BINOMIAL THEOREM

Binomial Expression :

An expression containing two terms, is called a binomial expression. For example $a + b/x$, $x + 1/y$, $a - y^2$ etc. are binomial expressions. Expansion of $(x + a)^n$ is called Binomial Theorem. Expression containing three terms is called trinomial. For example $x + y + z$. And in general expression containing more than two terms is called multinomial.

Terminology used in binomial theorem :

Factorial notation : $\lfloor\underline{n}$ or $n!$ is pronounced as factorial n and is defined as

$$n! = \begin{cases} n(n-1)(n-2)........3.2.1 & ; \text{ if } n \in N \\ 1 & ; \text{ if } n = 0 \end{cases}$$

Note : $n! = n\,.\,(n-1)!\,;\qquad n \in N$

Mathematical meaning of ^nC_r : The term ^nC_r denotes number of combinations of r things choosen from n distinct things mathematically, $^nC_r = \dfrac{n!}{(n-r)!\,r!}$, $n \in N, r \in W, 0 \le r \le n$

Note : Other symbols of of ^nC_r are $\binom{n}{r}$ and $C(n, r)$.

Properties related to ^nC_r :

(i) $^nC_r = {}^nC_{n-r}$

Note : If $^nC_x = {}^nC_y \Rightarrow$ Either $x = y$ or $x + y = n$

(ii) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iii) $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

(iv) $^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n(n-1)}{r(r-1)}\,{}^{n-2}C_{r-2}$

$= .... = \dfrac{n(n-1)(n-2).........(n-(r-1))}{r\,(r-1)(r-2).......2\,.1}$

(v) If n and r are relatively prime, then ^nC_r is divisible by n. But converse is not necessarily true.

Statement of binomial theorem :

$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + ... + {}^nC_r\, a^{n-r} b^r + ... + {}^nC_n\, a^0 b^n$

where $n \in N$

or $(a + b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r$

Note : If we put $a = 1$ and $b = x$ in the above binomial expansion, then

or $(1 + x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + ... + {}^nC_r\, x^r + ... + {}^nC_n\, x^n$

or $(1 + x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$

Solved Examples

Ex.1 Expand the following binomials :

(i) $(x-3)^5$ (ii) $\left(1-\dfrac{3x^2}{2}\right)^4$

Sol. (i) $(x-3)^5 = {}^5C_0x^5 + {}^5C_1x^4(-3)^1 + {}^5C_2x^3(-3)^2 + {}^5C_3x^2(-3)^3$

$+ {}^5C_4x(-3)^4 + {}^5C_5(-3)^5$

$= x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

(ii) $\left(1-\dfrac{3x^2}{2}\right)^4 = {}^4C_0 + {}^4C_1\left(-\dfrac{3x^2}{2}\right) + {}^4C_2\left(-\dfrac{3x^2}{2}\right)^2$

$+ {}^4C_3\left(-\dfrac{3x^2}{2}\right)^3 + {}^4C_4\left(-\dfrac{3x^2}{2}\right)^4$

$= 1 - 6x^2 + \dfrac{27}{2}x^4 - \dfrac{27}{2}x^6 + \dfrac{81}{16}x^8$

Observations :

(i) The number of terms in the binomial expansion $(a+b)^n$ is $n+1$.

(ii) The sum of the indices of a and b in each term is n.

(iii) The binomial coefficients (${}^nC_0$, ${}^nC_1$ ....${}^nC_n$) of the terms equidistant from the beginning and the end are equal, i.e. ${}^nC_0 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$ etc. $\{\because {}^nC_r = {}^nC_{n-r}\}$

(iv) The binomial coefficient can be remembered with the help of the following pascal's Triangle (also known as Meru Prastra provided by Pingla)

Index of the binomial	The binomial coefficient
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Regarding Pascal's Triangle, we note the following :

(a) Each row of the triangle begins with 1 and ends with 1.

(b) Any entry in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right.

Solved Examples

Ex.2 : The number of dissimilar terms in the expansion of $(1-3x+3x^2-x^3)^{20}$ is

(A) 21 (B) 31 (C) 41 (D) 61

Sol. $(1-3x+3x^2-x^3)^{20} = [(1-x)^3]^{20} = (1-x)^{60}$

Therefore number of dissimilar terms in the expansion of $(1-3x+3x^2-x^3)^{20}$ is **61.**

General term :

$(x+y)^n = {}^nC_0\,x^n y^0 + {}^nC_1\,x^{n-1}y^1 + \text{..........} + {}^nC_r\,x^{n-r}y^r + \text{.........} + {}^nC_n\,x^0y^n$

$(r+1)^{th}$ term is called general term and denoted by T_{r+1}.

$T_{r+1} = {}^nC_r\,x^{n-r}y^r$ & $\dfrac{T_{r+1}}{T_r} = \dfrac{(n-r+1)}{r}\left(\dfrac{y}{x}\right)$

Note :

The r^{th} term from the end is equal to the $(n-r+2)^{th}$ term from the begining, i.e. ${}^nC_{n-r+1}\,x^{r-1}y^{n-r+1}$

Solved Examples

Ex.3 Find (i) 28^{th} term of $(5x+8y)^{30}$

(ii) 7^{th} term of $\left(\dfrac{4x}{5}-\dfrac{5}{2x}\right)^9$

Sol. (i) $T_{27+1} = {}^{30}C_{27}(5x)^{30-27}(8y)^{27}$

$= \dfrac{30!}{3!\,27!}(5x)^3 \cdot (8y)^{27}$

(ii) 7th term of $\left(\dfrac{4x}{5}-\dfrac{5}{2x}\right)^9$

$T_{6+1} = {}^9C_6\left(\dfrac{4x}{5}\right)^{9-6}\left(-\dfrac{5}{2x}\right)^6$

$= \dfrac{9!}{3!\,6!}\left(\dfrac{4x}{5}\right)^3\left(\dfrac{5}{2x}\right)^6 = \dfrac{10500}{x^3}$

Solved Examples

Ex.4 The 7th term from the end in the expansion of $\left(x-\frac{2}{x^2}\right)^{10}$ is equal to-

[1] ${}^{10}C_4 2^4\left(\frac{1}{x^2}\right)$ [2] ${}^{10}C_4 2^4$

[3] $-{}^{10}C_3 2^3 x$ [4] None of these

Sol.[1] The 7th term from the end = 5th term from beginning

$$T_5 = {}^{10}C_4 x^6\left(-\frac{2}{x^2}\right)^4 = {}^{10}C_4.2^4\left(\frac{1}{x^2}\right)$$

Ex.5. The ratio of the coefficient of x^{15} to the term independent of x in $\left(x^2+\frac{2}{x}\right)^{15}$ is

(1) 12 : 32 (2) 1 : 32

(3) 32 : 12 (4) 32 : 1

Sol. [2] General term in the expansion is

${}^{15}C_r\left(x^2\right)^{15-r}\left(\frac{2}{x}\right)^r$ i.e., ${}^{15}C_r x^{30-3r}.2^r$

