COMPLEX NUMBERS

PRELIMINARY

(i) The numbers of the form $x + iy$ are known as complex numbers, where $x, y \in R$ and $i = \sqrt{-1}$.

i is an imaginary unit and it is known as **iota**.

(ii) Complex numbers are denoted by z . Let $z = x + iy$, then $Re(z) = x$ and $Im(z) = y$

(iii) $i^2 = -1$, $i^3 = -i$, $i^4 = 1$,

$i^{4n} = 1$, $i^{4n+1} = i, i^{4n+2} = i^2 = -1$, $i^{4n+3} = i^3 = -i$

(iv) The sum of four consecutive powers of i is always zero, i. e. $i^{4n} + i^{4n+1} + i^{4n+2} + i^{4n+3} = 0$

Note :

(i) $i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{1}$

(ii) $\sqrt{-x} \times \sqrt{-y} \neq \sqrt{xy}$

So for two real numbers x and y, $\sqrt{x \times y} = \sqrt{x} \times \sqrt{y}$ possible if both x, y are non-negative.

(iii) ' i ' is neither positive, zero nor negative, Due to this reason order relations are not defined for imaginary numbers.

Solved Examples

Ex.1 If n is a positive integer, then which of the following relations is false

(A) $i^{4n} = 1$ (B) $i^{4n-1} = i$

(C) $i^{4n+1} = i$ (D) $i^{-4n} = 1$

Sol. We know that $i^2 = -1$

$\Rightarrow (i^2)^2 = (-1)^2 = 1$

$\Rightarrow i^{4n} = 1^n$ and therefore $i^{4n-1} = -i$

Ans.(B)

Ex.2 If $i^2 = -1$, then the value of $\sum_{n=1}^{200} i^n$ is

(A) 50 (B) –50

(C) 0 (D) 100

Sol. $\sum_{n=1}^{200} i^n = i + i^2 + i^3 + ... + i^{200} = \frac{i(1-i^{200})}{1-i}$

(since G.P.)

$= \frac{i(1-1)}{1-i} = 0$ **Ans.(C)**

Ex.3 The value of $\frac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1$ is

(A) 2 (B) -2

(C) 1 (D) -1

Sol. Given expression

$$=\frac{i^{10}\left(i^{582}+i^{580}+i^{578}+i^{576}+i^{574}\right)}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1$$

$$=i^{10}-1=\left(i^2\right)^5-1=\left(-1\right)^5-1 \quad =-1-1=-2$$

Ans.(C)

Ex.4 Find the value of $[i]^{198}$

Sol. $[i]^{198}=[i^2]^{99}=[-1]^{99}=-1$

Ex.5 Find the value of $i^n+i^{n+1}+i^{n+2}+i^{n+3}$

Sol. $i^n+i^{n+1}+i^{n+2}+i^{n+3}$

$= \quad i^n[1+i+i^2+i^3]$

$= \quad i^n[1+i-1-i] = i^n[0]=0$

Ex.6 The sum of series $i^2+i^4+i^6+$ ....(2n + 1) terms is -

(A) 0 (B) 1

(C) n (D) – 1

Sol. Given series is a G.P. So, Sum of a G. P. is

$$=\frac{i^2[1-(i^2)^{2n+1}]}{1-i^2} = \frac{(-1)(1-(i)^{4n+2})}{1+1}$$

$$=\frac{(-1)(1+1)}{2} = -1$$

Ans.[D]

COMPLEX NUMBER

A number of the form $z=x+iy$ where $x, y \in R$ and $i=\sqrt{-1}$ is called a complex number where x is called as real part and y is called imaginary part of complex number and they are expressed as

Re (z) = x, Im (z) = y

Here if x = 0 the complex number is purely Imaginary and if y = 0 the complex number is purely Real.

A complex number may also be defined as an ordered pair of real numbers any may be denoted by the symbol (a, b). If we write z = (a, b) then a is called the real part and b the imaginary part of the complex number z.

Note :

(i) Inequalities in complex number are not defined because 'i' is neither positive, zero nor negative so $4+3i<1+2i$ or $i<0$ or $i>0$ is meaning less.

(ii) If two complex numbers are equal, then their real and imaginary parts are separately equal.

Thus if $a+ib=c+id \Rightarrow a=c$ and $b=d$

so if $z=0 \Rightarrow x+iy=0 \Rightarrow x=0$ and $y=0$

The student must note that

$x, y \in R$ and $x, y \neq 0$. Then if

$x+y=0 \Rightarrow x=y$ is correct

but $x+iy=0 \Rightarrow x=-iy$ is incorrect

Hence a real number cannot be equal to the imaginary number, unless both are zero.

(iii) The complex number 0 is purely real and purely imaginary both.

ALGEBRAIC OPERATIONS WITH COMPLEX NUMBERS

(i) **Addition :** $(a+ib)+(c+id)=(a+c)+i(b+d)$

(ii) **Subtraction :** $(a+ib)-(c+id)=(a-c)+i(b-d)$

(iii) **Multiplication :**

$$(a+ib)(c+id)=(ac-bd)+i(ad+bc)$$

(iv) **Division :** $\frac{a+ib}{c+id}=\frac{ac+bd}{c^2+d^2}+\frac{i(bc-ad)}{c^2+d^2}$ $(c+id \neq 0)$

Properties of Algebraic operations with Complex Number

Let z, z_1, z_2 and z_3 are any complex number then their algebraic operation satisfy following properties-

Commutativity : $z_1+z_2=z_2+z_1$ & $z_1 z_2=z_2 z_1$

Associativity : $(z_1+z_2)+z_3=z_1+(z_2+z_3)$ and $(z_1 z_2) z_3=z_1(z_2 z_3)$

Identity element : If O = (0, 0) and 1 = (1, 0) then $z+0=0+z=z$ and $z.1=1.z=z$. Thus 0 and 1 are the identity elements for addition and multiplication respectively.

Inverse element : Additive inverse of z is $-z$ and multiplicative inverse of z is $\frac{1}{z}$.

Cancellation Law :

$$\left.\begin{array}{l} z_1 + z_2 = z_1 + z_3 \\ z_2 + z_1 = z_3 + z_1 \end{array}\right\} \Rightarrow z_2 = z_3$$

and $z_1 \neq 0$ $\left.\begin{array}{l} z_1 z_2 = z_1 z_3 \\ z_2 z_1 = z_3 z_1 \end{array}\right\} \Rightarrow z_2 = z_3$

Distributivity : $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

and $(z_2 + z_3) z_1 = z_2 z_1 + z_3 z_1$

MULTIPLICATIVE INVERSE OF A NON-ZERO COMPLEX NUMBER

Multiplicative inverse of a non-zero complex number $z = x + iy$ is

$$z^{-1} = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

i.e. $z^{-1} = \dfrac{\text{Re}(z)}{|z|^2} + i\dfrac{-\text{Im}(z)}{|z|^2}$

Solved Examples

Ex.7 If $z = -3 + 2i$, then $1/z$ is equal to

(A) $\frac{1}{13}(3+2i)$ (B) $-\frac{1}{13}(3+2i)$

(C) $\frac{1}{\sqrt{13}}(3+2i)$ (D) $-\frac{1}{\sqrt{13}}(3+2i)$

Sol. $z^{-1} = -\frac{3}{13} - i\frac{2}{13} = -\frac{1}{13}(3+2i)$ **Ans.(B)**

EQUALITY OF COMPLEX NUMBERS

Two complex numbers are said to be equal if and only if their real parts and imaginary parts are separately equal i.e.

If $a + ib = c + id$, then $a = c$ & $b = d$

Solved Examples

Ex.8 The values of x and y satisfying the equation $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$ are

(A) $x = -1, y = 3$ (B) $x = 3, y = -1$

(C) $x = 0, y = 1$ (D) $x = 1, y = 0$

Sol. $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$

$\Rightarrow (4 + 2i)x + (9 - 7i)y - 3i - 3 = 10i$

Ans.(B)

Equating real and imaginary parts,

we get $2x - 7y = 13$ and $4x + 9y = 3$.

Hence $x = 3$ and $y = -1$.

Ex.9 If $a + ib = c + id$, then

(A) $a - c = i(b - d)$ (B) $a - ib = c - id$

(C) $a = d, b = c$ (D) none of these

Sol. If $a + ib = c + id$. Equating real and imaginary parts,

we get $a = c$ and $b = d$ $\Rightarrow -b = -d$.

Therefore $a - ib = c - id$ **Ans.(B)**

Ex.10 $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if $\theta =$

(A) $2n\pi \pm \frac{\pi}{3}$ (B) $n\pi + \frac{\pi}{3}$

(C) $n\pi \pm \frac{\pi}{3}$ (D) none of these

Sol. $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ will be purely imaginary, if the real part vanishes, i.e., $\frac{3-4\sin^2\theta}{1+4\sin^2\theta} = 0$

$\Rightarrow 3 - 4\sin^2\theta = 0$ (only if θ be real)

$\Rightarrow \sin\theta = \pm\frac{\sqrt{3}}{2} = \sin\left(\pm\frac{\pi}{3}\right)$

$\Rightarrow \theta = n\pi + (-1)^n\left(\pm\frac{\pi}{3}\right) = n\pi \pm \frac{\pi}{3}$.

