RELATIONS AND FUNCTIONS

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ORDERED PAIR

A pair of objects listed in a specific order is called an ordered pair. It is written by listing the two objects in specific order separating them by a comma abd enclosing the pair in parantheses.

In the ordered pair (a, b), a is called the first element and b is called the second element.

Two ordered pairs are set to be equal if their corresponding elements are equal. i.e.

(a, b) = (c, d) if a = c and b = d.

CARTESIAN PRODUCT

The set of all possible ordered pairs (a, b), where a $\in$ A and b $\in$ B i.e. $\{(a, b) ; a \in A \text{ and } b \in B\}$ is called the cartesian product of A to B and is denoted by A × B.

$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

$= \{(a_1, b_1),(a_1, b_2),(a_2, b_1),(a_2, b_2),(a_3, b_1),(a_3, b_2)\}$

If set $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2\}$ then

A × B and B × A can be written as :

$A \times B = \{(a, b) : a \in A$ and b $\in$ B and

$B \times A = \{(b, a) ; b \in B \text{ and } a \in A\}$

Clearly $A \times B \neq B \times A$ until A and B are equal

Note :

1. If number of elements in A : n(A) = m and n(B) = n then number of elements in (A × B) = m × n
2. Since A × B contains all such ordered pairs of the type (a, b) such that a $\in$ A & b $\in$ B, that means it includes all possibilities in which the elements of set A can be related with the elements of set B. Therefore, A × B is termed as largest possible relation defined from set A to set B, also known as universal relation from A to B.

   Similarly $A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$ is called ordered triplet.

Solved Examples

Ex.1 If A = {1, 2} and B = {3, 4}, then find A × B.

Sol. A × B = {(1, 3), (1, 4), (2, 3), (2, 4)}

Ex.2 Let A and B be two non-empty sets having elements in common, then prove that A × B and B × A have n^2 elements in common.

Sol. We have $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

On replacing C by B and D by A, we get

$\Rightarrow (A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$ It is given that AB has n elements so $(A \cap B) \times (B \cap A)$ has n^2 elements

But $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

$\therefore$ $(A \times B) \cap (B \times A)$ has n^2 elements

Hence $A \times B$ and $B \times A$ have n^2 elements in common

RELATION

A relation R from set A to B ($R : A \to B$) is a correspondence between set A to set B by which some or more elements of A are associated with some or more elements of B.

Therefore a relation R from A to B is a subset of $A \times B$. Thus, R is a relation from A to B $\Rightarrow R \subseteq A \times B$. The subsets is derived by describing a relationship between the first element and the second element of ordered pairs in $A \times B$ e.g. if $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{1, 2, 3, 4, 5\}$ and $R = \{(a, b) : a = b^2, a \in A, b \in B\}$ then $R = \{(1, 1), (4, 2), (9, 3)\}$. Here a R b $\Rightarrow$ 1 R 1, 4 R 2, 9 R 3.

NOTE :

(i) If a is related to b then symbolically it is written as a R b where a is pre-image and b is image

(ii) If a is not related to b then symbolically it is written as a $\not R$ b.

(iii) Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So total number of subsets of $A \times B$ i.e. number of relations from A to B is 2^{mn}.

(iv) A relation R from A to A is called a relation on A.

Solved Examples

Ex.3 If the number of elements in A is m and number of element in B is n then find

(i) The number of elements in the power set of $A \times B$.

(ii) number of relation defined from A to B

Sol. (i) Since $n(A) = m$; $n(B) = n$ then $n(A \times B) = mn$

So number of subsets of $A \times B = 2^{mn}$

$\Rightarrow \quad n(P(A \times B)) = 2^{mn}$

(ii) number of relation defined from A to B = 2^{mn}

Any relation which can be defined from set A to set B will be subset of $A \times B$

$\because$ $A \times B$ is largest possible relation $A \to B$

$\therefore$ no. of relation from $A \to B$ = no. of subsets of set $(A \times B)$

DOMAIN, CO-DOMAIN AND RANGE OF A RELATION :

Let R be a relation from a set A to a set B. Then the set of all first components of coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components of corrdinates of the ordered pairs in R is called the range of R.

Thus, Domain (R) = $\{a : (a, b) \in R\}$,

Co-domain (R) = B and

Range (R) = $\{b : (a, b) \in R\}$

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of Co-domain (B).