Coefficient of x^{15} is ${}^{15}C_5 2^5$ $(r=5)$

Coefficient of constant term is ${}^{15}C_{10}\, 2^{10}$ $(r=10)$

Ratio is 1 : 32. $({}^{15}C_5 = {}^{15}C_{10})$

Ex.6. The term independent of x in the expansion of $\left(\sqrt[6]{x}-\frac{1}{\sqrt[3]{x}}\right)^9$ is equal to-

[1] $-{}^9C_3$ [2] ${}^9C_4$

[3] ${}^9C_2$ [4] none of these

Sol. [1] $T_{r+1} = {}^9C_r\,(\sqrt[6]{x})^{9-r}\left(-\frac{1}{\sqrt[3]{x}}\right)^r$

$= {}^9C_r\,(-1)^r\, x^{\frac{9-r}{6}-\frac{r}{3}} = {}^9C_r\,(-1)^r\, x^{\left(\frac{9-3r}{6}\right)}$

Now, $\frac{9-3r}{6}=0 \quad\Rightarrow\quad r=3 \qquad T=-{}^9C_3$

Ex.7 If the second, third and fourth terms in the expansion of $(a+b)^n$ are 135, 30 and 10/3 respectively, then-

[1] n = 3 [2] n = 2

[3] n = 7 [4] n = 5

Sol. [4] $T_2 = {}^nC_1 ab^{n-1} = 135$ ......(i)

$T_3 = {}^nC_2 a^2 b^{n-2} = 30$ ......(ii)

$T_4 = {}^nC_3 a^3 b^{n-3} = \frac{10}{3}$ ......(iii)

Dividing (i) by (ii)

$$\frac{{}^nC_2 ab^{n-1}}{{}^nC_1 a^2 b^{n-2}} = \frac{135}{30}$$

$$\frac{n}{\frac{n}{2}(n-1)}\frac{b}{a} = \frac{9}{2} \quad ......(iv)$$

$$\frac{b}{a} = \frac{9}{4}(n-1) \quad ......(v)$$

Dividing (ii) and (iii)

$$\frac{\frac{n(n-1)}{2}}{\frac{n(n-1)(n-3)}{3.2}}.\frac{b}{a} = \frac{10}{3} = 9 \quad ......(vi)$$

Eliminating a and b from (v) and (vi)

$\Rightarrow \quad n = 5$

Ex.8 Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Sol. The general term in the expansion of $\left(9^{1/4}+8^{1/6}\right)^{1000}$ is

$$T_{r+1} = {}^{1000}C_r\left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{\frac{1}{6}}\right)^r = {}^{1000}C_r\; 3^{\frac{1000-r}{2}}\; 2^{\frac{r}{2}}$$

The above term will be rational if exponent of 3 and 2 are integers

It means $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, ..., 1000}

Hence, number of rational terms is 501

Ex.9 Find the coefficient of x^{32} and x^{-17} in $\left(x^4-\frac{1}{x^3}\right)^{15}$.

Sol. Let $(r+1)^{th}$ term contains x^m

$$T_{r+1} = {}^{15}C_r\,(x^4)^{15-r}\left(-\frac{1}{x^3}\right)^r = {}^{15}C_r\, x^{60-7r}\,(-1)^r$$

(i) for x^{32}, $60-7r=32$

$\Rightarrow 7r = 28 \qquad \Rightarrow r = 4$, so 5^{th} term.

$T_5 = {}^{15}C_4\, x^{32}\,(-1)^4$

Hence, coefficient of x^{32} is 1365

(ii) for x^{-17}, $60-7r=-17$

$\Rightarrow r = 11$, so 12^{th} term.

$T_{12} = {}^{15}C_{11}\, x^{-17}\,(-1)^{11}$

Hence, coefficient of x^{-17} is -1365

Middle term(s) :

(a) If n is even, there is only one middle term, which is $\left(\frac{n+2}{2}\right)^{th}$ term.

(b) If n is odd, there are two middle terms, which are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+1}{2}+1\right)^{th}$ terms.

Note : In any binomial expansion, the middle term(s) has greatest binomial coefficient.

In the expansion of $(a+b)^n$

If n	No. of greatest binomial coefficient	Greatest binomial coefficient
Even	1	$^nC_{n/2}$
Odd	2	$^nC_{(n-1)/2}$ and $^nC_{(n+1)/2}$

(Values of both these coefficients are equal)

Solved Examples

Ex.10 Find the middle term(s) in the expansion of

(i) $\left(1-\frac{x^2}{2}\right)^{14}$ (ii) $\left(3a-\frac{a^3}{6}\right)^9$

Sol. (i) $\left(1-\frac{x^2}{2}\right)^{14}$

Here, n is even, therefore middle term is $\left(\frac{14+2}{2}\right)^{th}$ term.

It means T_8 is middle term

$$T_8 = {}^{14}C_7\left(-\frac{x^2}{2}\right)^7 = -\frac{429}{16}\,x^{14}.$$

(ii) $\left(3a-\frac{a^3}{6}\right)^9$

Here, n is odd therefore, middle terms are $\left(\frac{9+1}{2}\right)^{th}$ & $\left(\frac{9+1}{2}+1\right)^{th}$.

It means T_5 & T_6 are middle terms

$$T_5 = {}^9C_4\,(3a)^{9-4}\left(-\frac{a^3}{6}\right)^4 = \frac{189}{8}\,a^{17}$$

$$T_6 = {}^9C_5\,(3a)^{9-5}\left(-\frac{a^3}{6}\right)^5 = -\frac{21}{16}\,a^{19}.$$

Ex.11 If the middle term in the expansion of $\left(x^2+\frac{1}{x}\right)^n$ is $924x^6$, then $n=$

[1] 10 [2] 12

[3] 14 [4] None of these

Sol. [2] Since n is even therefore $\left(\frac{n}{2}+1\right)^{th}$ term is middle term, hence $^nC_{n/2}\,(x^2)^{n/2}\left(\frac{1}{x}\right)^{n/2} = 924x^6$

$\Rightarrow x^{n/2} = x^6 \quad \Rightarrow n = 12.$

Ex. 12 The greatest coefficient in the expansion of $(1+x)^{2n}$ is

[1] $\frac{1.3.5.....(2n-1)}{n!}.2^n$ [2] $^{2n}C_{n-1}$

[3] $^{2n}C_{n+1}$ [4] None of these

Sol. [1] The greatest coefficient

= the coefficient of the middle term

$= {}^{2n}C_n = \frac{1.3.5.....(2n-1)}{n!}.2^n$

To Determine a Particular Term in the Expansion

In the expansion of $\left(x^{\alpha}\pm\frac{1}{x^{\beta}}\right)^n$, if x^m occurs in T_{r+1}, then r is given by

$n\alpha - r(\alpha+\beta) = m \quad \Rightarrow \quad r = \frac{n\alpha - m}{\alpha+\beta}$

Thus in above expansion if constant term i.e., the term which is independent of x, occurs in T_{r+1} then r is determined by

$n\alpha - r(\alpha+\beta) = 0 \quad \Rightarrow \quad r = \frac{n\alpha}{\alpha+\beta}$

Solved Examples

Ex.13 The term independent of x in the expansion of $\left(\frac{4}{3}x^2-\frac{3}{2x}\right)^9$ is –

(1) 5^{th} (2) 6^{th}

(3) 7^{th} (4) 8^{th}

Sol. Here comparing $\left(\frac{4}{3}x^2 - \frac{3}{2x}\right)^9$ with $\left(x^\alpha \pm \frac{1}{x^\beta}\right)^n$

We get $\alpha = 2$, $\beta = 1$, $n = 9$

$r = \frac{9(2)}{2+1} = 6$

$\therefore (6+1) = 7^{th}$ term is independent of x .

