 Ans.

& for $a_n = 5n + 2$

d = 5 Ans.

(iii) Sum of n terms of an AP is always quadratic in 'n'

Eg. : $S_n = 2n^2 + 3n$.

Eg. : $S_n = \frac{n}{4}(n + 1)$

we can find 'd' also from S_n.

d = 2 (coefficient of n^2)

for eg. : $2n^2 + 3n$, d = 2(2) = 4

Verification		$S_n = 2n^2 + 3n$
at	n = 1	$S_1 = 2 + 3 = 5 =$ first term
at	n = 2	$S_2 = 2(2)^2 + 3(2)$
		$= 8 + 6 = 14 \neq$ second term
		= sum of first two terms.

$\therefore$ second term = $S_2 - S_1 = 14 - 5 = 9$

$\therefore$ $d = a_2 - a_1 = 9 - 5 = 4$

Eg. : $S_n = \frac{n}{4}(n + 1)$

$S_n = \frac{n^2}{4} + \frac{n}{4}$

$\therefore$ $d = 2\left(\frac{1}{4}\right) = \frac{1}{2}$ Ans.

❖ EXAMPLES ❖

Ex.1 If the nth term of a progression be a linear expression in n, then prove that this progression is an AP.

Sol. Let the nth term of a given progression be given by

$T_n = an + b$, where a and b are constants.

Then, $T_{n-1} = a(n - 1) + b = [(an + b) - a]$

$\therefore (T_n - T_{n-1}) = (an + b) - [(an + b) - a] = a$, which is a constant.

Hence, the given progression is an AP.

Ex.2 Write the first three terms in each of the sequences defined by the following -

(i) $a_n = 3n + 2$ (ii) $a_n = n^2 + 1$

Sol. (i) We have,

$a_n = 3n + 2$

Putting n = 1, 2 and 3, we get

$a_1 = 3 \times 1 + 2 = 3 + 2 = 5,$

$a_2 = 3 \times 2 + 2 = 6 + 2 = 8,$

$a_3 = 3 \times 3 + 2 = 9 + 2 = 11$

Thus, the required first three terms of the sequence defined by $a_n = 3n + 2$ are 5, 8, and 11.

(ii) We have,

$a_n = n^2 + 1$

Putting n = 1, 2, and 3 we get

$a_1 = 1^2 + 1 = 1 + 1 = 2$

$a_2 = 2^2 + 1 = 4 + 1 = 5$

$a_3 = 3^2 + 1 = 9 + 1 = 10$

Thus, the first three terms of the sequence defined by $a_n = n^2 + 1$ are 2, 5 and 10.

Ex.3 Write the first five terms of the sequence defined by $a_n = (-1)^{n-1} . 2^n$

Sol. $a_n = (-1)^{n-1} \times 2^n$

Putting n = 1, 2, 3, 4, and 5 we get

$a_1 = (-1)^{1-1} \times 2^1 = (-1)^0 \times 2 = 2$

$a_2 = (-1)^{2-1} \times 2^2 = (-1)^1 \times 4 = -4$

$a_3 = (-1)^{3-1} \times 2^3 = (-1)^2 \times 8 \times 8$

$a_4 = (-1)^{4-1} \times 2^4 = (-1)^3 \times 16 = -16$

$a_5 = (-1)^{5-1} \times 2^5 = (-1)^4 \times 32 = 32$

Thus the first five term of the sequence are 2, –4, 8, –16, 32.

Ex.4 The n^{th} term of a sequence is 3n – 2. Is the sequence an A.P. ? If so, find its 10^{th} term.

Sol. We have $a_n = 3n - 2$

Clearly a_n is a linear expression in n. So, the given sequence is an A.P. with common difference 3.

Putting n = 10, we get

$a_{10} = 3 \times 10 - 2 = 28$

> **REMARK :** It is evident from the above examples that a sequence is not an A.P. if its nth term is not a linear expression in n.

Ex.5 Find the 12^{th}, 24^{th} and nth term of the A.P. given by 9, 13, 17, 21, 25, ...

Sol. We have,

a = First term = 9 and,

d = Common difference = 4

[$\because$ 13 – 9 = 4, 17 – 13 = 4, 21 – 7 = 4 etc.]

We know that the nth term of an A.P. with first term a and common difference d is given by

$a_n = a + (n - 1)\, d$

Therefore,

$a_{12} = a + (12 - 1)\, d$

$= a + 11d = 9 + 11 \times 4 = 53$

$a_{24} = a + (24 - 1)\, d$

$= a + 23\, d = 9 + 23 \times 4 = 101$

and, $a_n = a + (n - 1)\, d$

$= 9 + (n - 1) \times 4 = 4n + 5$

$a_{12} = 53$, $a_{24} = 101$ and $a_n = 4n + 5$

Ex.6 Which term of the sequence –1, 3, 7, 11, ... , is 95 ?