Ans.(C)

Ex.11 If $(x + iy)(2 - 3i) = 4 + i$, then-

(A) $x = -14/13, y = 5/13$

(B) $x = 5/13, y = 14/13$

(C) $x = 14/13, y = 5/13$

(D) $x = 5/13, y = -14/13$

Sol. $x + iy = \frac{4+i}{2-3i} = \frac{(4+i)(2+3i)}{13} = \frac{5+14i}{13}$

$\therefore$ $x = 5/13, y = 14/13$. **Ans.[B]**

Ex.12 If Complex Number $\frac{z-1}{z+1}$ is purely imaginary then locus of z is -

(A) a circle (B) a straight line

(C) a parabola (D) None of these

Sol. Let $z = x + iy$ then

$$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy}$$

$$= \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{x^2-1+iy(x-1)+iy(x+1)+y^2}{(x+1)^2+y^2}$$

$$= \frac{(x^2-1+y^2)+i[2xy]}{(x+1)^2+y^2}$$

If it is purely Imaginary

$$\frac{x^2-1+y^2}{(x+1)^2+y^2} = 0$$

$\Rightarrow \quad x^2 + y^2 - 1 = 0$

$\Rightarrow \quad x^2 + y^2 = 1$

which is the equation of a circle. **Ans.[A]**

CONJUGATE COMPLEX NUMBER

The complex numbers $z = (a, b) = a + ib$ and $\bar{z} = (a, -b) = a - ib$ where $b \neq 0$ are said to be complex conjugate of each other (Here the complex conjugate is obtained by just changing the sign of i) e.g.conjugate of $z = -3 + 4i$ is $\bar{z} = -3 - 4i$.

Note : Image of any complex number in x-axis is called its conjugate.

Properties of Conjugate Complex Number

Let $z = a + ib$ and $\bar{z} = a - ib$ then

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2a = 2\,\text{Re}(z) =$ purely real

(iii) $z - \bar{z} = 2ib = 2i\,\text{Im}(z) =$ purely imaginary

(iv) $z\bar{z} = a^2 + b^2 = |z|^2$

(v) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(vi) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(vii) $\overline{re^{i\theta}} = re^{-i\theta}$

(viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

(ix) $\overline{z^n} = (\bar{z})^n$

(x) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

(xi) $z + \bar{z} = 0$ or $z = -\bar{z}$

$\Rightarrow z = 0$ or z is purely imaginary

(xii) $z = \bar{z} \Rightarrow$ z is purely real

Solved Examples

Ex.13 The conjugate of $\frac{1}{3+4i}$ is -

(A) $(3-4i)$ (B) $\frac{1}{25}(3+4i)$

(C) $\frac{1}{25}(3-4i)$ (D) None of these

Sol. $\frac{1}{3+4i} = \frac{3-4i}{(3+4i)(3-4i)} = \frac{1}{25}(3-4i)$

$\Rightarrow$ conjugate of $\left(\frac{1}{3+4i}\right) = \frac{1}{25}(3+4i)$.

Ans. [B]

Ex.14 If z is a complex number such that $z^2 = (\bar{z})^2$, then

(A) z is purely real

(B) z is purely imaginary

(C) Either z is purely real or purely imaginary

(D) None of these

Sol. Let $z = x + iy$, then its conjugate $\bar{z} = x - iy$

Given that $z^2 = (\bar{z})^2$

$\Rightarrow \quad x^2 - y^2 + 2ixy = x^2 - y^2 - 2ixy$

$\Rightarrow 4ixy = 0.$

If $x \neq 0$ then $y = 0$ and if $y \neq 0$ then $x = 0$.

Ans.(C)

MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = x + iy$ is denoted as mod (z) or | z |, is defined as

$|z| = \sqrt{x^2 + y^2}$, where x = Re(z), y = Im(z). Sometimes, |z| is called absolute value of z. Note that $|z| \geq 0$.

For example, if z = 3 + 2i, then $|z| = \sqrt{3^2 + 2^2} = \sqrt{13}$.

Properties of modulus

(i) $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$, i.e., $x = 0, y = 0$

(ii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$.

(iii) $z\bar{z} = |z|^2$

(iv) $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$

(v) $|z_1 z_2| = |z_1|\,|z_2|$

(vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ (provide $z_2 \neq 0$)

(vii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

(viii) $|z_1 - z_2| \geq ||z_1| - |z_2||$

(ix) $|z^2| = |z|^2$ or $|z^n| = |z|^n$ also $|z_1 z_2 \ldots z_n| = |z_1|\,|z_2| \ldots |z_n|$

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(xi) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\bar{z}_2)$

(xii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1\bar{z}_2)$

Solved Examples

Ex.15 The modulus of $z = \frac{(1+i\sqrt{3})(\cos\theta + i\sin\theta)}{2(1-i)(\cos\theta - i\sin\theta)}$ is-

(A) $\frac{1}{3\sqrt{2}}$ (B) $\frac{1}{\sqrt{3}}$

(C) $\frac{1}{\sqrt{2}}$ (D) 1

Sol. $|z| = \frac{|1+i\sqrt{3}|\,|\cos\theta + i\sin\theta|}{2|1-i|\,|\cos\theta - i\sin\theta|}$

$= \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$ **Ans.[C]**

Ex.16 If for any complex number z, $|z - 4| < |z - 2|$, then

(A) R(z) > 2 (B) R(z) < 0

(C) R(z) > 0 (D) R(z) > 3

Sol. Let $z = x + iy$, then

$|z - 4| < |z - 2|$

$\Rightarrow (x-4)^2 + y^2 < (x-2)^2 + y^2$

$\Rightarrow -4x < -12 \Rightarrow x > 3$

$\Rightarrow R(z) > 3$ **Ans.[D]**

Ex.17 If $\left|\frac{z-3i}{z+3i}\right| = 1$ then the locus of z is -

(A) x axis

(B) $x - y = 0$

(C) Circle passing through origin

(D) y axis

Sol. Let $z = x + iy$ then

$\left|\frac{z-3i}{z+3i}\right| = 1 \Rightarrow |z - 3i| = |z + 3i|$

$\Rightarrow |x + iy - 3i| = |x + iy + 3i|$

$\Rightarrow \sqrt{x^2 + (y-3)^2} = \sqrt{x^2 + (y+3)^2}$

$\Rightarrow 12y = 0$

$\Rightarrow y = 0$, which is equation of x - axis

Ans.[A]

Ex.18 For any two complex numbers z_1 and z_2 and any real numbers a and b;

$|(az_1 - bz_2)|^2 + |(bz_1 + az_2)|^2 =$

(A) $(a^2 + b^2)(|z_1| + |z_2|)$

(B) $(a^2 + b^2)(|z_1|^2 + |z_2|^2)$

(C) $(a^2 + b^2)(|z_1|^2 - |z_2|^2)$

(D) none of these

Sol. $|(az_1 - bz_2)|^2 + |(bz_1 + az_2)|^2$

$= a^2|z_1|^2 + b^2|z_2|^2 - 2ab\,\text{Re}(z_1\bar{z}_2) + b^2|z_1|^2 + a^2|z_2|^2 + 2ab\,\text{Re}(z_1\bar{z}_2)$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ **Ans.(B)**

REPRESENTATION OF A COMPLEX NUMBER

(a) Cartesian Representation :

The complex number z = x + iy = (x, y) is represented by a point P whose coordinates are refered to rectangular axis x ox′ and yoy′, which are called real and imaginary axes respectively. Thus a complex number z is represented by a point in a plane, and corresponding to every point in this plane there exists a complex number such a plane is called Argand plane or Argand diagram or complex plane or gussian plane.

Note :

(i) Distance of any complex number from the origin is called the modulus of complex number and is denoted by |z|. Thus, $|z| = \sqrt{x^2 + y^2}$.

(ii) Angle of any complex number with positive direction of x-axis is called amplitude or argument of z. Thus, amp (z) = arg (z) = $\theta = \tan^{-1}\frac{y}{x}$.

(b) Polar Representation : If z = x + iy is a complex number then z = r (cos θ + i sin θ) is a polar form of complex number z where x = r cos θ, y = r sin θ and $r = \sqrt{x^2 + y^2} = |z|$.

(c) Exponential Form : If z = x + iy is a complex number then its exponential form is $z = r e^{i\theta}$ where r is modulas and θ is amplitude of complex number.

(d) Vector Representation : If z = x + iy is a complex number such that it represent point P(x, y) then its vector representation is $z = \vec{OP}$

Solved Examples

Ex.19 If A ≡ 1 + 2i, B ≡ –3 + i, C ≡ – 2 – 3i and D ≡ 2 – 2i are vertices of a quadrilateral, then it is a

(A) rectangle (B) parallelogram
(C) square (D) rhombus

Sol. ∵ A≡(1,2); B≡(–3, 1); C≡(–2,–3); D≡(2,–2)

∴ $AB^2 = 16 + 1 = 17$, $BC^2 = 1 + 16 = 17$

$CD^2 = 16 + 1 = 17$, $AC^2 = 9 + 25 = 34$

$BD^2 = 25 + 9 = 34$.