Ex.14 The co-efficient of x^{39} in the expansion of $(x^4 - 1/x^3)^{15}$ is

(1) 455 (2) –455

(3) 105 (4) None of these

Sol. From above formula

$r = \frac{15(4) - 39}{4+3} = 3$

$\therefore$ The required term $= T_4 = {}^{15}C_3 (x^4)^{12} (-1/x^3)^3$

$= -455\, x^{39}$

$\therefore$ coefficient of $x^{39} = -455$

Numerically greatest term in the expansion of (a + b)ⁿ, n Î N

Binomial expansion of $(a+b)^n$ is as follows :–

$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots\ldots + {}^nC_n\, a^0 b^n$

If we put certain values of a and b in RHS, then each term of binomial expansion will have certain value. The term having numerically greatest value is said to be numerically greatest term.

Let T_r and T_{r+1} be the r^{th} and $(r+1)^{th}$ terms respectively

$T_r = {}^nC_{r-1}\, a^{n-(r-1)}\, b^{r-1}$

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Now, $\left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}^nC_{r-1}} \frac{a^{n-r}\, b^r}{a^{n-r+1} b^{r-1}}\right| = \frac{n-r+1}{r} \cdot \left|\frac{b}{a}\right|$

Consider $\left|\frac{T_{r+1}}{T_r}\right| \ge 1$

$\left(\frac{n-r+1}{r}\right)\left|\frac{b}{a}\right| \ge 1$

$\frac{n+1}{r} - 1 \ge \left|\frac{a}{b}\right|$ $\qquad r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$

So greatest term will be T_{r+1} where $r = \left[\frac{n+1}{1 + \left|\frac{a}{x}\right|}\right]$

[.] denotes greatest integer function.

Note : If $\frac{n+1}{1+\left|\frac{a}{x}\right|}$ itself is a natural number, then $T_r = T_{r+1}$ and both the terms are numerically greatest.

Solved Examples

Ex.15 If the sum of the coefficients in expansion of $(1+2x)^n$ is 6561, the greatest term in the expansion for $x = \frac{1}{2}$ is

[1] 4^{th} [2] 5^{th} [3] 6^{th} [4] 7^{th}

Sol. [2]

Sum of the coefficient in the expansion of $(1+2x)^n = 6561$

$\Rightarrow (1+2x)^n = 6561$, when x = 1

$\Rightarrow 3^n = 6561 \Rightarrow 3^n = 3^8 \Rightarrow n = 8$

Now, $\frac{T_{r+1}}{T_r} = \frac{{}^8C_r (2x)^r}{{}^8C_{r-1} (2x)^{r-1}} = \frac{9-r}{r} \times 2x$

$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{9-r}{r} \quad \left[\because x = \frac{1}{2}\right]$

$\therefore \frac{T_{r+1}}{T_r} > 1 \Rightarrow \frac{9-r}{r} > 1 \Rightarrow 9 - r > r \Rightarrow 2r < 9 \Rightarrow r < 4\frac{1}{2}$

Hence, 5^{th} term is the greatest term.

Ex.16 The greatest term in the expansion of $(1+\sqrt{2})^{12}$ is -

[1] 7th [2] 8th [3] 6th [4] 5th

Sol. [2]

$\frac{T_r + 1}{T_r} = \frac{12 - r + 1}{r}(\sqrt{2}) \ge 1$

$13 - r \ge \frac{r}{\sqrt{2}} \qquad \Rightarrow \qquad 13 \ge \left(\frac{\sqrt{2}+1}{\sqrt{2}}\right) r$

$r \le \left(\frac{13\sqrt{2}}{\sqrt{2}+1}\right) \qquad \Rightarrow \qquad r \le 13\sqrt{2}\,(\sqrt{2} - 1)$

$r \le 13\,(2 - \sqrt{2})$

$r \le 13\,(0.586)$

$r \le 7.616 \quad \Rightarrow \quad r = 7$

So, 8th term is greatest.

Ex.17 Find the numerically greatest term in the expansion of $(3-5x)^{15}$ when $x = \frac{1}{5}$.

Sol. Let r^{th} and $(r+1)^{th}$ be two consecutive terms in the expansion of $(3-5x)^{15}$

$T_{r+1} \geq T_r$

${}^{15}C_r\, 3^{15-r}\,(|-5x|)^r \geq {}^{15}C_{r-1}\, 3^{15-(r-1)}\,(|-5x|)^{r-1}$

$\frac{(15)!}{(15-r)!\,r!}\,|-5x| \geq \frac{3.(15)!}{(16-r)!\,(r-1)!}$

$5 \,.\, \frac{1}{5}\,(16-r) \geq 3r$

$16 - r \geq 3r$

$4r \leq 16$

$r \leq 4$

Result :

If $(\sqrt{A}+B)^n = I + f$, where I and n are positive integers, n being odd and $0 < f < 1$, then $(I+f)\,f = k^n$ where $A - B^2 = k > 0$ and $\sqrt{A} - B < 1$. If n is an even integer, then $(I+f)(1-f) = k^n$

Ex.18 If n is positive integer, then prove that the integral part of $(7 + 4\sqrt{3})^n$ is an odd number.

Sol. Let $(7+4\sqrt{3})^n = I + f$ ....(i)

where I & f are its integral and fractional parts respectively.

It means $0 < f < 1$

Now, $0 < 7 - 4\sqrt{3} < 1$

$0 < (7-4\sqrt{3})^n < 1$

Let $(7-4\sqrt{3})^n = f'$ ....(ii)

$\Rightarrow 0 < f' < 1$

Adding (i) and (ii)

$I + f + f' = (7+4\sqrt{3})^n + (7-4\sqrt{3})^n$

$= 2\,[{}^nC_0\, 7^n + {}^nC_2\, 7^{n-2}\,(4\sqrt{3})^2 + ...]$

$I + f + f' =$ even integer

$\Rightarrow$ ($f + f'$ must be an integer)

$0 < f + f' < 2 \quad \Rightarrow \quad f + f' = 1$

$I + 1 =$ even integer therefore I is an odd integer.

Ex.19 Show that the integer just above $(\sqrt{3} + 1)^{2n}$ is divisible by 2^{n+1} for all $n \in N$.

Sol. Let $(\sqrt{3}+1)^{2n} = (4+2\sqrt{3})^n = 2^n\,(2+\sqrt{3})^n = I + f$ ....(i)

where I and f are its integral & fractional parts respectively

$0 < f < 1$.