Sol. Clearly, the given sequence is an A.P.

We have,

a = first term = –1 and,

d = Common difference = 4.

Let 95 be the n^{th} term of the given A.P. then,

$a_n = 95$

$\Rightarrow$ $a + (n - 1)\, d = 95$

$\Rightarrow$ $-1 + (n - 1) \times 4 = 95$

$\Rightarrow$ $-1 + 4n - 4 = 95$ $\Rightarrow$ $4n - 5 = 95$

$\Rightarrow$ $4n = 100$ $\Rightarrow$ $n = 25$

Thus, 95 is 25^{th} term of the given sequence.

Ex.7 Which term of the sequence 4, 9 , 14, 19, ... is 124 ?

Sol. Clearly, the given sequence is an A.P. with first term a = 4 and common difference d = 5.

Let 124 be the n^{th} term of the given sequence. Then, $a_n = 124$

$a + (n - 1)\, d = 124$

$\Rightarrow 4 + (n - 1) \times 5 = 124$

$\Rightarrow$ $n = 25$

Hence, 25^{th} term of the given sequence is 124.

Ex.8 The 10^{th} term of an A.P. is 52 and 16^{th} term is 82. Find the 32^{nd} term and the general term.

Sol. Let a be the first term and d be the common difference of the given A.P. Let the A.P. be $a_1, a_2, a_3, ... a_n, ...$

It is given that $a_{10} = 52$ and $a_{16} = 82$

$\Rightarrow$ $a + (10 - 1)\, d = 52$ and $a + (16 - 1)\, d = 82$

$\Rightarrow$ $a + 9d = 52$ ....(i)

and, $a + 15d = 82$ ....(ii)

Subtracting equation (ii) from equation (i), we get

$-6d = -30$ $\Rightarrow$ $d = 5$

Putting d = 5 in equation (i), we get

$a + 45 = 52$ $\Rightarrow$ $a = 7$

$\therefore$ $a_{32} = a + (32 - 1)\, d = 7 + 31 \times 5 = 162$

and, $a_n = a + (n - 1)\, d = 7\, (n - 1) \times 5 = 5n + 2$.

Hence $a_{32} = 162$ and $a_n = 5n + 2$.

Ex.9 Determine the general term of an A.P. whose 7^{th} term is –1 and 16^{th} term 17.

Sol. Let a be the first term and d be the common difference of the given A.P. Let the A.P. be $a_1, a_2, a_3, ... a_n, ...$

It is given that $a_7 = -1$ and $a_{16} = 17$

$a + (7 - 1)\, d = -1$ and, $a + (16 - 1)\, d = 17$

$\Rightarrow$ $a + 6d = -1$ ....(i)

and, $a + 15d = 17$ ....(ii)

Subtracting equation (i) from equation (ii), we get

$9d = 18$ $\Rightarrow$ $d = 2$

Putting d = 2 in equation (i), we get

$a + 12 = -1$ $\Rightarrow$ $a = -13$

Now, General term = a_n

$= a + (n - 1)\, d = -13 + (n - 1) \times 2 = 2n - 15$

Ex.10 If five times the fifth term of an A.P. is equal to 8 times its eight term, show that its 13^{th} term is zero.

Sol. Let $a_1, a_2, a_3, ... , a_n, ...$ be the A.P. with its first term = a and common difference = d.

It is given that $5a_5 = 8a_8$

$\Rightarrow\ 5(a + 4d) = 8(a + 7d)$

$\Rightarrow\ 5a + 20d = 8a + 56d \Rightarrow\ 3a + 36d = 0$

$\Rightarrow\ 3(a + 12d) = 0 \qquad \Rightarrow\ a + 12d = 0$

$\Rightarrow\ a + (13 - 1)d = 0 \qquad \Rightarrow\ a_{13} = 0$

Ex.11 If the m^{th} term of an A.P. be 1/n and n^{th} term be 1/m, then show that its $(mn)^{th}$ term is 1.