Now since AB = BC = CD and AC = BD

∴ ABCD is square. **Ans.[C]**

Ex.20 The polar form of – 1 + i is-

(A) $\sqrt{2}$ (cos π / 4 + i sin π / 4)

(B) $\sqrt{2}$ (cos 5π / 4 + i sin 5π / 4)

(C) $\sqrt{2}$ (cos 3π / 4 + i sin 3π /4)

(D) $\sqrt{2}$ (cos π / 4 – i sin π / 4)

Sol. ∵ $|-1+i| = \sqrt{2}$, amp(–1+i) = π–π/4 = 3π/4

∴ –1 + i = $\sqrt{2}$ (cos 3π / 4 + i sin 3π / 4)

Ans. [C]

Ex.21 The polar form of $\frac{1+7i}{(2-i)^2}$ is

(A) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(B) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

(C) $\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

(D) none of these

Sol. $\frac{1+7i}{(2-i)^2} = \frac{(1+7i)}{(3-4i)}\frac{(3+4i)}{(3+4i)} = \frac{-25+25i}{25} = -1+i$

Let z = x + iy = – 1 + i

∴ r cos θ = –1 and r sin θ = 1

∴ $\theta = \frac{3\pi}{4}$ and $r = \sqrt{2}$

Thus $\frac{1+7i}{(2-i)^2} = \sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

Ans.(A)

ARGUMENT OF A COMPLEX NUMBER

The amplitude or argument of a complex number z is the inclination of the directed line segment representing z, with real axis. If z = x + iy then

$$\text{amp}(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

The argument of any complex number is not unique. $2n\pi + \theta$ (n integer) is also argument of z for various values of n. The value of θ satisfying the inequality $-\pi < \theta \le \pi$ is called the principle value of the argument.

For finding the principle argument of any complex number first check that the complex number is in which quadrant and then find the angle θ and amplitude using the adjacent figure.

Note :

(i) If a complex number is multiplied by iota(i) its amplitude will be be increased by $\pi/2$ and will be decreased by $\pi/2$, if is multiplied by –i.

(ii) Amplitude of complex number in I and II quadrant is always positive and in III and IV is always negative.

(iii) Argument of zero is not defined.

(iv) Let z_1, z_2, z_3 be the affixes of P, Q, R respectively in the Argand Plane. Then from the figure the angle between PQ and PR is

$$\theta = \theta_2 - \theta_1 = \arg \overline{PR} - \arg \overline{PQ} = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

Properties of Arguments

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(ii) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$

(iii) $\arg(z^n) = n \arg(z)$

(iv) If $\arg(z) = 0 \Rightarrow$ z is a positive real number

(v) $\arg(z) + \arg(\bar{z}) = 0$

(vi) $\arg(z - \bar{z}) = \pm \pi/2$

(vii) $\arg(z) = \pi \Rightarrow$ z is a negative real number

(viii) $\arg(\bar{z}) = -\arg(z) = \arg(1/z)$

(ix) $\arg(-z) = \arg(z) \pm \pi$

(x) $\arg(i\,y) = \pi/2$ if $y > 0$

$= -\pi/2$ if $y < 0$ (where $y \in R$)

Solved Examples

Ex.22 Let z_1 and z_2 be two complex numbers with α and β as their principal arguments such that $\alpha + \beta > \pi$, then principal $\arg(z_1 z_2)$ is given by

(A) $\alpha + \beta + \pi$ (B) $\alpha + \beta - \pi$

(C) $\alpha + \beta - 2\pi$ (D) $\alpha + \beta$

Sol. We know that principal argument of a complex number lie between $-\pi$ and π, but $\alpha + \beta > \pi$, therefore principal $\arg(z_1 z_2) = \arg z_1 + \arg z_2 = \alpha + \beta$, is given by $\alpha + \beta - 2\pi$. **Ans.(C)**

Ex.23 The amplitude of the complex number $z = \sin\alpha + i(1 - \cos\alpha)$ is

(A) $2\sin\frac{\alpha}{2}$ (B) $\frac{\alpha}{2}$

(C) α (D) None of these

Sol. $z = \sin\alpha + i(1 - \cos\alpha)$

$$\Rightarrow \arg(z) = \tan^{-1}\left(\frac{1-\cos\alpha}{\sin\alpha}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right)$$

$$= \tan^{-1}\tan\left(\frac{\alpha}{2}\right) = \frac{\alpha}{2}$$ **Ans.(B)**

Ex.24 The amplitude of $\sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$

(A) $\frac{\pi}{5}$ (B) $\frac{2\pi}{5}$

(C) $\frac{\pi}{10}$ (D) $\frac{\pi}{15}$

Sol.

$$\sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$$

$$= 2\sin\frac{\pi}{10}\cos\frac{\pi}{10} + i2\sin^2\frac{\pi}{10}$$

$$= 2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$$

For amplitude, $\tan\theta = \dfrac{\sin\frac{\pi}{10}}{\cos\frac{\pi}{10}} = \tan\frac{\pi}{10}$

$\Rightarrow \theta = \frac{\pi}{10}$. **Ans.(C)**

Ex.25 The amplitude of $\frac{a+ib}{a-ib}$ is equal to-

(A) $\tan^{-1}\left(\frac{a^2-b^2}{a^2+b^2}\right)$ (B) $\tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)$

(C) $\tan^{-1}\left(\frac{2ab}{a^2+b^2}\right)$ (D) $\tan^{-1}\left(\frac{a^2-b^2}{2ab}\right)$

Sol. amp $\left(\frac{a+ib}{a-ib}\right)$ = amp (a + ib) – amp (a – ib)

$= \tan^{-1}\left(\frac{b}{a}\right) - \tan^{-1}\left(-\frac{b}{a}\right)$

$= \tan^{-1}\left[\frac{2(b/a)}{1-(b^2/a^2)}\right] = \tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)$

Ans.[B]

Ex.26 If $z = \frac{1}{i}$ then arg ($\bar{z}$) is -

(A) π (B) $-\frac{\pi}{2}$

(C) 0 (D) $\frac{\pi}{2}$

Sol. $z = \frac{1}{i} = \frac{1}{i} \times \frac{-i}{-i} = \frac{-i}{+1} = -i$

$\therefore$ $\bar{z}$ = i, which is the positive Imaginary quantity

$\therefore$ arg ($\bar{z}$) = $\frac{\pi}{2}$ **Ans.[D]**

Ex.27 If $z = \frac{3-i}{2+i} + \frac{3+i}{2-i}$ then arg (zi) is-

(A) $-\pi$ (B) π

(C) $\frac{\pi}{2}$ (D) $-\frac{\pi}{2}$

Sol. $z = \frac{3-i}{2+i} + \frac{3+i}{2-i}$

$= \frac{(3-i)(2-i)+(3+i)(2+i)}{(2+i)(2-i)}$

$\Rightarrow$ z = 2

$\Rightarrow$ (iz) = 2i,

which is the positive Imaginary quantity

$\therefore$ arg (iz) = $\frac{\pi}{2}$ **Ans.[D]**

DIFFERENT WAYS OF WRITING A COMPLEX NUMBER

$z = a + ib$ (Algebraic form)

$z = r(\cos\theta + i\sin\theta)$ (Polar form)

where r is modulus of complex number and θ is it's argument.

$z = r\,e^{i\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula)

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}-e^{i\theta}}{2i}$

Solved Examples

Ex.28 If $z = re^{i\theta}$, then $|e^{iz}|$ =

(A) $e^{r\sin\theta}$ (B) $e^{-r\sin\theta}$

(C) $e^{-r\cos\theta}$ (D) $e^{r\cos\theta}$

Sol. If $z = r\,e^{i\theta} = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $iz = ir(\cos\theta + i\sin\theta) = -r\sin\theta + ir\cos\theta$

or $e^{iz} = e^{(-r\sin\theta + ir\cos\theta)} = e^{-r\sin\theta}\,e^{ri\cos\theta}$

or $|e^{iz}| = |e^{-r\sin\theta}|\,|e^{ri\cos\theta}|$

$= e^{-r\sin\theta}[\cos^2(r\cos\theta) + \sin^2(r\cos\theta)]^{1/2}$

$= e^{-r\sin\theta}$ **Ans.(B)**

ROTATION THEOREM

$$\frac{z_2-z_0}{|z_2-z_0|} = \frac{z_1-z_0}{|z_1-z_0|}e^{i\theta}$$

or $$\frac{z_1-z_0}{|z_1-z_0|} = \frac{z_2-z_0}{|z_2-z_0|}e^{i(2\pi-\theta)} = \frac{z_2-z_0}{|z_2-z_0|}e^{-i\theta}$$

Solved Examples

Ex29 If the points z_1, z_2, z_3 are the vertices of an equilateral triangle in the complex plane, then the value of $z_1^2 + z_2^2 + z_3^2$ is equal to

(A) $\frac{z_1}{z_2}+\frac{z_2}{z_3}+\frac{z_3}{z_1}$ (B) $z_1z_2 + z_2z_3 + z_3z_1$

(C) $z_1z_2 - z_2z_3 - z_3z_1$ (D) $-\frac{z_1}{z_2}-\frac{z_2}{z_3}-\frac{z_3}{z_1}$

Sol. $\overline{AC} = \overline{AB}\ e^{i\pi/3}$

By rotating $\pi/3$ in clockwise sense

$\Rightarrow (z_3 - z_1) = (z_2 - z_1)e^{i\pi/3}$ ....(i)

Also $(z_1 - z_2) = (z_3 - z_2)\ e^{i\pi/3}$ ....(ii)

DIviding (i) by (ii) we get

$$\Rightarrow \frac{z_3 - z_1}{z_1 - z_2} = \frac{z_2 - z_1}{z_3 - z_2}$$

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

Ans.(B)

SQUARE ROOT OF A COMPLEX NUMBER

If $z = x + iy$

suppose $\sqrt{z} = \sqrt{x + iy} = a + ib$

$\Rightarrow x + iy = a^2 - b^2 + 2iab$

On comparing the real and imaginary parts

$x = a^2 - b^2$, $y = 2ab$

Now, $a^2 + b^2 = \sqrt{x^2 + y^2} = |z|$ .....(i)

$a^2 - b^2 = x$ .....(ii)

From Equation (i) and (ii)

$$a = \pm\sqrt{\frac{|z| + x}{2}},\ b = \pm\sqrt{\frac{|z| - x}{2}}$$

Solving these two equations we shall get the required square roots as follows :

$$\pm\left[\sqrt{\frac{|z| + x}{2}} + i\sqrt{\frac{|z| - x}{2}}\right] \text{ if } y > 0 \quad \text{and}$$

$$\pm\left[\sqrt{\frac{|z| + x}{2}} - i\sqrt{\frac{|z| - x}{2}}\right] \text{ if } y < 0$$

Note :

(i) The square root of i is $\pm\left(\frac{1+i}{\sqrt{2}}\right)$ (Here b = 1)

(ii) The square root of $-$ i is $\pm\left(\frac{1-i}{\sqrt{2}}\right)$ (Here b = -1)

(iii) The square root of ω is $\pm\,\omega^2$

(iv) The square root of ω^2 is $\pm\,\omega$

Solved Examples

Ex.30 The square roots of $7 + 24i$ is.

