PRINCIPLE OF MATHEMATICAL INDUCTION

INTRODUCTION

The process of drawing a valid general result from particular results is called the process of induction.

The principle of mathematical induction is a mathematical process which is used to establish the validity of a general result involving natural numbers.

STATEMENT

A sentence is called a statement if it is either true or false but not both.

For example, the sentence "Two plus five equals seven" is a statement because this sentence is true.

A statement concerning the natural number 'n' is generally denoted by P(n). For example, if P(n) denotes the statement : "n(n + 1) is an even number," then

P(3) is the statement : "3(3 + 1) is an even number"

and P(7) is the statement : "7(7 + 1) is an even number" etc.

Here P(3) and P(7) are both true.

THE PRINCIPLE OF MATHEMATICAL INDUCTION :

If P(n) is a statement ($n \in N$) such that :

(i) P(1) is true and

(ii) truth of P(k) implies the truth of P(k + 1).

Then by the principle of mathematical induction (P.M.I.) the statement P(n) is true for all $n \in N$.

Remark :

If a given statement P(n) is to be proved for n = m + 1, m + 2, m + 3............ for some $m \in N$, then we are required to prove that P(m + 1) is true instead of proving P(1) is true.

Solved Examples

Ex.1 By using P.M.I. prove that $10^n + 3.4^{n+2} + 5$ is divisible by 9, $n \in N$.

Sol. Given statement is true for n = 1 as 10 + 192 + 5 = 207 is divisible by 9.

Let us assume that the result is true for n = k

i.e. $10^k + 3.4^{k+2} + 5 = 9\lambda,\ \lambda \in N$.

Now for n = k + 1

$10^{k+1} + 3.4^{k+3} + 5 = 10(9\lambda - 3.4^{k+2} - 5) + 3.4^{k+3} + 5$

$= 90\lambda - 288.4^k - 45$

which is divisible by 9. so the result is true for n = k + 1

so by P.M.I. the result is true for all $n \in N$.

Ex.2 Let P(n) be the statement "7 divides $(2^{3n} - 1)$". What is P(n + 1) ?

Sol. P(n + 1) is the statement "7 divides $(2^{3(n+1)} - 1)$"

Clearly P(n + 1) is obtained by replacing n by (n + 1) in P(n).

Ex.3 If P(n) is the statement "$2^{3n} - 1$ is an integral multiple of 7", and if P(r) is true, prove that P(r + 1) is true.

Sol. Let P(r) be true. Then $2^{3r} - 1$ is an integral multiple of 7.

We wish to prove that P(r + 1) is true i.e. $2^{3(r+1)} - 1$ is an integral multiple of 7.

Now P(r) is true

$\Rightarrow 2^{3r} - 1$ is an integral multiple of 7

$\Rightarrow 2^{3r} - 1 = 7\lambda$ for some $\lambda \in N$

$\Rightarrow 2^{3r} = 7\lambda + 1$ ...(i)

Now $2^{3(r+1)} - 1 = 2^{3r} \cdot 2^3 - 1 = (7\lambda + 1) \times 8 - 1$

$\Rightarrow 2^{3(r+1)} - 1 = 56\lambda + 8 - 1 = 56\lambda + 7 = 7(8\lambda + 1)$

$\Rightarrow 2^{3(r+1)} - 1 = = 7\mu$, where $\mu = 8\lambda + 1 \in N$

$\Rightarrow 2^{3(r+1)} - 1$ is an integral multiple of 7

$\Rightarrow$ P(r + 1) is true

Ex.4 Prove by the principle of mathematical induction that for all $n \in N$:

$$1 + 4 + 7 + \ldots + (3n - 2) = \frac{1}{2}n(3n - 1)$$

Sol. Let P(n) be the statement given by

$$P(n) : 1 + 4 + 7 + \ldots + (3n - 2) = \frac{1}{2}n(3n - 1)$$

Step-I We have $P(1) : 1 = \frac{1}{2} \times (1) \times (3 \times 1 - 1)$

$\because 1 = \frac{1}{2} \times (1) \times (3 \times 1 - 1)$

So, P(1) is true

Step-II Let P(m) be true, then

$$1 + 4 + 7 + \ldots + (3m - 2) = \frac{1}{2}m(3m - 1) \quad ...(i)$$

We wish to show that P(m + 1) is true. For this we have to show that

$$1 + 4 + 7 + \ldots + (3m - 2) + [3(m + 1) - 2] = \frac{1}{2}(m + 1)(3(m + 1) - 1)$$

Now $1 + 4 + 7 + \ldots + (3m - 2) + [3(m + 1) - 2]$

$= \frac{1}{2}m(3m - 1) + [3(m + 1) - 2]$ [Using (i)]

$= \frac{1}{2}m(3m - 1) + (3m + 1) = \frac{1}{2}[3m^2 - m + 6m + 2]$

$= \frac{1}{2}[3m^2 + 5m + 2] = \frac{1}{2}(m + 1)(3m + 2)$

$= \frac{1}{2}(m + 1)[3(m + 1) - 1]$

$\therefore$ P(m + 1) is true

Thus P(m) is true $\Rightarrow$ P(m + 1) is true

Hence by the principle of mathematical induction the given result is true for all $n \in N$.