Sol. Let a and d be the first term and common difference respectively of the given A.P. Then,

$\frac{1}{n} = m^{th}$ term $\Rightarrow\ \frac{1}{n} = a + (m - 1)d \quad ....(i)$

$\frac{1}{m} = n^{th}$ term $\Rightarrow\ \frac{1}{m} = a + (n - 1)d \quad ....(ii)$

On subtracting equation (ii) from equation (i), we get

$$\frac{1}{n} - \frac{1}{m} = (m - n)d$$

$$\Rightarrow\ \frac{m-n}{mn} = (m - n)d \quad \Rightarrow\ d = \frac{1}{mn}$$

Putting $d = \frac{1}{mn}$ in equation (i), we get

$$\frac{1}{n} = a + \frac{(m-1)}{mn} \quad \Rightarrow\ a = \frac{1}{mn}$$

$\therefore\ (mn)^{th}$ term $= a + (mn - 1)d$

$$= \frac{1}{mn} + (mn - 1)\frac{1}{mn} = 1$$

Ex.12 If m times m^{th} term of an A.P. is equal to n times its nth term, show that the (m + n) term of the A.P. is zero.

Sol. Let a be the first term and d be the common difference of the given A.P. Then, m times m^{th} term = n times n^{th} term

$\Rightarrow\ ma_m = na_n$

$\Rightarrow\ m\{a + (m - 1)d\} = n\{a + (n - 1)d\}$

$\Rightarrow\ m\{a + (m - 1)d\} - n\{a + (n - 1)d\} = 0$

$\Rightarrow\ a(m - n) + \{m(m - 1) - n(n - 1)\}d = 0$

$\Rightarrow\ a(m - n) + (m - n)(m + n - 1)d = 0$

$\Rightarrow\ (m - n)\{a + (m + n - 1)d\} = 0$

$\Rightarrow\ a + (m + n - 1)d = 0$

$\Rightarrow\ a_{m+n} = 0$

Hence, the $(m + n)^{th}$ term of the given A.P. is zero.

Ex.13 If the p^{th} term of an A.P. is q and the q^{th} term is p, prove that its n^{th} term is (p + q – n).

Sol Let a be the first term and d be the common difference of the given A.P. Then,

p^{th} term $= q \Rightarrow a + (p - 1)d = q \quad ....(i)$

q^{th} term $= p \Rightarrow a + (q - 1)d = p \quad ....(ii)$

Subtracting equation (ii) from equation (i), we get

$(p - q)d = (q - p) \Rightarrow d = -1$

Putting d = – 1 in equation (i), we get

$a = (p + q - 1)$

n^{th} term $= a + (n - 1)d$

$= (p + q - 1) + (n - 1) \times (-1) = (p + q - n)$

Ex.14 If p^{th}, q^{th} and r^{th} terms of an A.P. are a, b, c respectively, then show that

(i) $a(q - r) + b(r - p) + c(p - q) = 0$

(ii) $(a - b)r + (b - c)p + (c - a)q = 0$

Sol. Let A be the first term and D be the common difference of the given A.P. Then,

$a = p^{th}$ term $\Rightarrow a = A + (p - 1)D \quad ....(i)$

$b = q^{th}$ term $\Rightarrow b = A + (q - 1)D \quad ....(ii)$

$c = r^{th}$ term $\Rightarrow c = A + (r - 1)D \quad ....(iii)$

(i) : We have,

$a(q - r) + b(r - p) + c(p - q)$

$= \{A + (p - 1)D\}(q - r)$

$+ \{A + (q - 1)\}(r - p)$

$+ \{A + (r - 1)D\}(p - q)$

[Using equations (i), (ii) and (iii)]

$= A\{(q - r) + (r - p) + (p - q)\}$

$+ D\{(p - 1)(q - r) + (q - 1)(r - p)$

$+ (r - 1)(p - q)\}$

$= A\{(q - r) + (r - p) + (p - q)\}$

$+ D\{(p - 1)(q - r) + (q - 1)(r - p)$

$+ (r - 1)(p - q)\}$

$= A . 0 + D\{p(q - r) + q(r - p)$

$+ r(p - q) - (q - r) - (r - p) - (p - q)\}$

$= A . 0 + D . 0 = 0$

(ii) : On subtracting equation (ii) from equation (i), equation (iii) from equation (ii) and equation (i) from equation (iii), we get

$a - b = (p - q)D$, $(b - c) = (q - r)D$ and $c - a = (r - p)D$

$\therefore\ (a - b)r + (b - c)p + (c - a)q$

$= (p - q)Dr + (q - r)Dp + (r - p)Dq$

$= D \{(p - q) r + (q - r) p + (r - p) q\}$

$= D \times 0 = 0$

Ex.15 Determine the 10^{th} term from the end of the A.P. 4, 9, 14, ..., 254.