Now $0 < \sqrt{3} - 1 < 1$

$0 < (\sqrt{3}-1)^{2n} < 1$

Let $(\sqrt{3}-1)^{2n} = (4-2\sqrt{3})^n$

$= 2^n\,(2-\sqrt{3})^n = f'$. ....(ii)

$0 < f' < 1$

adding (i) and (ii)

$I + f + f' = (\sqrt{3}+1)^{2n} + (\sqrt{3}-1)^{2n}$

$= 2^n\,[(2+\sqrt{3})^n + (2-\sqrt{3})^n] = 2.2^n\,[{}^nC_0\, 2^n + {}^nC_2\, 2^{n-2}\,(\sqrt{3})^2 + ...]$

$I + f + f' = 2^{n+1}\,k$ (where k is a positive integer)

$0 < f + f' < 2 \quad \Rightarrow \quad f + f' = 1$

$I + 1 = 2^{n+1}\,k$.

$I + 1$ is the integer just above $(\sqrt{3}+1)^{2n}$ and which is divisible by 2^{n+1}.

Ex20. Show that $9^n + 7$ is divisible by 8, where n is a positive integer.

Sol. $9^n + 7 = (1+8)^n + 7$

$= {}^nC_0 + {}^nC_1\,.\,8 + {}^nC_2\,.\,8^2 + ... + {}^nC_n\, 8^n + 7.$

$= 8.\,C_1 + 8^2.\,C_2 + ... + C_n\,.\,8^n + 8.$

$= 8\lambda$, where λ is a positive integer

Hence, $9^n + 7$ is divisible by 8.

Ex.21 What is the remainder when 5^{99} is divided by 13.

Sol. $5^{99} = 5.5^{98} = 5.\,(25)^{49} = 5\,(26-1)^{49}$

$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1\,(26)^{48} + ... + {}^{49}C_{48}\,(26)^1 - {}^{49}C_{49}\,(26)^0]$

$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1\,(26)^{48} + ... + {}^{49}C_{48}\,(26)^1 - 1]$

$= 5\,[{}^{49}C_0\,(26)^{49} - {}^{49}C_1\,(26)^{48} + ... + {}^{49}C_{48}\,(26)^1 - 13] + 60$

$= 13\,(k) + 52 + 8$ (where k is a positive integer)

$= 13\,(k+4) + 8$

Hence, remainder is 8.

Ex.22 Find the last two digits of the number $(17)^{10}$.

Sol. $(17)^{10} = (289)^5 = (290-1)^5$

$= {}^5C_0\,(290)^5 - {}^5C_1\,(290)^4 + \ldots\ldots + {}^5C_4\,(290)^1 - {}^5C_5\,(290)^0$

$= {}^5C_0\,(290)^5 - {}^5C_1\,.\,(290)^4 + \ldots\ldots{}^5C_3\,(290)^2 + 5 \times 290 - 1$

= A multiple of 1000 + 1449

Hence, last two digits are 49

Note : We can also conclude that last three digits are 449.

Ex.23 Which number is larger $(1.01)^{1000000}$ or 10,000 ?

Sol. By Binomial Theorem

$$\begin{aligned}(1.01)^{1000000} &= (1+0.01)^{1000000}\\ &= 1 + {}^{1000000}C_1\,(0.01) + \text{other positive terms}\\ &= 1 + 1000000 \times 0.01 + \text{other positive terms}\\ &= 1 + 10000 + \text{other positive terms}\end{aligned}$$

Hence $(1.01)^{1000000} > 10,000$

Some standard expansions :

(i) Consider the expansion

$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, y^r = {}^nC_0\, x^n\, y^0 + {}^nC_1\, x^{n-1}\, y^1 + \ldots\ldots + {}^nC_r\, x^{n-r}\, y^r + \ldots\ldots + {}^nC_n\, x^0\, y^n$ ....(i)

(ii) Now replace $y \to -y$ we get

$(x-y)^n = \sum_{r=0}^{n} {}^nC_r\,(-1)^r\, x^{n-r}\, y^r = {}^nC_0\, x^n\, y^0 - {}^nC_1\, x^{n-1}\, y^1 + \ldots + {}^nC_r\,(-1)^r x^{n-r}\, y^r + \ldots + {}^nC_n\,(-1)^n\, x^0\, y^n$ ....(ii)

(iii) Adding (i) & (ii), we get

$(x+y)^n + (x-y)^n = 2[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2} y^2 + \ldots\ldots]$

(iv) Subtracting (ii) from (i), we get

$(x+y)^n - (x-y)^n = 2[{}^nC_1\, x^{n-1}\, y^1 + {}^nC_3\, x^{n-3}\, y^3 + \ldots\ldots]$

Binomial Coefficient & Their Properties

In the expansion of $(1+x)^n$; i.e. $(1+x)^n = {}^nC_0 + {}^nC_1 x + \ldots\ldots + {}^nC_r x^r + \ldots\ldots + {}^nC_n x^n$

The coefficients ${}^nC_0, {}^nC_1, {}^nC_n$ of various powers of x, are called binomial coefficients and they are written as $C_0, C_1, C_2, \ldots\ldots, C_n$

Hence

* $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots\ldots + C_r x^r + \ldots\ldots + C_n x^n$ .....(1)

Where $C_0 = 1,\ C_1 = n,\ C_2 = \dfrac{n(n-1)}{2!}$

* $C_r = \dfrac{n(n-1)\ldots\ldots(n-r+1)}{r!},\ C_n = 1$

Now, we shall obtain some important expressions involving binomial coefficients-

* **sum of coefficient :** putting x = 1 in (1), we get

$C_0 + C_1 + C_2 + \ldots\ldots + C_n = 2^n$ .....(2)

* **Sum of coefficients with alternate signs :** putting $x = -1$ in (1)

We get $C_0 - C_1 + C_2 - C_3 + \ldots\ldots = 0$ .....(3)

* **Sum of coefficients of even and odd terms :** from (3) , we have

$C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots$ (4)

i.e. sum of coefficients of even and odd terms are equal.

from (2) and (4)

$\Rightarrow C_0 + C_2 + \ldots\ldots = C_1 + C_3 + \ldots\ldots = 2^{n-1}$

* **Sum of products of coefficients :** Replacing x by 1/x in (1) We get

$\left(1+\dfrac{1}{x}\right)^n = C_0 + \dfrac{C_1}{x} + \dfrac{C_2}{x^2} + \ldots\ldots \dfrac{C_n}{x^n} + \ldots\ldots$ .....(5)

Multiplying (1) by (5), we get

$\dfrac{(1+x)^{2n}}{x^n} = (C_0 + C_1 x + C_2 x^2 + \ldots\ldots)\left(C_0 + \dfrac{C_1}{x} + \dfrac{C_2}{x^2} + \ldots\ldots\right)$

Now, comparing coefficients of x^r on both the sides, we get

$C_0 C_r + C_1 C_{r+1} + \ldots\ldots + C_{n-r} C_n = {}^{2n}C_{n-r}$

$= \dfrac{2n!}{(n+r)!\,(n-r)!}$ .....(6)

* **Sum of squares of coefficients :** putting $r = 0$ in (6) we get $C_0^{\,2} + C_1^{\,2} + C_2^{\,2} + ...+ C_n^{\,2} = \frac{2n!}{n!\,n!}$