(A) $\pm(4 + 3i)$ (B) $\pm(3 + 4i)$

(C) $\pm(2 + 3i)$ (D) $\pm(4 - 3i)$

Sol. Here $|z| = 25$, $x = 7$,

Hence square root =

$$\pm\left[\left(\frac{25 + 7}{2}\right)^{1/2} + i\left(\frac{25 - 7}{2}\right)^{1/2}\right] = \pm(4 + 3i)$$

Ans.(A)

MISCELLANEOUS RESULTS

(i) If ABC is an equilateral triangle having vertices z_1, z_2, z_3 then $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ or $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$.

(ii) If z_1, z_2, z_3, z_4 are vertices of parallelogram then $z_1 + z_3 = z_2 + z_4$.

(iii) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers represented by points P and Q respectively in Argand Plane then -

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= |(x_2 - x_1) + i(y_2 - y_1)| = |z_2 - z_1|$$

(iv) If a point P divides AB in the ratio of m : n, then $z = \frac{mz_2 + nz_1}{m + n}$ where z_1, z_2 and z represents the point A, B and P respectively.

(v) $|z - z_1| = |z - z_2|$ represents a perpendicular bisector of the line segmentjoining the points z_1 and z_2.

(vi) Let P be any point on a circle whose centre C and radius r, let the affixes of P and C be z and z_0 then $|z - z_0| = r$.

(a) Again if $|z - z_0| < r$ represent interior of the circle of radius r.

(b) $|z - z_0| > r$ represent exterior of the circle of radius r.

(vii) Let z_1, z_2, z_3 be the affixes of P, Q, R respectively in the Argand Plane. Then from the figure the angle between PQ and PR is.

$\theta = \theta_2 - \theta_1$

$= \arg. \vec{PR} - \arg \vec{PQ}$

$= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

(a) If z_1, z_2, z_3 are collinear, thus $\theta = 0$

therefore $\frac{z_3 - z_1}{z_2 - z_1}$ is purely real.

(b) If z_1, z_2, z_3 are such that PR ⊥ PQ,

$\theta = \pi/2$ So $\frac{z_3 - z_1}{z_2 - z_1}$ is purely imaginary.

Solved Examples

Ex.31 The points represented by the complex numbers $1 + i, -2 + 3i, \frac{5}{3}i$ on the Argand diagram are

(A) Vertices of an equilateral triangle

(B) Vertices of an isosceles triangle

(C) Collinear

(D) None of these

Sol. Let $z_1 = 1 + i$, $z_2 = -2 + 3i$ and $z_3 = 0 + \frac{5}{3}i$

Then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 3 & 1 \\ 0 & 5/3 & 1 \end{vmatrix}$

$= 1\left(3 - \frac{5}{3}\right) + 1(2) + 1\left(\frac{-10}{3}\right)$

$= \frac{4}{3} + 2 - \frac{10}{3} = \frac{4 + 6 - 10}{3} = 0$ **Ans.(C)**

Ex.32 If the complex numbers, z_1, z_2, z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3 =$

(A) 0 (B) 1

(C) –1 (D) None of these

Sol. Let the complex number z_1, z_2, z_3 denote the vertices A, B, C of an equilateral triangle ABC. Then, if O be the origin we have **OA** $= z_1$, **OB** $= z_2$, **OC** $= z_3$

Therefore $|z_1| = |z_2| = |z_3| \Rightarrow OA = OB = OC$

i.e., O is then circumcentre of ΔABC

Hence $z_1 + z_2 + z_3 = 0$. **Ans.(A)**

Ex.33 If $|z| = 2$, then the points representing the complex numbers $-1 + 5z$ will lie on a

(A) Circle (B) Straight line

(C) Parabola (D) None of these

Sol. Let $\omega = -1 + 5z$, then $\omega + 1 = 5z$

$\Rightarrow |\omega + 1| = 5|z| = 5 \times 2 = 10$

($\because |z| = 2$, given value)

Thus ω lies on a circle. **Ans.(A)**

Ex.34 The equation $z\bar{z} + (2-3i)z + (2+3i)\bar{z} + 4 = 0$ represents a circle of radius

(A) 2 (B) 3

(C) 4 (D) 6

Sol. Here $a = 2 - 3i$, $\bar{a} = 2 + 3i$ and $b = 4$.

Hence radius $= \sqrt{a\bar{a} - b} = \sqrt{(2-3i)(2+3i) - 4} = 3$

QUADRATIC EQUATIONS

POLYNOMIAL

Algebraic expression containing many terms is called Polynomial.

e.g $4x^4 + 3x^3 - 7x^2 + 5x + 3$, $3x^3 + x^2 - 3x + 5$

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots a_{n-1}x^{n-1} + a_nx^n$

where x is a variable, $a_0, a_1, a_2 \ldots\ldots\ldots a_n \in C$.

1. **Real Polynomial :** Let a_0, $a_1, a_2 \ldots\ldots a_n$ be real numbers and x is a real variable.

   Then $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$ is called real polynomial of real variable x with real coefficients.

   eg. $-3x^3 - 4x^2 + 5x - 4$, $x^2 - 2x + 1$ etc. are real polynomials.

2. **Complex Polynomial:** If $a_0, a_1, a_2 \ldots a_n$ be complex numbers and x is a varying complex number, then $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots a_nx^n$ is called a complex polynomial of complex variable x with complex coefficients.

   eg.- $3x^2 - (2+4i)x + (5i-4)$, $x^3 - 5ix^2 + (1+2i)x + 4$ etc. are complex polynomials.

3. **Degree of Polynomial :** Highest Power of variable x in a polynomial is called as a degree of polynomial.

   e.g. $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots a_{n-1}x^{n-1} + a_nx^n$ is n degree polynomial.

   $f(x) = 4x^3 + 3x^2 - 7x + 5$ is 3 degree polynomial

   $f(x) = 3x - 4$ is single degree polynomial or Linear polynomial.

   $f(x) = bx$ is odd Linear polynomial

QUADRATIC EXPRESSION

A Polynomial of degree two of the form $ax^2 + bx + c\ (a \neq 0)$ is called a quadratic expression in x.

e.g $3x^2 + 7x + 5$, $x^2 - 7x + 3$

General form :– $f(x) = ax^2 + bx + c$

where $a, b, c \in C$ and $a \neq 0$

QUADRATIC EQAUTION

An equation $ax^2 + bx + c = 0$ (where $a \neq 0$, and $a,b,c \in R$), is called a quadratic equation. Here a, b and c are called coefficient of the equation. This equation always has two roots. Let the roots be α and β.

1. **Roots of a Quadratic Equation**

   The values of variable x which satisfy the quadratic equation is called as Roots (also called solutions or zeros) of a Quadratic Equation.

SOLUTION OF QUADRATIC EQUATION

1. Factorization Method :

Let $ax^2 + bx + c = a(x-\alpha)(x-\beta) = 0$

Then $x = \alpha$ and $x = \beta$ will satisfy the given equation.

Hence factorize the equation and equating each to zero gives roots of equation.

e.g. $3x^2 - 2x - 1 = 0 \equiv (x-1)(3x+1) = 0$

$x = 1, -\frac{1}{3}$

2. Hindu Method {Sri Dharacharya Method} (Discriminant Formula) :

By completing the perfect square as

$ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Adding and substracting $\left(\frac{b}{2a}\right)^2$

$$\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right] = 0$$

Which gives, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Hence the Quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ has two roots, given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note : Every quadratic equation has two and only two roots.

Solved Examples

Ex.1 The roots of the equation $x^2 - 2x - 8 = 0$ are -

(A) $-4, 2$ (B) $4, -2$

(C) $4, 2$ (D) $-4, -2$

Sol. Quadratic Equation $x^2 - 2x - 8 = 0$

After factorization $(x-4)(x+2) = 0$

$\Rightarrow x = 4, -2$

Ans.[B]

Ex.2 The roots of the equation $x^2 - 4x + 1 = 0$ are -

(A) $2 \pm \sqrt{3}$ (B) $2, 4$

(C) $-2 \pm \sqrt{3}$ (D) $\sqrt{3} \pm 2$

Sol. Here a = 1, b = 4, c = 1

Using Hindu Method

$$x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Ans.[A]

NATURE OF ROOTS

In Quadratic equation $ax^2 + bx + c = 0$, the term $b^2 - 4ac$ is called discriminant of the equation, which plays an important role in finding the nature of the roots. It is denoted by Δ or D.