Sol. We have,

l = Last term = 254 and,

d = Common difference = 5,

10^{th} term from the end $= l - (10 - 1) d$

$= l - 9d = 254 - 9 \times 5 = 209$.

➤ ARITHMETIC MEAN (A.M.)

If three or more than three terms are in A.P., then the numbers lying between first and last term are known as Arithmetic Means between them.i.e.

The A.M. between the two given quantities a and b is A so that a, A, b are in A.P.

i.e. $A - a = b - A \Rightarrow A = \frac{a+b}{2}$

Note : A.M. of any n positive numbers $a_1, a_2 ......a_n$ is

$$A = \frac{a_1 + a_2 + a_3 + ...a_n}{n}$$

n AM's between two given numbers

If in between two numbers 'a' and 'b' we have to insert n AM $A_1, A_2, ...A_n$ then $a, A_1, A_2, A_3....A_n$, b will be in A.P. The series consist of (n + 2) terms and the last term is b and first term is a.

$a + (n + 2 - 1) d = b$

$d = \frac{b-a}{n+1}$

$A_1 = a + d$, $A_2 = a + 2d,....A_n = a + nd$ or $A_n = b - d$

Note :

(i) Sum of n AM's inserted between a and b is equal to n times the single AM between a and b i.e. $\sum_{r=1}^{n} A_r = nA$ where

$$A = \frac{a+b}{2}$$

(ii) between two numbers

$$= \frac{\text{sum of m AM's}}{\text{sum of n AM's}} = \frac{m}{n}$$

➤ SUPPOSITION OF TERMS IN A.P.

(i) When no. of terms be odd then we take three terms are as: a – d, a, a + d five terms are as – 2d, a – d, a, a + d, a + 2d

Here we take middle term as 'a' and common difference as 'd'.

(ii) When no. of terms be even then we take 4 term are as : a – 3d, a – d, a + d, a + 3d

6 term are as = a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d

Here we take 'a – d, a + d' as middle terms and common difference as '2d'.

Note :

(i) If no. of terms in any series is odd then only one middle term is exist which is $\left(\frac{n+1}{2}\right)^{th}$ term where n is odd.

(ii) If no. of terms in any series is even then middle terms are two which are given by

$(n/2)^{th}$ and $\left\{\left(\frac{n}{2}\right)+1\right\}^{th}$ term where n is even.

➤ SOME PROPERTIES OF AN A.P.

(i) If each term of a given A.P. be increased, decreased, multiplied or divided by some non zero constant number then resulting series thus obtained will also be in A.P.

(ii) In an A.P., the sum of terms equidistant from the beginning and end is constant and equal to the sum of first and last term.

(iii) Any term of an AP (except the first term) is equal to the half of the sum of terms equidistant from the term i.e.

$$a_n = \frac{1}{2}(a_{n-k} + a_{n+k}), k < n$$

(iv) If in a finite AP, the number of terms be odd, then its middle term is the AM between the first and last term and its sum is equal to the product of middle term and no. of terms.

SOME STANDARD RESULTS

(i) Sum of first n natural numbers

$$\Rightarrow \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

(ii) Sum of first n odd natural numbers

$$\Rightarrow \sum_{r=1}^{n} (2r-1) = n^2$$

(iii) Sum of first n even natural numbers

$$= \sum_{r=1}^{n} 2r = n\,(n+1)$$

(iv) Sum of squares of first n natural numbers

$$= \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

(v) Sum of cubes of first n natural numbers

$$= \sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$$

(vi) If for an A.P. p^{th} term is q, q^{th} term is p then m^{th} term is = p + q – m

(vii) If for an AP sum of p terms is q, sum of q terms is p, then sum of (p + q) term is : **(p+q)**.

(viii) If for an A.P. sum of p terms is equal to sum of q terms then sum of (p + q) terms is zero.

❖ EXAMPLES ❖

Ex.16 The sum of three numbers in A.P. is –3, and their product is 8. Find the numbers.

Sol. Let the numbers be (a – d), a, (a + d). Then,

Sum = – 3 ⇒ (a – d) + a (a + d) = – 3

⇒ 3a = – 3

⇒ a = – 1

Product = 8

⇒ (a – d) (a) (a + d) = 8

⇒ a $(a^2 – d^2)$ = 8

⇒ (–1) $(1 – d^2)$ = 8

⇒ $d^2 = 9$ ⇒ d = ± 3

If d = 3, the numbers are –4, –1, 2. If d = – 3, the numbers are 2, – 1, –4.

Thus, the numbers are –4, –1, 2, or 2, – 1, – 4.

Ex.17 Find four numbers in A.P. whose sum is 20 and the sum of whose squares is 120.