* putting $r = 1$ in (6), we get

$C_0\,C_1 + C_1\,C_2 + C_2\,C_3 + ...+ C_{n-1}\,C_n = {}^{2n}C_{n-1}$

$= \frac{2n!}{(n+1)!(n-1)!}$ .....(7)

* putting $r = 2$ in (6), we get

$C_0\,C_2 + C_1\,C_3 + C_2\,C_4 + ...+ C_{n-2}\,C_n = {}^{2n}C_{n-2}$

$= \frac{2n!}{(n+2)!(n-2)!}$ .....(8)

* Differentiating both sides of (1) w.r.t. x, we get

$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ...+ nC_n x^{n-1}$

Now putting $x = 1$ and $x = -1$ respectively

$C_1 + 2C_2 + 3C_3 + ...+ nC_n = n.2^{n-1}$ .....(9)

and $C_1 - 2C_2 + 3C_3 - .... = 0$ .....(10)

* adding (2) and (9)

$C_0 + 2C_1 + 3C_2 + ...+ {}^{(n+1)}C_n = 2^{n-1}\,(n+2)$ .....(11)

* Integrating (1) w.r.t. x between the limits 0 to 1, we get,

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + ... + \frac{C_nx^{n+1}}{n+1}\right]_0^1$$

$$\Rightarrow C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1} \quad ....(12)$$

* Integrating (1) w.r.t. x between the limits -1 to 0, we get,

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^0 = \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + ... + \frac{C_nx^{n+1}}{n+1}\right]_{-1}^0$$

$$\Rightarrow C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + ... + \frac{(-1)^n.C_n}{n+1} = \frac{1}{(n+1)} \quad ....(13)$$

Solved Examples

Ex.24 If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + c_nx^n$, then show that

(i) $C_0 + 3C_1 + 3^2C_2 + ... + 3^n\,C_n = 4^n$.

(ii) $C_0 + 2C_1 + 3.\,C_2 + ... + (n + 1)\,C_n = 2^{n-1}\,(n+2)$.

(iii) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + ... + (-1)^n\,\frac{C_n}{n+1} = \frac{1}{n+1}$.

Sol.

(i) $(1 + x)^n = C_0 + C_1\,x + C_2x^2 + ... + C_nx^n$

put $x = 3$

$C_0 + 3\,.\,C_1 + 3^2\,.\,C_2 + ... + 3^n\,.\,C_n = \mathbf{4^n}$

(ii) **I. <u>Method : By Summation</u>**

L.H.S. $= {}^nC_0 + 2.\,{}^nC_1 + 3\,.\,{}^nC_2 + ... + (n + 1).\,{}^nC_n$.

$= \sum_{r=0}^{n}(r+1)\,.\,{}^nC_r = \sum_{r=0}^{n} r.\,{}^nC_r + \sum_{r=0}^{n}{}^nC_r$

$= n\sum_{r=0}^{n}{}^{n-1}C_{r-1} + \sum_{r=0}^{n}{}^nC_r = n\,.\,2^{n-1} + 2^n = 2^{n-1}\,(n+2)$.

RHS

II. <u>Method : By Differentiation</u>

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

Multiplying both sides by x,

$x(1 + x)^n = C_0x + C_1x^2 + C_2x^3 + ... + C_n\,x^{n+1}$.

Differentiating both sides

$(1 + x)^n + x\,n\,(1 + x)^{n-1} = C_0 + 2.\,C_1x + 3\,.\,C_2x^2 + ... + (n + 1)C_nx^n$.

putting $x = 1$, we get

$C_0 + 2.C_1 + 3\,.\,C_2 + ... + (n + 1)\,C_n = 2^n + n\,.\,2^{n-1}$

$C_0 + 2.C_1 + 3\,.\,C_2 + ... + (n + 1)\,C_n = 2^{n-1}\,(n + 2)$

Proved

(iii) **I <u>Method : By Summation</u>**

L.H.S. $= C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + ... + (-1)^n.\,\frac{C_n}{n+1}$

$= \sum_{r=0}^{n}(-1)^r\,.\,\frac{{}^nC_r}{r+1}$

$= \frac{1}{n+1}\sum_{r=0}^{n}(-1)^r\,.\,{}^{n+1}C_{r+1}\quad \left\{\frac{n+1}{r+1}.\,{}^nC_r = {}^{n+1}C_{r+1}\right\}$

$= \frac{1}{n+1}\,[{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - ....+ (-1)^n\,.\,{}^{n+1}C_{n+1}]$

$= \frac{1}{n+1}\,[-{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + ... + (-1)^n\,.\,{}^{n+1}C_{n+1} + {}^{n+1}C_0]$

$= \frac{1}{n+1} =$ R.H.S. , since

$\left\{-{}^{n+1}C_0 + {}^{n+1}C_1 - {}^{n+1}C_2 + ... + (-1)^n\;{}^{n+1}C_{n+1} = 0\right\}$

II Method : By Integration

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$.

Integrating both sides, within the limits – 1 to 0.

$$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^{0}$$

$$= \left[C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + ... + C_n\frac{x^{n+1}}{n+1}\right]_{-1}^{0}$$

$$\frac{1}{n+1} - 0 = 0 -$$

$$\left[-C_0 + \frac{C_1}{2} - \frac{C_2}{3} + ... + (-1)^{n+1}\frac{C_n}{n+1}\right]$$

$$C_0 - \frac{C_1}{2} + \frac{C_2}{3} - .... + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$$

Proved

Ex.25 If C_0, C_1, C_2, ...C_n are binomial coefficients then

$\frac{1}{n!0!} + \frac{1}{(n-1)!} + \frac{1}{(n-2)!2!} + ... + \frac{1}{0!n!}$ is equal to-

[1] 2^n [2] $\frac{2^{n-1}}{n!}$

[3] $\frac{2^n}{n!}$ [4] none of these

Sol. [3]

$C_0 + C_1 + C_2 + ... + C_n = 2^n$

$$\frac{n!}{(n-1)!} + \frac{n!}{(n-1)!1!} + \frac{n!}{(n-1)!2!} + ... + \frac{n!}{0!n!} = \frac{2^n}{n!}$$

(divide it by n !)

Ex.26 If C_0, C_1, C_2 ....C_n are binomial coefficients, then $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... \frac{C_n}{n+1}$ is equal to-

[1] $\frac{2^{n+1}+1}{n+1}$ [2] $\frac{2^{n+1}-1}{n+1}$

[3] $\frac{2^{n+1}}{n+1}$ [4] none of these

Sol. [2]

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+C_nx^n$

Integrate it with respect to x

$$\frac{(1+x)^{n+1}}{n+1} = C_0x + \frac{C_1x^2}{2} + C_2\frac{x^3}{3} + ... + C_n\frac{x^{n+1}}{n+1} + K$$

Put x = 0 then, $K = \frac{1}{n+1}$, Now put x = 1

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{(n+1)}$$

Ex.27 If $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$, then prove that

(i) $C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$

(ii) $C_0C_2 + C_1C_3 + C_2C_4 + ... + C_{n-2}C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$

(iii) $1. C_0^2 + 3 . C_1^2 + 5. C_2^2 + ... + (2n + 1) . C_n^2 . = 2n. {}^{2n-1}C_n + {}^{2n}C_n$.