(A) Suppose a, b, c $\in$ R and a $\neq$ 0 then

(i) If $D > 0 \Rightarrow$ Roots are Real and unequal

(ii) If $D = 0 \Rightarrow$ Roots are Real and equal and each equal to $-b/2a$

(iii) If $D < 0 \Rightarrow$ Roots are imaginary and unequal or complex conjugate.

(B) Suppose a, b, c $\in$ Q, a $\neq$ 0 then

(i) If $D > 0$ and D is perfect square

$\Rightarrow$ Roots are unequal and Rational

(ii) If $D > 0$ and D is not perfect square

$\Rightarrow$ Roots are irrational and unequal

CONJUGATE ROOTS

The Irrational and complex roots of a quadratic equation are always occurs in pairs. Therefore (a, b, c,$\in$Q)

If	One Root	then	Other Root
	$\alpha + i\beta$		$\alpha - i\beta$
	$\alpha + \sqrt{\beta}$		$\alpha - \sqrt{\beta}$

Solved Examples

Ex.3 The roots of the equation $x^2 - 2\sqrt{2}x + 1 = 0$ are

(A) Imaginary and different

(B) Real and different

(C) Real and equal

(D) Rational and different

Sol. The discriminant of the equation $\left(-2\sqrt{2}\right)^2 - 4(1)(1) = 8 - 4 = 4 > 0$ and a perfect square, so roots are real and different but we can't say that roots are rational because coefficients are not rational therefore.

$$\alpha, \beta = \frac{2\sqrt{2} \pm \sqrt{\left(2\sqrt{2}\right)^2 - 4}}{2} = \frac{2\sqrt{2} \pm 2}{2} = \sqrt{2} \pm 1$$ this is irrational

$\therefore$ The roots are real and different.

Ans. [B]

Ex.4 If the roots of the equation $x^2 + 2x + p = 0$ are real then the value of p is

(A) $p \le 1$ (B) $p \le 2$

(C) $p \le 3$ (D) None of these

Sol. Here a = 1, b = 2, c = p

$\therefore$ discriminant $= (2)^2 - 4(1)(p) \ge 0$

(since roots are real)

$= 4 - 4p \ge 0 \Rightarrow 4 \ge 4p$

$\Rightarrow p \le 1$

Ans. [A]

Ex.5 The roots of the quadratic equation $x^2 - 2(a + b)x + 2(a^2 + b^2) = 0$ are -

(A) Rational and different

(B) Rational and equal

(C) Irrational and different

(D) Imaginary and different

Sol. 1, B = –2 (a+b), C = 2 $(a^2 + b^2)$

$B^2 - 4AC = 1[2(a + b)]^2 - 4(1)(2a^2 + 2b^2)$

$= 4a^2 + 4b^2 + 8ab - 8a^2 - 8b^2$

$= -4a^2 - 4b^2 + 8ab$

$= -4(a-b)^2 < 0$

So roots are imaginary and different.

Ans.[D]

Ex.6 The roots of the equation $(b+c)x^2 - (a+b+c)x + a = 0$ are $(a,b,c \in Q)$ -

(A) Real and different

(B) Rational and different

(C) Imaginary and different

(D) Real and equal

Sol. The discriminant of the equation is

$(a+b+c)^2 - 4(b+c)(a)$

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 4(b+c)a$

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 4ab - 4ac$

$= a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$

$(a-b-c)^2 > 0$

So roots are rational and different. **Ans.[B]**

SUM AND PRODUCT OF ROOTS

If α and β are the roots of quadratic equation $ax^2 + bx + c = 0$, then,

(i) Sum of Roots

$$S = \alpha + \beta = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

(ii) Product of Roots

$$P = \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

e.g. In equation

$3x^2 + 4x - 5 = 0$

Sum of roots S $= -\frac{4}{3}$,

Product of roots P $= -\frac{5}{3}$

1. Relation between Roots and Coefficients

If roots of quadratic equation $ax^2 + bx + c = 0$ $(a \ne 0)$ are α and β then :

(i) $(\alpha-\beta)=\sqrt{(\alpha+\beta)^2-4\alpha\beta}=\pm\dfrac{\sqrt{b^2-4ac}}{a}=\dfrac{\pm\sqrt{D}}{a}$

(ii) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\dfrac{b^2-2ac}{a^2}$

(iii) $\alpha^2-\beta^2=(\alpha+\beta)\sqrt{(\alpha+\beta)^2-4\alpha\beta}$

$$=-\frac{b\sqrt{b^2-4ac}}{a^2}=\frac{\pm\sqrt{D}}{a}$$

(iv) $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=-\dfrac{b(b^2-3ac)}{a^3}$

(v) $\alpha^3-\beta^3=(\alpha-\beta)^3+3\alpha\beta(\alpha-\beta)$

$$=\sqrt{(\alpha+\beta)^2-4\alpha\beta}\left\{(\alpha+\beta)^2-\alpha\beta\right\}$$

$$=\frac{(b^2-ac)\sqrt{b^2-4ac}}{a^3}$$

(vi) $\alpha^4+\beta^4=\left\{(\alpha+\beta)^2-2\alpha\beta\right\}^2-2\alpha^2\beta^2$

$$=\left(\frac{b^2-2ac}{a^2}\right)^2-2\frac{c^2}{a^2}$$

(vii) $\alpha^4-\beta^4=(\alpha^2-\beta^2)(\alpha^2+\beta^2)$

$$=\frac{-b(b^2-2ac)\sqrt{b^2-4ac}}{a^4}$$

(viii) $\alpha^2+\alpha\beta+\beta^2=(\alpha+\beta)^2-\alpha\beta$

(ix) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}=\dfrac{\alpha^2+\beta^2}{\alpha\beta}=\dfrac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

(x) $\alpha^2\beta+\beta^2\alpha=\alpha\beta(\alpha+\beta)$

(xi) $\left(\dfrac{\alpha}{\beta}\right)^2+\left(\dfrac{\beta}{\alpha}\right)^2=\dfrac{\alpha^4+\beta^4}{\alpha^2\beta^2}=\dfrac{(\alpha^2+\beta^2)^2-2\alpha^2\beta^2}{\alpha^2\beta^2}$

Solved Examples

Ex.7 If the product of the roots of the quadratic equation $mx^2-2x+(2m-1)=0$ is 3 then the value of m is -

(A) 1 (B) 2
(C) –1 (D) 3

Sol. Product of the roots c/a = 3 = $\dfrac{2m-1}{m}$

$\therefore 3m-2m=-1 \quad \Rightarrow m=-1$

Ans. [C]

Ex.8 If the equation $(k-2)x^2-(k-4)x-2=0$ has difference of roots as 3 then the value of k is-

(A) 1,3 (B) 3,3/2
(C) 2, 3/2 (D) 3/2, 1

Sol. $(\alpha-\beta)=\sqrt{(\alpha+\beta)^2-4\alpha\beta}$

Now $\alpha+\beta=\dfrac{(k-4)}{(k-2)}$, $\alpha\beta=\dfrac{-2}{k-2}$

$$\therefore (\alpha-\beta)=\sqrt{\left(\frac{k-4}{k-2}\right)^2+\frac{8}{(k-2)}}$$

$$=\frac{\sqrt{k^2+16-8k+8(k-2)}}{(k-2)}$$

$$\Rightarrow 3=\frac{\sqrt{k^2+16-8k+8k-16}}{(k-2)}$$

$\Rightarrow 3k-6=\pm k$

$\therefore k=3, 3/2$

Ans.[B]

Ex.9 If α, β are roots of the equation $ax^2+bx+c=0$ then the value of $\dfrac{1}{(a\alpha+b)^2}+\dfrac{1}{(a\beta+b)^2}$ is -

(A) $\dfrac{b^2-2ac}{ac}$ (B) $\dfrac{2ac-b^2}{ac}$
(C) $\dfrac{b^2-2ac}{a^2c^2}$ (D) $\dfrac{b^2}{a^2c}$

Sol. Since α,β are the root of the ax^2+bx+c

then $a\alpha^2+b\alpha+c=0$

$\Rightarrow \alpha(a\alpha+b)+c=0$

$\Rightarrow (a\alpha+b)=-c/\alpha$ ...(1)

Similarly

$(a\beta+b)=-c/\beta$ ...(2)

$$\therefore \frac{1}{(a\alpha+b)^2}+\frac{1}{(a\beta+b)^2}=\frac{1}{(-c/\alpha)^2}+\frac{1}{(-c/\beta)^2}$$

$$\Rightarrow \frac{\alpha^2}{c^2}+\frac{\beta^2}{c^2}=\frac{\alpha^2+\beta^2}{c^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{c^2}$$

$$=\frac{b^2/a^2-2c/a}{c^2}=\frac{b^2-2ac}{a^2c^2}$$

Ans.[C]

FORMATION OF AN EQUATION WITH GIVEN ROOTS

A quadratic equation whose roots are α and β is given by

$(x-\alpha)(x-\beta)=0$

$\Rightarrow x^2-\alpha x-\beta x+\alpha\beta=0$

$\Rightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$

i.e., x^2 – (Sum of roots)x + Product of roots = 0

1. Transformation of an Eqaution

If α,β are roots of the equation $ax^2+bx+c=0$ then the equation whose roots are

(i) $\frac{1}{\alpha},\frac{1}{\beta}$ is $cx^2+bx+a=0$ (Replace x by $\frac{1}{x}$)

(ii) $-\alpha,-\beta$ is $ax^2-bx+c=0$ (Replace x by $-x$)

(iii) $k+\alpha,k+\beta$ is $a(x-k)^2+b(x-k)+c=0$

{Replace x by $(x-k)$}

(iv) $\alpha^n,\beta^n\ (n\in N)$ is $a(x^{1/n})^2+b(x^{1/n})+c=0$

(Replace x by $x^{1/n}$)

(v) $\alpha^{1/n},\beta^{1/n}(n\in N)$ is $a(x^n)^2+b(x^n)+c=0$

(Replace x by x^n)

(vi) $k\alpha,k\beta$ is $ax^2+kbx+k^2c=0$

(Replace x by $\frac{x}{k}$)

(vii) $\frac{\alpha}{k},\frac{\beta}{k}$ is $k^2ax^2+kbx+c=0$

(Replace x by kx)

2. Symmetric Expressions

The symmetric expressions of the roots α, β of an equation are those expressions in α and β, which do not change by interchanging α and β. To find the value of such an expression, we generally express that in terms of $\alpha+\beta$ and $\alpha\beta$.