Sol. Let the numbers be (a – 3d), (a – d), (a + d), (a + 3d), Then

Sum = 20

⇒ (a – 3d) + (a – d) + (a + d) + (a + 3d) = 20

⇒ 4a = 20

⇒ a = 5

Sum of the squares = 120

$(a – 3d)^2 + (a – d)^2 + (a + d)^2 + (a + 3d)^2 = 120$

⇒ $4a^2 + 20d^2 = 120$

⇒ $a^2 + 5d^2 = 30$

⇒ $25 + 5d^2 = 30$ [∵ a = 5]

⇒ $5d^2 = 5$ ⇒ d = ± 1

If d = 1, then the numbers are 2, 4, 6, 8. If d = – 1, then the numbers are 8, 6, 4, 2. Thus, the numbers are 2, 4, 6, 8 or 8, 6, 4, 2.

Ex.18 Divide 32 into four parts which are in A.P. such that the product of extremes is to the product of means is 7 : 15.

Sol. Let the four parts be (a – 3d), (a – d), (a + d) and (a + 3d). Then,

Sum = 32

⇒ (a – 3d) + (a – d) + (a + d) + (a + 3d) = 32

⇒ 4a = 32 ⇒ a = 8

It is given that $\frac{(a-3d)(a+3d)}{(a-d)(a+d)} = \frac{7}{15}$

$$\Rightarrow \frac{a^2-9d^2}{a^2-d^2} = \frac{7}{15} \Rightarrow \frac{64-9d^2}{64-d^2} = \frac{7}{15}$$

⇒ $128d^2 = 512$

⇒ $d^2 = 4$ ⇒ d = ± 2

Thus, the four parts are a – d, a – d, a + d and a + 3d i.e. 2, 6, 10 and 14.

Ex.19 Find the sum of 20 terms of the A.P. 1, 4, 7, 10, ...

Sol. Let a be the first term and d be the common difference of the given A.P. Then, we have a = 1 and d = 3.

We have to find the sum of 20 terms of the given A.P.

Putting a = 1, d = 3, n = 20 in

$S_n = \frac{n}{2}$ [2a + (n – 1) d], we get

$S_{20} = \frac{20}{2}$ [2 × 1 + (20 – 1) × 3]

= 10 × 59 = 590

Ex.20 Find the sum of first 30 terms of an A.P. whose second term is 2 and seventh term is 22.

Sol. Let a be the first term and d be the common difference of the given A.P. Then,

$a_2 = 2$ and $a_7 = 22$

⇒ a + d = 2 and a + 6d = 22

Solving these two equations, we get

a = – 2 and d = 4.

$S_n = \frac{n}{2}$ [2a + (n – 1) d]

∴ $S_{30} = \frac{30}{2}$ [2 × (–2) + (30 – 1) × 4]

⇒ 15 (–4 + 116) = 15 × 112

= 1680

Hence, the sum of first 30 terms is 1680.

Ex.21 Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3.

Sol. Clearly, the numbers between 250 and 1000 which are divisible by 3 are 252, 255, 258, ..., 999. This is an A.P. with first term a = 252, common difference = 3 and last term = 999. Let there be n terms in this A.P. Then,

⇒ $a_n = 999$

⇒ a + (n – 1)d = 999

⇒ 252 + (n – 1) × 3 = 999 ⇒ n = 250

∴ Required sum = $S_n = \frac{n}{2}$ [a + l]

$= \frac{250}{2}$ [252 + 999] = 156375

Ex.22 How many terms of the series 54, 51, 48, ... be taken so that their sum is 513 ? Explain the double answer.

Sol. ∵ a = 54, d = – 3 and $S_n = 513$

⇒ $\frac{n}{2}$ [2a + (n – 1) d] = 513

⇒ $\frac{n}{2}$ [108 + (n – 1) × – 3] = 513

⇒ $n^2 - 37n + 342 = 0$

⇒ (n – 18) (n – 19) = 0 ⇒ n = 18 or 19

Here, the common difference is negative, So, 19^{th} term is a_{19} = 54 + (19 – 1) × – 3 = 0.

Thus, the sum of 18 terms as well as that of 19 terms is 513.

Ex.23 If the m^{th} term of an A.P. is $\frac{1}{n}$ and the n^{th} term is $\frac{1}{m}$, show that the sum of mn terms is $\frac{1}{2}$(mn + 1).