Ex.5 If x and y are any two distinct integers, then prove by mathematical induction that $(x^n - y^n)$ is divisible by (x – y) for all $n \in N$.

Sol. Let P(n) be the statement given by

P(n) : $(x^n - y^n)$ is divisible by (x – y)

Step-I P(1) : $(x^1 - y^1)$ is divisible by (x – y)

$\because x^1 - y^1 = (x - y)$ is divisible by (x – y)

$\therefore$ P(1) is true

Step-II Let P(m) be true, then

$(x^m - y^m)$ is divisible by (x – y)

$\Rightarrow (x^m - y^m) = \lambda(x - y)$ for some $\lambda \in Z$ ...(i)

We shall now show that P(m + 1) is true. For this it is sufficient to show that $(x^{m+1} - y^{m+1})$ is divisible by (x – y).

Now $x^{m+1} - y^{m+1} = x^{m+1} - x^m y + x^m y - y^{m+1}$

$= x^m(x - y) + y(x^m - y^m)$

$= x^m(x - y) + y\lambda(x - y)$ [Using (i)]

$= (x - y)(x^m + y\lambda)$ which is divisible by (x – y)

So P(m + 1) is true

Thus P(m) is true $\Rightarrow$ P(m + 1) is true

Hence by the principle of mathematical induction P(n) is true for all $n \in N$

i.e. $(x^n - y^n)$ is divisible by (x – y) for all $n \in N$

Ex.6 Prove by the principle of mathematical induction that for all $n \in N$, 3^{2n} when divided by 8 the remainder is always 1.

Sol. Let P(n) be the statement given by

P(n) : 3^{2n} when divided by 8 the remainder is 1

or P(n) : $3^{2n} = 8\lambda + 1$ for some $\lambda \in N$

Step-I P(1) : $3^2 = 8\lambda + 1$ for some $\lambda \in N$

$\because\ 3^2 = 8 \times 1 + 1 = 8\lambda + 1$ where $\lambda = 1$

$\therefore$ P(1) is true

Step-II Let P(m) be true then

$3^{2m} = 8\lambda + 1$ for some $\lambda \in N$

We shall now show that P(m + 1) is true for which we have to show that $3^{2(m+1)}$ when divided by 8 the remainder is 1 i.e. $3^{2(m+1)} = 8\mu + 1$ for some $\mu \in N$

Now $3^{2(m+1)} = 3^{2m} \cdot 3^2 = (8\lambda + 1) \times 9$ [Using (i)]

$\Rightarrow$ P(m + 1) is true

Thus P(m) is true $\Rightarrow$ P(m + 1) is true

Hence by the principle of mathematical induction P(n) is true for all $n \in N$ i.e. 3^{2n} when divided by 8 the remainder is always 1.

Ex.7 Prove by the principle of mathematical induction that $n < 2^n$ for all $n \in N$.

Sol. Let P(n) be the statement given by P(n) : $n < 2^n$

Step-I P(1) : $1 < 2^1$

$\because\ 1 < 2^1$

$\therefore$ P(1) is true

Step-II Let P(m) be true, then $m < 2^m$

we shall now show that P(m + 1) is true for which we will have to prove that $(m + 1) < 2^{m+1}$

Now P(m) is true

$\Rightarrow m < 2^m$

$\Rightarrow 2m < 2.2^m \Rightarrow 2m < 2^{m+1} \Rightarrow (m + m) < 2^{m+1}$

$\Rightarrow m+1 \le m+m < 2^{m+1}$ $[\because 1 \le m \therefore m+1 \le m+m]$

$\Rightarrow (m + 1) < 2^{m+1}$

$\Rightarrow$ P(m + 1) is true

Thus P(m) is true $\Rightarrow$ P(m + 1) is true

So by the principle of mathematical induction P(n) is true for all $n \in N$ i.e. $n < 2^n$ for all $n \in N$

Ex.8 Prove that : $1 + 2 + 3 + \ldots + n < \frac{(2n+1)^2}{8}$ for all $n \in N$.