Some examples of symmetric expressions are–

(i) $\alpha^2+\beta^2$ (ii) $\alpha^2+\alpha\beta+\beta^2$

(iii) $\frac{1}{\alpha}+\frac{1}{\beta}$ (iv) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

(v) $\alpha^2\beta+\beta^2\alpha$ (vi) $\left(\frac{\alpha}{\beta}\right)^2+\left(\frac{\beta}{\alpha}\right)^2$

(vii) $\alpha^3+\beta^3$ (viii) $\alpha^4+\beta^4$

Solved Examples

Ex.10 If α,β are the root of a quadratic equation $x^2-3x+5=0$ then the equation whose roots are $(\alpha^2-3\alpha+7)$ and $(\beta^2-3\beta+7)$ is

(A) $x^2+4x+1=0$ (B) $x^2-4x-1=0$

(C) $x^2-4x+4=0$ (D) $x^2+2x+3=0$

Sol. Since α,β are the roots of equation $x^2-3x+5=0$

So $\alpha^2-3\alpha+5=0$; $\beta^2-3\beta+5=0$

$\therefore \alpha^2-3\alpha=-5$ ..... (i)

$\beta^2-3\beta=-5$ ..... (ii)

Putting the value from (i) and (ii) in $(\alpha^2-3\alpha+7)$ and $(\beta^2-3\beta+7)$

We get $(-5+7)$ and $(-5+7)$

$\therefore$ 2 and 2 are the roots

$\therefore$ The required equation is $x^2-4x+4=0$

Ans.[C]

Ex.11 The quadratic equation whose one root is $\frac{1}{2+\sqrt{5}}$ will be

(A) $x^2+4x-1=0$ (B) $x^2-4x-1=0$

(C) $x^2+4x+1=0$ (D) None of these

Sol. Given root $=\frac{1}{2+\sqrt{5}}=\sqrt{5}-2$

So the other root $=-\sqrt{5}-2$. Then sum of the roots $=-4$, product of the roots $=-1$

Hence the equation is $x^2+4x-1=0$

Ans.[A]

Ex.12 The equation whose roots are 3 and 4 will be-

(A) $x^2 + 7x + 12 = 0$

(B) $x^2 - 7x + 12 = 0$

(C) $x^2 - x + 12 = 0$

(D) $x^2 + 7x - 12 = 0$

Sol. The quadratic equation is given by

x^2 – (sum of the roots) x + (product of roots) =0

∴ The required equation

$= x^2 - (3+4)\,x + 3.4 = 0$

$= x^2 - 7x + 12 = 0$ **Ans. [B]**

Ex.13 The quadratic equation with rational coefficients whose one root is $2 + \sqrt{3}$ is -

(A) $x^2 - 4x + 1 = 0$ (B) $x^2 + 4x + 1 = 0$

(C) $x^2 + 4x - 1 = 0$ (D) $x^2 + 2x + 1 = 0$

Sol. The required equation is

$x^2 - \{(2+\sqrt{3}) + (2-\sqrt{3})\}\,x$

$+ (2+\sqrt{3})(2-\sqrt{3}) = 0$

or $x^2 - 4x + 1 = 0$

Ans. [A]

Ex.14 If α,β are roots of the equation

$x^2 - 5x + 6 = 0$ then the equation whose roots are $\alpha + 3$ and $\beta + 3$ is-

(A) $x^2 - 11x + 30 = 0$

(B) $(x-3)^2 - 5(x-3) + 6 = 0$

(C) Both (1) and (2)

(D) None

Sol. Let $\alpha + 3 = x$

∴ $\alpha = x - 3$ (Replace x by x – 3)

So the required equation is

$(x-3)^2 - 5(x-3) + 6 = 0$ ...(1)

$\Rightarrow x^2 - 6x + 9 - 5x + 15 + 6 = 0$

$\Rightarrow x^2 - 11x + 30 = 0$ ...(2)

Ans.[C]

Ex.15 If α,β are roots of the equation $2x^2 + x - 1 = 0$ then the equation whose roots are $1/\alpha$, $1/\beta$ will be -

(A) $x^2 + x - 2 = 0$ (B) $x^2 + 2x - 8 = 0$

(C) $x^2 - x - 2 = 0$ (D) None of these

Sol. From the given equation

$\alpha + \beta = -1/2$, $\alpha\beta = -1/2$

The required equation is-

$$x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - \left(\frac{\alpha+\beta}{\alpha\beta}\right)x + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - \left(\frac{-1/2}{-1/2}\right)x + \frac{1}{-1/2} = 0$$

$\Rightarrow x^2 - x - 2 = 0$ **Ans.[C]**

Short cut : Replace x by 1/x

$\Rightarrow 2(1/x)^2 + 1/x - 1 = 0 \Rightarrow x^2 - x - 2 = 0$

ROOTS UNDER PARTICULAR CASES

For the quadratic equation $ax^2 + bx + c = 0$

(i) If $b = 0$ $\Rightarrow$ roots are of equal magnitude but of opposite sign

(ii) If $c = 0$ $\Rightarrow$ one root is zero other is $-b/a$

(iii) If $b = c = 0$ $\Rightarrow$ both root are zero

(iv) If $a = c$ $\Rightarrow$ roots are reciprocal to each other

(v) If $\left.\begin{matrix} a > 0 \;\; c < 0 \\ a < 0 \;\; c > 0 \end{matrix}\right\}$ $\Rightarrow$ If Roots are of opposite signs

(vi) If $\left.\begin{matrix} a > 0, b > 0, c > 0 \\ a < 0, b < 0, c < 0 \end{matrix}\right\}$ $\Rightarrow$ both roots are negative.

(vii) If $\left.\begin{matrix} a > 0, b < 0, c > 0 \\ a < 0, b > 0, c < 0 \end{matrix}\right\}$ $\Rightarrow$ both roots are positive.

(viii) If sign of a = sign of b $\neq$ sign of c

$\Rightarrow$ Greater root in magnitude is negative.

(ix) If sign of b = sign of c ≠ sign of a

$\Rightarrow$ Greater root in magnitude is positive.

(x) If a+b+c = 0

$\Rightarrow$ one root is 1 and second root is c/a.

(xi) If a = b = c = 0 then equation will become an identity and will be satisfy by every value of x.

Solved Examples

Ex.16 The roots of the equation $x^2 - 3x - 4 = 0$ are–

(A) Opposite and greater root in magnitude is positive

(B) Opposite and greater root in magnitude is negative

(C) Reciprocal to each other

(D) None of these

Sol. The roots of the equation $x^2 - 3x - 4 = 0$ are of opposite sign and greater root is positive

($\because$ a >0 , b< 0, c < 0) **Ans.[A]**

Ex.17 The roots of the equation $2x^2 - 3x + 2 = 0$ are -

(A) Negative of each other

(B) Reciprocal to each other

(C) Both roots are zero

(D) None of these

Sol. The roots of the equations $2x^2 - 3x + 2 = 0$ are reciprocal to each other because here a = c

Ans.[B]

Ex.18 If equation $\frac{x^2 - bx}{ax - c} = \frac{k-1}{k+1}$ has equal and opposite roots then the value of k is -

(A) $\frac{a+b}{a-b}$ (B) $\frac{a-b}{a+b}$

(C) $\frac{a}{b}+1$ (D) $\frac{a}{b}-1$

Sol. Let the roots are α & $-\alpha$.

given equation is

$(x^2 - bx)(k+1) = (k-1)(ax - c)$

$\Rightarrow x^2(k+1) - bx(k+1) = ax(k-1) - c(k-1)$

$\Rightarrow x^2(k+1) - bx(k-1) - ax(k-1) + c(k-1) = 0$

Now sum of roots = 0 ($\because \alpha - \alpha = 0$)

$\therefore$ b (k+1) + a (k–1) = 0

$\Rightarrow k = \frac{a-b}{a+b}$ **Ans.[B]**

Ex.19 The real values of a for which the quadratic equation $2x^2 - (a^3 + 8a - 1)x + a^2 - 4a = 0$ possesses roots of opposite signs are given by

(A) $a > 5$ (B) $0 < a < 4$

(C) $a > 0$ (D) $a > 7$

Sol. The roots of the given equation will be of opposite signs if they are real and their product is negative, i.e., Discriminant ≥ 0 and product of roots < 0.