Sol. Let a be the first term and d be the common difference of the given A.P. Then,

$a_m = \frac{1}{n} \Rightarrow a + (m-1)\,d = \frac{1}{n}$ ...(i)

and $a_n = \frac{1}{m} \Rightarrow a + (n-1)\,d = \frac{1}{m}$ ...(ii)

Subtracting equation (ii) from equation (i), we get

$(m - n)\,d = \frac{1}{n} - \frac{1}{m}$

$\Rightarrow (m-n)\,d = \frac{m-n}{mn} \Rightarrow d = \frac{1}{mn}$

Putting $d = \frac{1}{mn}$ in equation (i), we get

$a + (m-1)\frac{1}{mn} = \frac{1}{n}$

$\Rightarrow a + \frac{1}{n} - \frac{1}{mn} = \frac{1}{n} \Rightarrow a = \frac{1}{mn}$

Now, $S_{mn} = \frac{mn}{2}$ {2a + (mn – 1) × d}

$\Rightarrow S_{mn} = \frac{mn}{2}\left[\frac{2}{mn} + (mn-1)\times\frac{1}{mn}\right]$

$\Rightarrow S_{mn} = \frac{1}{2}(mn+1)$

Ex.24 If the term of m terms of an A.P. is the same as the sum of its n terms, show that the sum of its (m + n) terms is zero.

Sol. Let a be the first term and d be the common difference of the given A.P. Then,

$S_m = S_n$

$\Rightarrow \frac{m}{2}[2a+(m-1)d] = \frac{n}{2}[2a+(n-1)d]$

$\Rightarrow 2a(m-n) + \{m(m-1) - n(n-1)\} d = 0$

$\Rightarrow 2a(m-n) + \{(m^2 - n^2) - (m-n)\} d = 0$

$\Rightarrow (m-n)[2a + (m+n-1)d] = 0$

$\Rightarrow 2a + (m+n-1) d = 0$

$\Rightarrow 2a + (m+n-1) d = 0 \;\; [\because m - n \neq 0]$ ....(i)

Now, $S_{m+n} = \frac{m+n}{2}\{2a + (m+n-1)d\}$

$S_{m+n} = \frac{m+n}{2} \times 0 = 0$ [Using equation (i)]

Ex.25 The sum of n, 2n, 3n terms of an A.P. are S_1, S_2, S_3 respectively. Prove that $S_3 = 3(S_2 - S_1)$.

Sol. Let a be the first term and d be the common difference of the given A.P. Then,

S_1 = Sum of n terms

$\Rightarrow S_1 = \frac{n}{2}\{2a + (n-1)d\}$ ....(i)

S_2 = Sum of 2n terms

$\Rightarrow S_2 = \frac{2n}{2}[2a + (2n-1)d]$ ....(ii)

and, S_3 = Sum of 3n terms

$\Rightarrow S_3 = \frac{3n}{2}[2a + (3n-1)d]$ ....(iii)

Now, $S_2 - S_1$

$= \frac{2n}{2}[2a + (2n-1)d] - \frac{n}{2}[2a + (n-1)d]$

$S_2 - S_1 = \frac{n}{2}[2\{2a + (2n-1)d\} - \{2a + (n-1)d\}]$

$= \frac{n}{2}[2a + (3n-1)d]$

$\therefore 3(S_2 - S_1) = \frac{3n}{2}[2a + (3n-1)d] = S_3$

[Using (iii)]

Hence, $S_3 = 3(S_2 - S_1)$

Ex.26 The sum of n terms of three arithmetical progression are S_1, S_2 and S_3. The first term of each is unity and the common differences are 1, 2 and 3 respectively. Prove that $S_1 + S_3 = 2S_2$.

Sol. We have,

S_1 = Sum of n terms of an A.P. with first term 1 and common difference 1

$= \frac{n}{2}[2 \times 1 + (n-1)1] = \frac{n}{2}[n+1]$

S_2 = Sum of n terms of an A.P. with first term 1 and common difference 2

$= \frac{n}{2}[2 \times 1 + (n-1) \times 2] = n^2$

S_3 = Sum of n terms of an A.P. with first term 1 and common difference 3

$= \frac{n}{2}[2 \times 1 + (n-1) \times 3] = \frac{n}{2}(3n-1)$

Now, $S_1 + S_3 = \frac{n}{2}(n+1) + \frac{n}{2}(3n-1)$

$= 2n^2$ and $S_2 = n^2$

Hence $S_1 + S_3 = 2S_2$

Ex.27 The sum of the first p, q, r terms of an A.P. are a, b, c respectively. Show that

$$\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{b}{r}(p-q) = 0$$

Sol. Let A be the first term and D be the common difference of the given A.P. Then,

a = Sum of p terms $\Rightarrow a = \frac{p}{2}[2A + (q-1)D]$

$\Rightarrow \frac{2a}{p} = [2A + (p-1)D]$ ....(i)

b = Sum of q terms

$\Rightarrow b = \frac{q}{2}[2A + (q-1)D]$

$\Rightarrow \frac{2b}{q} = [2A + (q-1)D]$ ....(ii)

and, c = Sum of r terms

$\Rightarrow c = \frac{r}{2}[2A + (r-1)D]$

$\Rightarrow \frac{2c}{r} = [2A + (r-1)D]$ ....(iii)