Sol. Let P(n) be the statement given by

$$P(n) : 1 + 2 + 3 + \ldots + n < \frac{(2n+1)^2}{8}$$

Step-I We have

$$P(1) : 1 < \frac{(2\times1+1)^2}{8} \qquad \because 1 < \frac{(2\times1+1)^2}{8} = \frac{9}{8}$$

$\therefore$ P(1) is true

Step-II Let P(m) be true, then

$$1 + 2 + 3 + \ldots + m < \frac{(2m+1)^2}{8} \quad \ldots(i)$$

We shall now show that P(m + 1) is true i.e.

$$1 + 2 + 3 + \ldots + m + (m + 1) < \frac{[2(m+1)+1]^2}{8}$$

Now P(m) is true

$$\Rightarrow 1 + 2 + 3 + \ldots + m < \frac{(2m+1)^2}{8}$$

$$\Rightarrow 1+2+3+\ldots+m+(m+1) < \frac{(2m+1)^2}{8} + (m+1)$$

$$\Rightarrow 1 + 2 + 3 + \ldots + m + (m + 1) < \frac{(2m+1)^2 + 8(m+1)}{8}$$

$$\Rightarrow 1+2+3+\ldots+m+(m+1) < \frac{(4m^2+12m+9)}{8}$$

$$\Rightarrow 1 + 2 + 3 + \ldots + m + (m + 1) < \frac{(2m+3)^2}{8} = \frac{[2(m+1)+1]^2}{8}$$

$\therefore$ P(m + 1) is true

Thus P(m) is true $\Rightarrow$ P(m + 1) is true

Hence by the principle of mathematical induction P(n) is true for all $n \in N$

Ex.9 Prove by the principle of mathematical induction that for all $n \in N$,

$\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$

$$= \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

Sol. Let P(n) be the statement given by

$P(n) : \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$

$$= \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

Step-I We have $P(1) : \sin\theta$

$$= \frac{\sin\left(\frac{1+1}{2}\right)\theta \sin\left(\frac{1\times\theta}{2}\right)}{\sin\frac{\theta}{2}}$$

$$\because \sin\theta = \frac{\sin\left(\frac{1+1}{2}\right)\theta \cdot \sin\left(\frac{1\times\theta}{2}\right)}{\sin\frac{\theta}{2}}$$

$\therefore$ P(1) is true

Step-II Let P(m) be true, then

$$\sin\theta + \sin 2\theta + \ldots + \sin m\theta = \frac{\sin\left(\frac{m+1}{2}\right)\theta \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}} \quad \text{...(i)}$$

We shall now show that P(m + 1) is true

i.e. $\sin\theta + \sin 2\theta + \ldots + \sin m\theta + \sin(m+1)\theta$

$$= \frac{\sin\left(\frac{(m+1)+1}{2}\right)\theta \sin\left(\frac{m+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

We have $\sin\theta + \sin 2\theta + \ldots + \sin m\theta + \sin(m+1)\theta$

$$= \frac{\sin\left(\frac{m+1}{2}\right)\theta \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}} + \sin(m+1)\theta \quad \text{[Using (i)]}$$

$$= \frac{\sin\left(\frac{m+1}{2}\right)\theta \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}} + 2\sin\left(\frac{m+1}{2}\right)\theta \cos\left(\frac{m+1}{2}\right)\theta$$

$$= \sin\left(\frac{m+1}{2}\right)\theta \left\{\frac{\sin\left(\frac{m\theta}{2}\right)}{\sin\frac{\theta}{2}} + 2\cos\left(\frac{m+1}{2}\right)\theta\right\}$$

$$= \sin\left(\frac{m+1}{2}\right)\theta \left\{\frac{\sin\left(\frac{m\theta}{2}\right) + 2\sin\frac{\theta}{2}\cos\left(\frac{m+1}{2}\right)\theta}{\sin\frac{\theta}{2}}\right\}$$

$$= \sin\left(\frac{m+1}{2}\right)\theta \left\{\frac{\sin\left(\frac{m\theta}{2}\right) + \sin\left(\frac{m+2}{2}\right)\theta - \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}}\right\}$$

$$= \frac{\sin\left(\frac{m+1}{2}\right)\theta \sin\left(\frac{m+2}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left\{\frac{(m+1)+1}{2}\right\}\theta \sin\left(\frac{m+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

$\therefore$ P(m + 1) is true

Thus, P(m) is true $\Rightarrow$ P(m + 1) is true

Hence by principle mathematical induction P(n) is true for all $n \in N$

Self Practice Problems :

(1) By using P.M.I. prove that $1.3 + 3.5 + 5.7 + \ldots + (2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$, $n \in N$.

(2) Prove that $12^n + 25^{n-1}$ is divisible by 13 for $n \in N$, by using the principle of mathematical induction.

(3) Prove the following by the principle of mathematical induction :

$7 + 77 + 777 + \ldots + 777\ldots7 = \frac{7}{81}$

$(10^{n+1} - 9n - 10)$ n - digits

(4) Prove the following by the principle of mathematical induction :

$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$ for all $n \geq 2, n \in N$.

(5) Prove the following by the principle of mathematical induction :

$\sin x + \sin 3x + \ldots + \sin(2n-1)x = \frac{\sin^2 nx}{\sin x}$