Sol.

(i) $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_n x^n$. ....(i)

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_nx^0$....(ii)

Multiplying (i) and (ii)

$(C_0 + C_1x + C_2x^2 + ... + C_nx^n)(C_0x^n + C_1x^{n-1} + ... + C_nx^0) = (1 + x)^{2n}$

Comparing coefficient of x^n,

$C_0^2 + C_1^2 + C_2^2 + ... + C_n^2 = {}^{2n}C_n$

(ii) From the product of (i) and (ii) comparing coefficients of x^{n-2} or x^{n+2} both sides,

$C_0C_2 + C_1C_3 + C_2C_4 + ... + C_{n-2}C_n = {}^{2n}C_{n-2}$ or ${}^{2n}C_{n+2}$.

(iii) **I Method : By Summation**

L.H.S. $= 1. C_0^2 + 3. C_1^2 + 5. C_2^2 + ... + (2n + 1) C_n^2$.

$$= \sum_{r=0}^{n}(2r+1)\,{}^nC_r^2 = \sum_{r=0}^{n}2.r\,.\,({}^nC_r)^2 + \sum_{r=0}^{n}({}^nC_r)^2$$

$$= 2\sum_{r=1}^{n}.\,n\,.\,{}^{n-1}C_{r-1}\,{}^nC_r + {}^{2n}C_n$$

$(1 + x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + ...{}^nC_n x^n$ ....(i)

$(x + 1)^{n-1} = {}^{n-1}C_0 x^{n-1} + {}^{n-1}C_1 x^{n-2} + ...+ {}^{n-1}C_{n-1}x^0$ ....(ii)

Multiplying (i) and (ii) and comparing coeffcients of x^n.

${}^{n-1}C_0 . {}^nC_1 + {}^{n-1}C_1 . {}^nC_2 + ... + {}^{n-1}C_{n-1} . {}^nC_n = {}^{2n-1}C_n$

$$\sum_{r=0}^{n}{}^{n-1}C_{r-1}\,.\,{}^nC_r = {}^{2n-1}C_n$$

Hence, required summation is $2n. {}^{2n-1}C_n + {}^{2n}C_n$ = R.H.S.

II Method : By Differentiation

$(1 + x^2)^n = C_0 + C_1x^2 + C_2x^4 + C_3x^6 + ...+ C_n\, x^{2n}$

Multiplying both sides by x

$x(1 + x^2)^n = C_0x + C_1x^3 + C_2x^5 + ... + C_nx^{2n+1}$.

Differentiating both sides

$x . n (1 + x^2)^{n-1} . 2x + (1 + x^2)^n = C_0 + 3. C_1x^2 + 5. C_2\, x^4 + ... + (2n + 1)\, C_n\, x^{2n}$ ....(i)

$(x^2 + 1)^n = C_0\, x^{2n} + C_1\, x^{2n-2} + C_2\, x^{2n-4} + ... + C_n$ ....(ii)

Multiplying (i) & (ii)

$(C_0 + 3C_1x^2 + 5C_2x^4 + ... + (2n + 1)\, C_n\, x^{2n})\,(C_0\, x^{2n} + C_1x^{2n-2} + ... + C_n)$

$= 2n\, x^2\, (1 + x^2)^{2n-1} + (1 + x^2)^{2n}$

comparing coefficient of x^{2n},

$C_0^{\,2} + 3C_1^{\,2} + 5C_2^{\,2} + ...+ (2n + 1)\, C_n^{\,2} = 2n\, .\, {}^{2n-1}C_{n-1} + {}^{2n}C_n$.

$C_0^{\,2} + 3C_1^{\,2} + 5C_2^{\,2} + ...+ (2n + 1)\, C_n^{\,2} = 2n\, .\, {}^{2n-1}C_n + {}^{2n}C_n$. Proved

Ex.28 The value of

$3\, {}^nC_0 - 8\, {}^nC_1 + 13\, {}^nC_2 - 18\, {}^nC_3 + ... +$ n terms is -

[1] 0 [2] 3^n

[3] 5^n [4] none of these

Sol. [1]

$3\, {}^nC_0 - 8\, {}^nC_1 + 13\, {}^nC_2 - 18\, {}^nC_3 + ...$
$3\, ({}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + ...) - 5\, ({}^nC_1 - 2\, {}^nC_2 + 3\, {}^nC_3 - ....)$

$\Rightarrow\ 3(0) - 5\,(0) = 0$

Ex.29 If C_r stands for ${}^nC_r$, then the value of $2C_0 - \frac{2^2}{2}C_1 + \frac{2^3}{3}\, C_2 + ... + (-1)^{n+1}\, \frac{2^{n+1}}{(n+1)}$ is equal to-

[1] 0, if n is even [2] 0, if n is odd

[3] $\frac{1}{n+1}$, if n is even [4] $\frac{1}{n+1}$, if n is odd

Sol. [2] $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...+ C_nx^n$

$$\frac{(1+x)^{n+1}-1}{n+1} = C_0x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + ... + C_n\frac{x^{n+1}}{n+1}$$

Put x = –2 $\quad \frac{(-1)^{n+1}-1}{n+1}$

$$= -2C_0 + C_1\frac{2^2}{2} - C_2\frac{2^3}{3} + ... + C_n\frac{(-2)^{n+1}}{(n+1)}$$

If n is odd, then L.H.S. = 0.

Ex.30 Find the summation of the following series –

(i) ${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^nC_m$

(ii) ${}^nC_3 + 2\, .\, {}^{n+1}C_3 + 3.\, {}^{n+2}C_3 + ... + n\, .\, {}^{2n-1}C_3$

Sol.

(i) **I Method :** Using property, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^nC_m$

$= \underbrace{{}^{m+1}C_{m+1} + {}^{m+1}C_m} + {}^{m+2}C_m + ... + {}^nC_m$

$\{\because {}^mC_m = {}^{m+1}C_{m+1}\}$

$= \underbrace{{}^{m+2}C_{m+1} + {}^{m+2}C_m} + ... + {}^nC_m$

$= {}^{m+3}C_{m+1} + ... + {}^nC_m = {}^nC_{m+1} + {}^nC_m = \mathbf{{}^{n+1}C_{m+1}}$

II Method

${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + ... + {}^nC_m$

The above series can be obtained by writing the coefficient of x^m in

$(1 + x)^m + (1 + x)^{m+1} + ... + (1 + x)^n$

Let $S = (1 + x)^m + (1 + x)^{m+1} + ... + (1 + x)^n$

$$= \frac{(1+x)^m\left[(1+x)^{n-m+1}-1\right]}{x} = \frac{(1+x)^{n+1}-(1+x)^m}{x}$$

$=$ coefficient of x^m in $\frac{(1+x)^{n+1}}{x} - \frac{(1+x)^m}{x} = {}^{n+1}C_{m+1} + 0 = {}^{n+1}C_{m+1}$

(ii) ${}^nC_3 + 2\, .\, {}^{n+1}C_3 + 3\, .\, {}^{n+2}C_3 + ... + n\, .\, {}^{2n-1}C_3$