Solved Examples

Ex.4 Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$ be two sets and let R be a relation from A to B defined by the phrase "$(x, y) \in R \Rightarrow x > y$". Find relation R and its domain and range.

Sol. Under relation R, we have 3R2, 5R2, 5R4, 7R4 and 7R6

i.e. $R = \{(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)\}$

$\therefore$ Dom (R) = $\{3, 5, 7\}$ and range (R) = $\{2, 4, 6\}$

Ex.5 Let $A = \{2, 3, 4, 5, 6, 7, 8, 9\}$. Let R be the relation on A defined by

$\{(x, y) : x \in A, y \in A$ and x divides y$\}$.

Find domain and range of R.

Sol. The relation R is

$R = \{(2, 2), (2, 4), (2, 6), (2, 8), (3, 3), (3, 6), (3, 9), (4, 4), (4, 8), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)\}$

Domain of R = $\{2, 3, 4, 5, 6, 7, 8, 9\}$ = A

Range of R = $\{2, 3, 4, 5, 6, 7, 8, 9\}$ = A

Ex.6 Let R be the relation on the set N of natural numbers defined by

$R : \{(x, y)\} : x + 3y = 12 \;\; x \in N, y \in N\}$ Find

(i) R **(ii)** Domain of R **(iii)** Range of R

Sol. (i) We have, $x + 3y = 12 \Rightarrow x = 12 - 3y$

Putting $y = 1, 2, 3$, we get $x = 9, 6, 3$ respectively

For $y = 4$, we get $x = 0 \notin N$. Also for $y > 4$, $x \notin N$

$\therefore$ $R = \{(9, 1), (6, 2), (3, 3)\}$

(ii) Domain of $R = \{9, 6, 3\}$

(iii) Range of $R = \{1, 2, 3\}$

INVERSE OF A RELATION

Let A, B be two sets and let R be a relation from a set A to B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$, Clearly,

$(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$ Also,

Dom of R = Range of R^{-1} and

Range of R = Dom of R^{-1}

Solved Examples

Ex.7 Let A be the set of first ten natural numbers and let R be a relation on A defined by $(x, y) \in R \Leftrightarrow x + 2y = 10$ i.e., $R = \{(x, y) : x \in A, y \in A \text{ and } x + 2y = 10\}$. Express R and R^{-1} as sets of ordered pairs. Determine also :

(i) Domains of R and R^{-1}

(ii) Range of R and R^{-1}

Sol. We have $(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \frac{10-x}{2}$, $x, y \in A$

where $A = \{1,2,3,4,5,6,7,8,9,10\}$

Now, $x = 1 \Rightarrow y = \frac{10-1}{2} = \frac{9}{2} \notin A$

This shows that 1 is not related to any element in A. Similarly we can observe that 3, 5, 7, 9 and 10 are not related to any element of a under the defined relation. Further we find that

for $x = 2$, $y = \frac{10-2}{2} = 4 \in A$ $\therefore$ $(2, 4) \in R$

for $x = 4$, $y = \frac{10-4}{2} = 3 \in A$ $\therefore$ $(4, 3) \in R$

for $x = 6$, $y = \frac{10-6}{2} = 2 \in A$ $\therefore$ $(6, 2) \in R$

for $x = 8$, $y = \frac{10-8}{2} = 1 \in A$ $\therefore$ $(8, 1) \in R$

Thus $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$

$\Rightarrow$ $R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, Dom (R) = $\{2, 4, 6, 8\}$ = Range (R^{-1})

and Range (R) = $\{4, 3, 2, 1\}$ = Dom (R^{-1})

TYPES OF RELATIONS

In this section we intend to define various types of relations on a given set A.

(i) Void relation : Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

(ii) Universal relation : Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

(iii) Identity relation : Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A. In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

(iv) Reflexive relation : A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus, R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

Note :

Every identity relation is reflexive but every reflexive relation in not identity.

(v) Symmetric relation : A relation R on a set A is said to be a symmetric relation iff $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$. i.e. $a R b \Rightarrow b R a$ for all $a, b \in A$. **(vi) Transitive relation :** Let A be any set. A relation R on A is said to be a transitive relation iff $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$ i.e. $a R b$ and $b R c \Rightarrow a R c$ for all $a, b, c \in A$

Note :

(i) Every null relation is a symmetric and transitive relation.

(ii) Every singleton relation is a transitive relation.

(iii) Universal and identity relations are reflexive, symmetric as well as transitive.