$\Rightarrow (a^3 + 8a - 1)^2 - 8(a^2 - 4a) \geq 0$ and $\frac{a^2 - 4a}{2} < 0$

$\Rightarrow a^2 - 4a < 0$

$\left[\because a^2 - 4a < 0 \Rightarrow (a^3 + 8a - 1)^2 - 8(a^2 - 4a) \geq 0\right]$

$\Rightarrow 0 < a < 4$ **Ans.[B]**

CONDITION FOR COMMON ROOTS

1. Only One Root is Common : Let α be the common root of quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ then

$\therefore a_1\alpha^2 + b_1\alpha + c_1 = 0$

$a_2\alpha^2 + b_2\alpha + c_2 = 0$

By Cramer's rule :

$$\frac{\alpha^2}{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}} = \frac{\alpha}{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

or

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$\therefore \alpha = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}, \alpha^2 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \alpha \neq 0.$

$\therefore$ The condition for only one Root common is

$(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$

2. Both roots are common : Then required conditions is $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Note: Two different quadratic equation with rational coefficient cannot have single common root which is complex or irrational, as imaginary and surd roots always occur in pair.

Solved Examples

Ex.20 If one root of the equations $x^2 + 2x + 3k = 0$ and $2x^2 + 3x + 5k = 0$ is common then the values of k is -

(A) 1, 2 (B) 0, –1
(C) 1, 3 (D) None of these

Sol. Since one root is common, let the root is α.

$$\frac{\alpha^2}{10k - 9k} = \frac{\alpha}{6k - 5k} = \frac{1}{3 - 4}$$

$\alpha^2 = -k$ ...(1)

$\alpha = -k$ ...(2)

$\therefore \alpha^2 = k^2$

$\Rightarrow k^2 = -k \Rightarrow k^2 + k = 0$

$\Rightarrow k(k + 1) = 0$

$\Rightarrow k = 0$ and $k = -1$ **Ans. [B]**

Ex.21 If the equations $2x^2 + x + k = 0$ and $x^2 + x/2 - 1 = 0$ have 2 common roots then the value of k is-

(A) 1 (B) 3
(C) –1 (D) –2

Sol. Since the given equation have two roots in common so from the condition

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\Rightarrow \frac{2}{1} = \frac{1}{1/2} = \frac{k}{-1}$$

$\therefore k = -2$ **Ans.[D]**

Ex.22 If $x^2 + x - 1 = 0$ and $2x^2 - x + \lambda = 0$ have a common root then –

(A) $\lambda^2 - 7\lambda + 1 = 0$ (B) $\lambda^2 + 7\lambda - 1 = 0$
(C) $\lambda^2 + 7\lambda + 1 = 0$ (D) $\lambda^2 - 7\lambda - 1 = 0$

Sol. Let the common root is α then

$\alpha^2 + \alpha - 1 = 0$

$2\alpha^2 - \alpha + \lambda = 0$

By cross multiplication

$$\frac{\alpha^2}{\lambda - 1} = \frac{\alpha}{-2 - \lambda} = \frac{1}{-1 - 2}$$

$$\alpha^2 = \frac{\lambda - 1}{-3} = \frac{1 - \lambda}{3}, \quad \alpha = \frac{2 + \lambda}{3}$$

$$\left(\frac{2 + \lambda}{3}\right)^2 = \frac{1 - \lambda}{3} \Rightarrow \lambda^2 + 7\lambda + 1 = 0$$ **Ans.[C]**

Ex. 23 If $x^2 + x - 1 = 0$ and $2x^2 - x + k = 0$ have a common root then

(A) $k^2 - 7k + 1 = 0$ (B) $k^2 + 7k + 1 = 0$
(C) $k^2 + 7k - 1 = 0$ (D) $k^2 - 7k - 1 = 0$

Sol. Let the common root is α then

$\alpha^2 + \alpha - 1 = 0$

$2\alpha^2 - \alpha + k = 0$

By cross multiplication

$$\frac{\alpha^2}{k - 1} = \frac{\alpha}{-2 - k} = \frac{1}{-1 - 2}$$

$$\alpha^2 = \frac{k - 1}{-3} = \frac{1 - k}{3}, \alpha = \frac{2 + k}{3}$$

$$\left(\frac{2 + k}{3}\right)^2 = \frac{1 - k}{3} \Rightarrow k^2 + 7k + 1 = 0$$ **Ans. [B]**

QUADRATIC EXPRESSION

The expression $ax^2 + bx + c$ is said to be a real quadratic expression in x. Where a, b and c are real $a \neq 0$. Let $y = ax^2 + bx + c$

$$\Rightarrow y = a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right\}$$

$$\Rightarrow \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2 \quad .....(1)$$

where $D = b^2 - 4ac$

Equation (1) represents a parabola with vertex at $A\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$, and axis of this parabola is parallel to y axis and it is $x = -b/2a$.

GREATEST AND LEAST VALUE OF QUADRATIC EXPRESSION

(i) If $a > 0$, then the quadratic expression $y = ax^2 + bx + c$ has no greatest value but it has least value $\frac{4ac - b^2}{4a}$ at $x = -\frac{b}{2a}$

(ii) If $a < 0$, then the quadratic expression $y = ax^2 + bx + c$ has no least value but it has greatest value $\frac{4ac - b^2}{4a}$ at $x = -\frac{b}{2a}$

Solved Examples

Ex.24 The range of the values of $\frac{x}{x^2+4}$ for all real value of x is

(A) $\frac{-1}{2} \le y \le \frac{1}{2}$ (B) $\frac{-1}{4} \le y \le \frac{1}{4}$

(C) $\frac{-1}{6} \le y \le \frac{1}{6}$ (D) None of these

Sol. Let $y = \frac{x}{x^2+4}$

$\Rightarrow x^2y - x + 4y = 0$

Now, $x \in R \Rightarrow B^2 - 4AC \ge 0 \Rightarrow 1 - 4y.4y \ge 0$

$= (4y-1)(4y+1) \le 0$

$\therefore \frac{-1}{4} \le y \le \frac{1}{4}$ **Ans.[B]**

Ex.25 If the roots of the quadratic equation $x^2 - 4x - \log_3 a = 0$ are real, then the least value of a is

(A) 81 (B) $\frac{1}{81}$

(C) $\frac{1}{64}$ (D) None of these

Sol. Since the roots of the given equation are real.

$\therefore$ Discriminant $\ge 0 \Rightarrow 16 + 4\log_3 a \ge 0$

$\Rightarrow \log_3 a \ge -4 \Rightarrow a \ge 3^{-4} \Rightarrow a \ge \frac{1}{81}$

Hence, the least value of a is $\frac{1}{81}$ **Ans.[B]**

Ex.26 The minimum value of the expression $4x^2 + 2x + 1$ is-

(A) 1/4 (B) 1/2

(C) 3/4 (D) 1

Sol. Since a = 4 > 0 therefore its minimum value is

$$= \frac{4(4)(1) - (2)^2}{4(4)} = \frac{16-4}{16} = \frac{12}{16} = \frac{3}{4}$$

Ans.[C]

Ex.27 The maximum value of $5 + 20x - 4x^2$ for all real value of x is-

(A) 10 (B) 20

(C) 25 (D) 30

Sol. Since a = –4 < 0 therefore its maximum value is

$$= \frac{4(-4)(5) - (20)^2}{4(-4)} = \frac{-80-400}{-16} = \frac{-480}{-16} = 30$$

Ans.[D]

FEW GRAPHS OF QUADRATIC EXPRESSION

(i) $y = x^2$ (a = 1, b = 0, c = 0)

Vertex (0, 0)

Axis of the parabola x = 0

This parabola opens in upward direction.

(ii) $y = -x^2 + 2x + 1$ (a = – 1, b = 2, c = 1)

vertex (1,2)

Axis of the parabola x = 1

This parabola opens downward.

Note: If a > 0, shape of parabola is upward.

And if a < 0, shape of parabola is downward.

Values of x where curve crosses x-axis will be roots of the equation $ax^2 + bx + c = 0$ as y = 0 on these points. If a curve is above x-axis for all x, then it means that it does not intersect x-axis or we can say equation $ax^2 + bx + c = 0$ have imaginary roots. This gives rise to following cases:

Cases:

Let $f(x) = ax^2 + bx + c$ where $a, b, c, \in R$ $(a \neq 0)$.

For some value of x, f(x) may be positive, negative or zero. This gives rise to the following cases:

(i) $a > 0$ and $b^2 - 4ac < 0$

$\Leftrightarrow f(x) > 0, \ \forall x \in R$

In this case whole of the parabola lies above the x - axis.

(ii) $a > 0$ and $b^2 - 4ac = 0 \Leftrightarrow f(x) \geq 0, \ \forall x \in R$

In this case whole of the parabola lies above the x-axis except at one point where it touches the x – axis. The point is $\left(-\frac{b}{2a}, 0\right)$.

(iii) $a > 0$ and $b^2 - 4ac > 0$

Let f(x) = 0 have two real roots α and β

(say $\alpha < \beta$)

then $f(x) > 0, \ \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$

and $f(x) < 0, \ \forall x \in (\alpha, \beta)$

(iv) $a < 0$ and $b^2 - 4ac < 0$

$\Leftrightarrow f(x) < 0, \ \forall x \in R$

In this case whole of the parabola lies below the x– axis.

(v) $a < 0$ and $b^2 - 4ac = 0$

$\Leftrightarrow f(x) \leq 0, \ \forall x \in R$

In this case whole of the parabola lies below the x– axis, except at one point

where it touches the x–axis.

The point is $\left(-\frac{b}{2a}, 0\right)$.

(vi) $a < 0$ and $b^2 - 4ac > 0$

Let f(x) = 0 have two real roots α and β $(\alpha < \beta)$.