The above series can be obatined by writing the coefficient of x^3 in

$(1 + x)^n + 2\, .\, (1 + x)^{n+1} + 3\, .\, (1 + x)^{n+2} + ... + n\, .\, (1 + x)^{2n-1}$

Let $S = (1 + x)^n + 2\, .\, (1 + x)^{n+1} + 3.\, (1 + x)^{n+2} + ... + n\, (1 + x)^{2n-1}$ .....(i)

$(1 + x)S = (1 + x)^{n+1} + 2\, (1 + x)^{n+2} + ... + (n - 1)\, (1 + x)^{2n-1} + n(1 + x)^{2n}$ ....(ii)

Subtracting (ii) from (i)

$-xS = (1+x)^n + (1+x)^{n+1} + (1+x)^{n+2} + \ldots\ldots\ldots\ldots + (1+x)^{2n-1} - n(1+x)^{2n}$

$= \dfrac{(1+x)^n\left[(1+x)^n - 1\right]}{x} - n\,(1+x)^{2n}$

$S = \dfrac{-(1+x)^{2n} + (1+x)^n}{x^2} + \dfrac{n(1+x)^{2n}}{x}$

$x^3 : S$ (coefficient of x^3 in S)

$x^3 : \dfrac{-(1+x)^{2n} + (1+x)^n}{x^2} + \dfrac{n(1+x)^{2n}}{x}$

Hence, required summation of the series is $-{}^{2n}C_5 + {}^{n}C_5 + n \,.\, {}^{2n}C_4$

Ex.31 The value of ${}^{4n}C_0 + {}^{4n}C_4 + {}^{4n}C_8 + \ldots\ldots\ldots + {}^{4n}C_{4n}$ is-

[1] 2^{4n} [2] 2^{4n-2}

[3] $2^{2n-1} + 2^{4n-2}$ [4] $2^{2n-1}(-1)^n + 2^{4n-2}$

Sol. [4]

$(1+x)^{4n} = {}^{4n}C_0 + {}^{4n}C_1 x + {}^{4n}C_2 x^2 + {}^{4n}C_3 x^3 + {}^{4n}C_4 x^4 + \ldots\ldots\ldots + {}^{4n}C_n x^{4n}$

Put x = 1 and x = –1, then adding.

$2^{4n-1} = {}^{4n}C_0 + {}^{4n}C_2 + {}^{4n}C_4 + \ldots\ldots. + {}^{4n}C_{4n}$ ......(i)

Now put, x = i

$(1+i)^{4n} = {}^{4n}C_0 + {}^{4n}C_1 i - {}^{4n}C_2 - {}^{4n}C_3 i + {}^{4n}C_4 + \ldots\ldots. + {}^{4n}C_{4n}$

Compare real and imaginary part, we get

$(-1)^n\,(2)^{2n} = {}^{4n}C_0 - {}^{4n}C_2 + {}^{4n}C_2 + {}^{4n}C_4 - {}^{4n}C_6 + \ldots. + {}^{4n}C_{4n}$ ......(ii)

Adding (i) and (ii), we get

$\Rightarrow {}^{4n}C_0 + {}^{4n}C_4 + \ldots.. + {}^{4n}C_{4n} = (-1)^n\,(2)^{2n-1} + 2^{4n-2}$

Ex.32 $aC_0 + (a+b)\,C_1 + (a+2b)\,C_2 + \ldots\ldots. + (a+nb)\,C_n$ is equal to-

[1] $(2a+nb)\,2^n$ [2] $(2a+nb)\,2^{n-1}$

[3] $(na+2b)\,2^n$ [4] $(na+2b)\,2^{n-1}$

Sol. [2]

$a\,(C_0 + C_1 + C_2 + \ldots\ldots\ldots + C_n) + b(C_1 + 2C_2 + \ldots\ldots + {}^{n}C_n)$

$= a.2^n + b.\,n\,2^{n-1} = 2^n\left(\dfrac{2a+nb}{2}\right) = (2a+nb)\,2^{n-1}$

Multinomial theorem :

As we know the Binomial Theorem –

$(x+y)^n = \sum_{r=0}^{n} {}^{n}C_r\; x^{n-r}\,y^r = \sum_{r=0}^{n} \dfrac{n!}{(n-r)!\,r!}\; x^{n-r}\,y^r$

putting $n - r = r_1$, $r = r_2$

therefore, $(x+y)^n = \sum_{r_1+r_2=n} \dfrac{n!}{r_1!\;r_2!}\; x^{r_1}\,.\,y^{r_2}$

Total number of terms in the expansion of $(x+y)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 = n$ i.e. ${}^{n+2-1}C_{2-1} = {}^{n+1}C_1 = n+1$

In the same fashion we can write the multinomial theorem

$(x_1 + x_2 + x_3 + \ldots\ldots\ldots\ldots\, x_k)^n$

$= \sum_{r_1+r_2+\ldots+r_k=n} \dfrac{n!}{r_1!\,r_2!\ldots\,r_k!}\; x_1^{r_1}\,.\,x_2^{r_2}\ldots x_k^{r_k}$

Here total number of terms in the expansion of $(x_1 + x_2 + \ldots\ldots\ldots. + x_k)^n$ is equal to number of non-negative integral solution of $r_1 + r_2 + \ldots\ldots. + r_k = n$ i.e. ${}^{n+k-1}C_{k-1}$

Solved Examples

Ex.33 Find the coefficient of $a^2\,b^3\,c^4\,d$ in the expansion of $(a - b - c + d)^{10}$

Sol. $(a - b - c + d)^{10} = \sum_{r_1+r_2+r_3+r_4=10} \dfrac{(10)!}{r_1!\,r_2!\,r_3!\,r_4!}\,(a)^{r_1}\,(-b)^{r_2}\,(-c)^{r_3}\,(d)^{r_4}$

we want to get $a^2\,b^3\,c^4\,d$ this implies that $r_1 = 2$, $r_2 = 3$, $r_3 = 4$, $r_4 = 1$

$\therefore$ coeff. of $a^2\,b^3\,c^4\,d$ is $\dfrac{(10)!}{2!\,3!\,4!\,1!}\,(-1)^3\,(-1)^4$

$= -12600$

Ex.34 In the expansion of $\left(1 + x + \dfrac{7}{x}\right)^{11}$, find the term independent of x.

Sol. $\left(1 + x + \dfrac{7}{x}\right)^{11} = \sum_{r_1+r_2+r_3=11} \dfrac{(11)!}{r_1!\,r_2!\,r_3!}\,(1)^{r_1}\,(x)^{r_2}\left(\dfrac{7}{x}\right)^{r_3}$

The exponent 11 is to be divided among the base variables 1, x and $\dfrac{7}{x}$ in such a way so that we get x^0.