(iv) Anti-symmetric Relation

Let A be any set. A relation R on set A is said to be an antisymmetric relation iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$ e.g. Relations "being subset of"; "is greater than or equal to" and "identity relation on any set A" are antisymmetric relations.

(v) Equivalence relation : A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b \in A$

Solved Examples

Ex.8 Which of the following are identity relations on set $A = \{1, 2, 3\}$.

$R_1 = \{(1, 1), (2, 2)\}$, $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$, $R_3 = \{(1, 1), (2, 2), (3, 3)\}$.

Sol. The relation R_3 is idenity relation on set A.

R_1 is not identity relation on set A as $(3, 3) \notin R_1$.

R_2 is not identity relation on set A as $(1, 3) \in R_2$

Ex.9 Which of the following are reflexive relations on set $A = \{1, 2, 3\}$.

$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$, $R_2 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$.

Sol. R_1 is a reflexive relation on set A.

R_2 is not a reflexive relation on A because $2 \in A$ but $(2, 2) \notin R_2$.

Ex.10 Prove that on the set N of natural numbers, the relation R defined by $x R y \Rightarrow x$ is less than y is transitive.

Sol. Because for any $x, y, z \in N$ $x < y$ and $y < z \Rightarrow x < z$ $\Rightarrow x R y$ and $y R z \Rightarrow x R z$. so R is transitive.

Ex.11 Let T be the set of all triangles in a plane with R a relation in T given by $R = \{(T_1, T_2) : T_1$ is congruent to $T_2\}$. Show that R is an equivalence relation.

Sol. Since a relation R in T is said to be an equivalenece relation if R is reflexive, symmetric and transitive.

(i) Since every triangle is congruent to itself

$\therefore$ R is reflexive

(ii) $(T_1, T_2) \in R \Rightarrow T_1$ is congruent to T_2

$\Rightarrow T_2$ is congruent to T_1 $\Rightarrow (T_2, T_1) \in R$

Hence R is symmetric

(iii) Let $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$

$\Rightarrow T_1$ is congruent to T_2 and T_2 is congruent to T_3

$\Rightarrow T_1$ is congruent to T_3 $\Rightarrow (T_1, T_3) \in R$

$\therefore$ R is transitive

Hence R is an equivalence relation.

Ex.12 Show that the relation R in R defined as $R = \{(a, b) : a \le b\}$ is transitive.

Sol. Let $(a, b) \in R$ and $(b, c) \in R$

$\therefore$ $(a \le b)$ and $b \le c \Rightarrow a \le c$

$\therefore$ $(a, c) \in R$

Hence R is transitive.

Ex.13 Show that the relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ is symmetric.

Sol. Let $(a, b) \in R$ $[\because (1, 2) \in R]$

$\therefore$ $(b, a) \in R$ $[\because (2, 1) \in R]$

Hence R is symmetric.

Ex.14 If $X = \{x_1, x_2, x_3\}$ and $y = (x_1, x_2, x_3, x_4, x_5\}$ then find which is a reflexive relation of the following :

(a) $R_1 : \{(x_1, x_1), (x_2, x_2)$

(b) $R_1 : \{(x_1, x_1), (x_2, x_2), (x_3, x_3)$

(c) $R_3 : \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_1, x_3), (x_2, x_4)$

(d) $R_3 : \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4)$

Sol. (a) non-reflexive because $(x_3, x_3) \notin R_1$

(b) Reflexive

(c) Reflexive

(d) non-reflexive because $x_4 \notin X$

Ex.15 If x = {a, b, c} and y = {a, b, c, d, e, f} then find which of the following relation is symmetric relation :

R_1 : { } i.e. void relation

R_2 : {(a, b)}

R_3 : {(a, b), (b, a)(a, c)(c, a)(a, a)}

Sol. R_1 is symmetric relation because it has no element in it.

R_2 is not symmetric because (b, a) ∈ R_2 & R_3 is symmetric.

Ex.16 If x = {a, b, c} and y = (a, b, c, d, e} then which of the following are transitive relation.

(a) R_1 = { }

(b) R_2 = {(a, a)}

(c) R_3 = {(a, a}.(c, d)}

(d) R_4 = {(a, b), (b, c)(a, c),(a, a), (c, a)}

Sol. (a) R_1 is transitive relation because it is null relation.

(b) R_2 is transitive relation because all singleton relations are transitive.

(c) R_3 is transitive relation

(d) R_4 is also transitive relation