Then $f(x) < 0, \ \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0, \ \forall x \in (\alpha, \beta)$

LOCATION OF ROOTS
(Interval in which roots lie)

In some problems we want the roots of the equation $ax^2 + bx + c = 0$ to lie in a given interval. For this we impose conditions on a, b and c. Since $a \neq 0$, we can take $f(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$.

(i) If both the roots are positive i.e., they lie in $(0, \infty)$, then the sum of the roots as well as the product of the roots must be positive.

$\Rightarrow \alpha + \beta = -\frac{b}{a} > 0$ and $\alpha\beta = \frac{c}{a} > 0$ with $b^2 - 4ac \geq 0$.

Similarly, if both the roots are negative i.e. they lie in $(-\infty, 0)$ then the sum of the roots must be negative and the product of the roots must be positive.

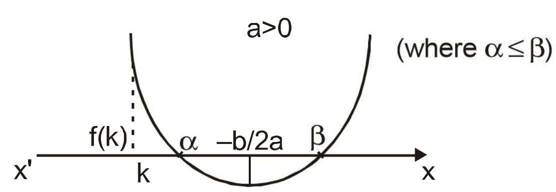
i.e. $\alpha + \beta = -\frac{b}{a} < 0$ and $\alpha\beta = \frac{c}{a} > 0$ with $b^2 - 4ac \geq 0$

Both the roots are of the same sign if a and c are of same sign. Now if b has the same sign that of a, both roots are negative or else both roots are positive.

If a and c are of opposite sign both roots are of opposite sign.

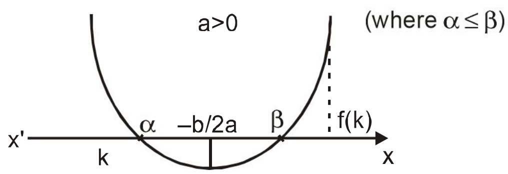
(ii) Both the roots are greater than given number k if the following three conditions are satisfied

$D \geq 0, \quad -\frac{b}{2a} > k$ and $f(k) > 0$.



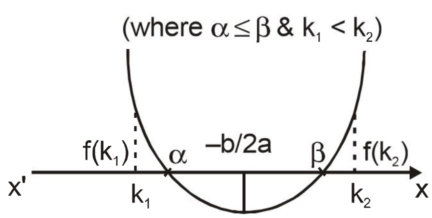
(iii) Both the roots will be less than a given number k if the following conditions are satisfied:

$$D \ge 0, \quad -\frac{b}{2a} < k \text{ and } f(k) > 0$$

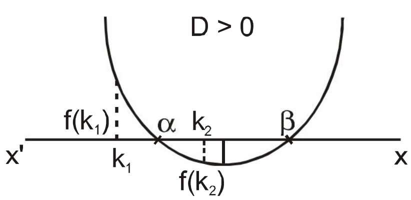


(iv) Both the roots will lie in the given interval (k_1, k_2) if the following conditions are satisfied:

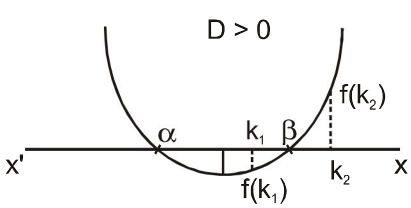
$$D \ge 0, \quad k_1 < -\frac{b}{2a} < k_2 \text{ and } f(k_1) > 0, \quad f(k_2) > 0$$



(v) Exactly one of the root lies in the given interval (k_1, k_2) if $f(k_1).f(k_2) < 0$

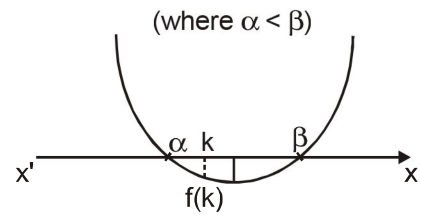


$f(k_1) > 0$ & $f(k_2) < 0$



$f(k_1) < 0$ & $f(k_2) > 0$

(vi) A given number k will lie between the roots if $f(k) < 0$, $D > 0$.



In particular, the roots of the equation will be of opposite signs if 0 lies between the roots $\Rightarrow f(0) < 0$.

Solved Examples

Ex.28 If f(x) is a quadratic expression which is positive for all real values of x and $g(x) = f(x) + f'(x) + f''(x)$, then for any real value of x-

(A) $g(x) < 0$ (B) $g(x) > 0$
(C) $g(x) = 0$ (D) $g(x) \ge 0$

Sol. Let $f(x) = ax^2 + bx + c$, then

$g(x) = (ax^2 + bx + c) + 2ax + b + 2a$

$= ax^2 + (b + 2a)x + (c + b + 2a)$

$\because f(x) > 0$, therefore $b^2 - 4ac < 0$ and $a > 0$.

Now for g(x),

Discriminant $= (b+2a)^2 - 4a(c+b+2a)$

$= b^2 - 4ac - 4a^2 < 0$

$(\because b^2 - 4ac < 0, -4a^2 < 0)$

Therefore signs of g(x) and a are same i.e. $g(x) > 0$. **Ans.[B]**

Ex.29 For real values of x, $2x^2 + 5x - 3 > 0$, if-

(A) $x < -2$ (B) $x > 0$
(C) $x > 1$ (D) None of these

Sol. Discriminant $b^2 - 4ac = 25 + 24 = 49 > 0$

$\Rightarrow$ Roots are real.

$\Rightarrow$ The given expression is positive for those real values of x for which $x \notin (-3, 1/2)$, because $a = 2 > 0$.

$\Rightarrow x > 1$ is true. **Ans.[C]**

THEORY OF EQUATIONS

(i) If $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$, $(a_0, a_1, a_2, ..., a_n \in R$ and $a_n \neq 0)$ then $p(x) = 0$ has exactly n roots. (real/complex)

(ii) Imaginary roots always occur in conjugate pairs. If $\beta \neq 0$ and $\alpha + i\beta$ is a root of p(x), then $\alpha - i\beta$ is also a root.

(iii) A polynomial equation in x of odd degree has at least one real root (moreover it has odd no. of real roots).

(iv) If $x_1, ..., x_n$ are the roots of $p(x) = 0$, then p(x) can be written in the form $p(x) \equiv a_n(x - x_1)...(x - x_n)$.

(v) If α is a root of $p(x) = 0$, then $(x - \alpha)$ is a factor of p(x) and vice - versa.

(vi) If $x_1, ..., x_n$ are the roots of $p(x) \equiv a_n x^n + ... + a_0 = 0$, $a_n \neq 0$.

Then $\sum_{i=1}^{n} x_i = -\frac{a_{n-1}}{a_n}$,

$\sum_{1 \le i < j \le n} x_i x_j = \frac{a_{n-2}}{a_n}$,

$x_1 x_2 ... x_n = (-1)^n \frac{a_o}{a_n}$.

(vii) If equation $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0$ has more than n distinct roots then f(x) is identically zero.

(viii) If p(a) and p(b) $(a < b)$ are of opposite sign, then $p(x) = 0$ has odd number of real roots in (a, b), i.e. it has at least one real root in (a, b) and if p(a) and p(b) are of same sign then $p(x) = 0$ has even number of real roots in (a, b).

(ix) If coefficients of p(x) (polynomial in x written in descending order) have 'm' changes in signs, then $p(x) = 0$ have at the most 'm' positive real roots and if $p(-x)$ have 't' changes in sign, then $p(x) = 0$ have at most 't' negative real roots. By this we can find maximum number of real roots.

Some Important Points

(i) Every equation of n^{th} degree $(n \ge 1)$ has exactly n roots and if the equation has more than n roots, it is an identity.

(ii) If roots of quadratic equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ are in the same ratio $\left(\text{i.e. } \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}\right)$ then $\boxed{\frac{b_1^2}{b_2^2} = \frac{a_1c_1}{a_2c_2}}$

(iii) If one root is k times the other root of quadratic equation $a_1x^2 + b_1x + c_1 = 0$ then $\boxed{\frac{(k+1)^2}{k} = \frac{b^2}{ac}}$

QUADRATIC EXPRESSION IN TWO VARIABLE

The general form of a quadratic expression in two variables x, y is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

The condition that this expression may be resolved into two linear rational factors is

$$\boxed{abc + 2fgh - af^2 - bg^2 - ch^2 = 0}$$

or

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Example : For what value of m the expression $y^2 + 2xy + 2x + my - 3$ can be resolved into two rational factors?

Sol. Here a = 0, b = 1, c = – 3

h = 1, g = 1, g = m/2

So $\Delta = 0 \Rightarrow \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & m/2 \\ 1 & m/2 & -3 \end{vmatrix} = 0$

$\Rightarrow (m/2 + 3) + (m/2 - 1) = 0$

$\Rightarrow m + 2 = 0$

$\Rightarrow m = -2$

Solved Examples

Ex.30 For what value of m the expression $y^2 + 4xy + 4x + my - 2$ can be resolved into two rational factors-

(A) 1 (B) –1

(C) 2 (D) –2

Sol. Here a = 0 , b = 1, c = –2

h = 2 , g = 2, f = m/2

So $\Delta = 0 \Rightarrow \begin{vmatrix} 0 & 2 & 2 \\ 2 & 1 & m/2 \\ 2 & m/2 & -2 \end{vmatrix} = 0$

$\Rightarrow \quad 2(4+m) + 2(m-2) = 0$

$\Rightarrow \quad 4m + 4 = 0$

$\Rightarrow \quad m = -1$ **Ans. [B]**