Multiplying equations (i), (ii) and (iii) by (q – r), (r – p) and (p – q) respectively and adding, we get

$$\frac{2a}{p}(q-r) + \frac{2b}{q}(r-p) + \frac{2c}{r}(p-q)$$

$= [2A + (p-1)D](q-r) + [2A + (q-1)D](r-p) + [(2A + (r-1)D](p-q)$

$= 2A(q - r + r - p + p - q) + D[(p-1)(q-r) + (q-1)(r-p) + (r-1)(p-q)]$

$= 2A \times 0 + D \times 0 = 0$

Ex.28 The ratio of the sum use of n terms of two A.P.'s is (7n + 1) : (4n + 27). Find the ratio of their m^{th} terms.

Sol. Let a_1, a_2 be the first terms and d_1, d_2 the common differences of the two given A.P.'s .Then the sums of their n terms are given by

$$S_n = \frac{n}{2}[2a_1 + (n-1)d_1], \text{ and}$$

$$S_n' = \frac{n}{2}[2a_2 + (n-1)d_2]$$

$$\therefore \frac{S_n}{S_n'} = \frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2}$$

It is given that $\frac{S_n}{S_n'} = \frac{7n+1}{4n+27}$

$$\Rightarrow \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27} \quad ....(i)$$

To find the ratio of the m^{th} terms of the two given A.P.'s, we replace n by (2m – 1) in equation (i). Then we get

$$\therefore \frac{2a_1 + (2m-2)d_1}{2a_2 + (2m-2)d_2} = \frac{7(2m-1)+1}{4(2m-1)+27}$$

$$\Rightarrow \frac{a_1 + (m-1)d_1}{a_2 + (m-1)d_2} = \frac{14m-6}{8m+23}$$

Hence the ratio of the m^{th} terms of the two A.P.'s is (14m – 6) : (8m + 23)

Ex.29 The ratio of the sums of m and n terms of an A.P. is $m^2 : n^2$. Show that the ratio of the m^{th} and n^{th} terms is (2m – 1) : (2n – 1).

Sol. Let a be the first term and d the common difference of the given A.P. Then, the sums of m and n terms are given by

$$S_m = \frac{m}{2}[2a + (m-1)d], \text{ and}$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

respectively. Then,

$$\frac{S_m}{S_n} = \frac{m^2}{n^2} \Rightarrow \frac{\frac{m}{2}[2a + (m-1)d]}{\frac{n}{2}[2a + (n-1)d]} = \frac{m^2}{n^2}$$

$$\Rightarrow \frac{2a + (m-1)d}{2a + (n-1)d} = \frac{m}{n}$$

$\Rightarrow [2a + (m-1)d]\,n = \{2a + (n-1)d\}\,m$

$\Rightarrow 2a(n-m) = d\{(n-1)m - (m-1)n\}$

$\Rightarrow 2a(n-m) = d(n-m)$

$\Rightarrow d = 2a$

Now, $\frac{T_m}{T_n} = \frac{a + (m-1)d}{a + (n-1)d}$

$$= \frac{a + (m-1)2a}{a + (n-1)2a} = \frac{2m-1}{2n-1}$$

Ex.30 If 4 AM's are inserted between 1/2 and 3 then find 3rd AM.

Sol. Here $d = \frac{3 - \frac{1}{2}}{4+1} = \frac{1}{2}$

$\therefore A_3 = a + 3d \Rightarrow \frac{1}{2} + 3 \times \frac{1}{2} = 2$

Ex.31 n AM's are inserted between 2 and 38. If third AM is 14 then n is equal to.

Sol. Here $2 + 3d = 14 \quad \Rightarrow d = 4$

$\therefore 4 = \frac{38-2}{n+1}$

$\Rightarrow 4n + 4 = 36 \Rightarrow n = 8$

Ex.32 Four numbers are in A.P. If their sum is 20 and the sum of their square is 120, then find the middle terms.