Therefore, possible set of values of (r_1, r_2, r_3) are $(11, 0, 0)$, $(9, 1, 1)$, $(7, 2, 2)$, $(5, 3, 3)$, $(3, 4, 4)$, $(1, 5, 5)$

Hence the required term is

$$\frac{(11)!}{(11)!}(7^0) + \frac{(11)!}{9!\,1!\,1!}\,7^1 + \frac{(11)!}{7!\,2!\,2!}\,7^2 + \frac{(11)!}{5!\,3!\,3!}\,7^3 + \frac{(11)!}{3!\,4!\,4!}\,7^4 + \frac{(11)!}{1!\,5!\,5!}\,7^5$$

$$= 1 + \frac{(11)!}{9!\,2!}\cdot\frac{2!}{1!\,1!}\,7^1 + \frac{(11)!}{7!\,4!}\cdot\frac{4!}{2!\,2!}\,7^2 + \frac{(11)!}{5!\,6!}\cdot\frac{6!}{3!\,3!}\,7^3 + \frac{(11)!}{3!\,8!}\cdot\frac{8!}{4!\,4!}\,7^4 + \frac{(11)!}{1!\,10!}\cdot\frac{(10)!}{5!\,5!}\,7^5$$

$$= 1 + {}^{11}C_2\,.\,{}^{2}C_1\,.\,7^1 + {}^{11}C_4\,.\,{}^{4}C_2\,.\,7^2 + {}^{11}C_6\,.\,{}^{6}C_3\,.\,7^3 + {}^{11}C_8\,.\,{}^{8}C_4\,.\,7^4 + {}^{11}C_{10}\,.\,{}^{10}C_5\,.\,7^5$$

$$= 1 + \sum_{r=1}^{5} {}^{11}C_{2r}\,.\,{}^{2r}C_r\,.\,7^r$$

Binomial Theorem for any Index

When n is a negative integer or a fraction then the expansion of a binomial is possible only when

(i) Its first term is 1, and

(ii) Its second term is numerically less than 1.

Thus when $n \notin N$ and $|x| < 1$, then it states

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots + \frac{n(n-1)(n-r+1)}{r!}x^r + \ldots\ldots\ldots\infty$$

1. General Term :

* $T_{r+1} = \dfrac{n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)}{r!}x^r$

Note :

(i) In this expansion the coefficient of different terms can not be expressed as ${}^nC_0, {}^nC_1, {}^nC_2\ldots\ldots$ because n is not a positive integer.

(ii) In this case there are infinite terms in the expansion.

2. Some Important Expansions :

If $|x| < 1$ and $n \in Q$ but $n \notin N$, then

(a) $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots\ldots + \dfrac{n(n-1)\ldots\ldots(n-r+1)}{r!}x^r + \ldots.$

(b) $(1-x)^n = 1 - nx + \dfrac{n(n-1)}{2!}x^2 - \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots. + \dfrac{n(n-1)\ldots..(n-r+1)}{r!}(-x)^r + \ldots..$

(c) $(1-x)^{-n} = 1 + nx + \dfrac{n(n+1)}{2!}x^2 + \dfrac{n(n+1)(n+2)}{3!}x^3 + \ldots. + \dfrac{n(n+1)\ldots..(n+r-1)}{r!}x^r + \ldots..$

(d) $(1+x)^{-n} = 1 - nx + \dfrac{n(n+1)}{2!}x^2 - \dfrac{n(n+1)(n+2)}{3!}x^3 + \ldots. + \dfrac{n(n+1)\ldots..(n+r-1)}{r!}(-x)^r + \ldots..$

By putting n = 1, 2, 3 in the above results (c) and (d), we get the following results-

* $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots + x^r + \ldots\ldots.$

General term $T_{r+1} = x^r$

* $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots (-x)^r + \ldots\ldots$

General term $T_{r+1} = (-x)^r$

* $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots.. + (r+1)x^r + \ldots..$

General term $T_{r+1} = (r+1)x^r$

* $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots. + (r+1)(-x)^r + \ldots\ldots$

General term $T_{r+1} = (r+1)(-x)^r$

* $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots + \dfrac{(r+1)(r+2)}{2!}x^r + \ldots.$

General term $= \dfrac{(r+1)(r+2)}{2!}x^r$

* $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots\ldots\ldots + \dfrac{(r+1)(r+2)}{2!}(-x)^r + \ldots\ldots..$

General term $= \dfrac{(r+1)(r+2)}{2!}(-x)^r$

Solved Examples

Ex.35 If x is very large and n is a negative integer or a proper fraction, then an approximate value of $\left(\frac{1+x}{x}\right)^n$ is equal to-

[1] $1 + \frac{x}{n}$ [2] $1 + \frac{n}{x}$

[3] $1 + \frac{1}{x}$ [4] $n\left(1+\frac{1}{x}\right)$

Sol. [2] $\left(1+\frac{1}{x}\right)^n = 1+\frac{n}{x}+\frac{n(n-1)}{1.2}\left(\frac{1}{x}\right)^2+\ldots\ldots$

but as x is very large terms therefore after 2nd term other terms can be ignored.

Ex.36 If $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{\sqrt{4-x}}$ is approximately equal to $a+bx$ for small values of x, then $(a,b) =$

[1] $\left(1,\frac{35}{24}\right)$ [2] $\left(1,-\frac{35}{24}\right)$

[3] $\left(2,\frac{35}{12}\right)$ [4] $\left(2,-\frac{35}{12}\right)$

Sol. [2] $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{2\left[1-\frac{x}{4}\right]^{1/2}}$

$$= \frac{\left[1+\frac{1}{2}(-3x)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}(-3x)^2+\ldots\ldots\right]+\left[1+\frac{5}{3}(-x)+\frac{5}{3}\frac{2}{3}\frac{1}{2}(-x)^2+\ldots\ldots\right]}{2\left[1+\frac{1}{2}\left(-\frac{x}{4}\right)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}\left(-\frac{x}{4}\right)^2+\ldots\ldots\right]}$$

$$= \frac{\left[1-\frac{19}{12}x+\frac{53}{144}x^2-\ldots\ldots\right]}{\left[1-\frac{x}{2}-\frac{1}{8}x^2-\ldots\ldots\right]} = 1-\frac{35}{24}x+\ldots\ldots$$

Neglecting higher powers of x, then

$a + bx = 1 - \frac{35}{24}x \quad \Rightarrow \quad a = 1,\ b = -\frac{35}{24}$.

Ex.37 Prove that the coefficient of x^r in $(1-x)^{-n}$ is ${}^{n+r-1}C_r$

Sol. $(r + 1)^{th}$ term in the expansion of $(1 - x)^{-n}$ can be written as

$$T_{r+1} = \frac{-n(-n-1)(-n-2)\ldots\ldots(-n-r+1)}{r!}(-x)^r$$

$$= (-1)^r\frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{r!}(-x)^r$$

$$= \frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{r!}x^r$$

$$= \frac{(n-1)!\,n(n+1)\ldots\ldots(n+r-1)}{(n-1)!\,r!}x^r$$

Hence, coefficient of x^r is $\frac{(n+r-1)!}{(n-1)!\,r!} = {}^{n+r-1}C_r$ Proved

Ex.38 If x is so small such that its square and higher powers may be neglected, then find the value of $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$

Sol. $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}}$

$$= \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2}$$

$$= \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right) = \frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right)$$

$$= 1 - \frac{x}{8} - \frac{19}{12}x = 1 - \frac{41}{24}x$$