Sol. Let the numbers are a – 3d, a – d, a + d, a + 3d

given a – 3d + a – d + a + d + a + 3d = 20

$\Rightarrow 4a = 20 \Rightarrow a = 5$

and $(a-3d)^2 + (a-d)^2 + (a+d)^2 + (a+3d)^2 = 120$

$4a^2 + 20d^2 = 120$

$4 \times 5^2 + 20d^2 = 120$

$d^2 = 1 \Rightarrow d = \pm 1$

Hence numbers are 2, 4, 6, 8

Ex.33 Find the common difference of an AP, whose first term is 5 and the sum of its first four terms is half the sum of the next four terms.

Sol. ATQ

$$a_1 + a_2 + a_3 + a_4 = \frac{1}{2}(a_5 + a_6 + a_7 + a_8)$$

$\Rightarrow 2[a_1 + a_2 + a_3 + a_4] = a_5 + a_6 + a_7 + a_8$

$\Rightarrow 2[a_1 + a_2 + a_3 + a_4] + (a_1 + a_2 + a_3 + a_4)$

$= [a_1 + a_2 + a_3 + a_4] + (a_5 + a_6 + a_7 + a_8)$

(adding both side $a_1 + a_2 + a_3 + a_4$)

$\Rightarrow \; 3(a_1 + a_2 + a_3 + a_4) = a_1 + ... + a_8 \Rightarrow 3S_4 = S_8$

$\Rightarrow \; 3\left[\frac{4}{2}(2\times 5+(4-1)\,d\right] = \left[\frac{8}{2}(2\times 5+(8-1)\,d\right]$

$\Rightarrow \; 3[10 + 3d] = 2[10 + 7d]$

$\Rightarrow \; 30 + 9d = 20 + 14d \Rightarrow 5d = 10 \quad \Rightarrow d = 2$

Ex.34 If the n^{th} term of an AP is $(2n + 1)$ then find the sum of its first three terms.

Sol. $\because\; a_n = 2n + 1$

$a_1 = 2(1) + 1 = 3$

$a_2 = 2(2) + 1 = 5$

$a_3 = 2(3) + 1 = 7$

$\therefore\; a_1 + a_2 + a_3 = 3 + 5 + 7 = 15$

Ex.35 Which term of the sequence 20, $19\frac{1}{4}$, $18\frac{1}{2}$, $17\frac{3}{4}$, ... is the first negative terms ?

Sol. The given sequence is an A.P. in which first term a = 20 and common difference $d = -\frac{3}{4}$.

Let a_n is the first negative term

then $a_n < 0$

$\Rightarrow a + (n-1)\,d < 0 \Rightarrow 20 + (n-1)\left(-\frac{3}{4}\right) < 0$

$\Rightarrow 20 < (n-1)\,\frac{3}{4} \Rightarrow 80 < 3\,(n-1)$

$\Rightarrow 80 < 3n - 3 \Rightarrow 83 < 3n \Rightarrow n > \frac{83}{3}$ or $n > 27\frac{2}{3}$

$\because$ 28 is the natural number just greater than $27\frac{2}{3}$

$\therefore$ n = 28 Ans.

IMPORTANT POINTS TO BE REMEMBERED

1. A succession of numbers formed and arranged according to some definite law is called a sequence.

For example :

(a) 3, 7, 11, 15 ....

(b) 2, 4, 8, 16 ....

2. Each number of the sequence is called a term of the sequence. A sequence is said to be finite or infinite according as the number of terms in it is finite or infinite.

3. If the terms of a sequence are connected by the sign of addition (+), we get a series

For example :

3 + 7 + 11 + 15 + ...

4. If the terms of a series constantly increase or decrease in numerical value, the series is called a progression.

5. A series is said to be in A.P. if the difference of each term after the first term and the proceeding term is constant. The constant difference is called common difference.

For Example : -

1 + 3 + 5 + 7 + 9 + ... is an A.P. with common difference 2.

6. General form of an A.P. is

$a + (n-1)d = a_n$

7. Sum of n terms of an A.P. is

$S_n = \frac{n}{2}\,[2a + (n-1)d] \; = \frac{n}{2}(a + a_n)$

8. nth term (a_n) = sum of n terms – sum of (n–1) terms of same AP

i.e. $a_n = S_n - S_{n-1}$

9. The n^{th} term is linear in 'n' and d = coefficient of n.

10. The sum of n terms is quadratic in 'n' and d = double of coefficient of n.

11. $S_1 = a$ = (first term of A.P.)

S_2 = sum of first two terms.

12. Sum of infinite terms $= \begin{cases} \infty \text{ if } d > 0 \\ -\infty \text{ if } d < 0 \end{cases}$