STRAIGHT LINE

EQUATION OF STRAIGHT LINE

A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called the equation of straight line. Every linear equation in two variable x and y always represents a straight line

eg. 3x + 4y = 5

General form of straight line is given by

ax + by + c = 0

EQUATION OF THE AXES

- (1) Equation of x-axis: y=0
- (2) Equation a line parallel to x axis at a distance 'a' from it ⇒ y=a
- (3) Equation of y-axis: x=0
- (4) Equation of a line parallel to y axis at a distance 'a' from it ⇒ x = a

SLOPE OFALINE

The slope of a line is equal to the tangent of the angle which it makes with the positive side of x-axis and it is generally denoted by m.

Thus if a line makes an angle θ with x-axis then its $|slope = m = tan \theta|$



The slope of a line joining two points (x_1, y_1) and

$$(x_2, y_2)$$
 is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$

Note :

- (i) -∞<m ≤∞
- (ii) Slope of x axis or a line parallel to x axis is $\tan 0^\circ = 0$
- (iii) Slope of y axis or a line parallel to y axis is $\tan 90^\circ = \infty$

DIFFERENT FORMS OF THE EQUATION OF STRAIGHT LINE

1. General form = ax + by + c = 0where a, b, c are any real numbers and a, b are not zero both at a time. Particular case; In ax + by + c = 0 $a = 0 \Rightarrow by + c = 0$ which is a line parallel to x axis $b = 0 \Rightarrow ax + c = 0$ which is a line parallel to y axis $c = 0 \Rightarrow ax + by = 0$ which is a line passing through origin.

Solved Examples

- **Ex.1** What is the slope of a line whose inclination with the positive direction of x-axis is :
 - (i) 0° (ii) 90° (iii) 120° (iv) 150°
- **Sol.** (i) Here $\theta = 0^{\circ}$

Slope = $\tan \theta = \tan 0^{\circ} = 0$.

- (ii) Here $\theta = 90^{\circ}$
 - \therefore The slope of line is not defined.
- (iii) Here $\theta = 120^{\circ}$ \therefore Slope = tan θ = tan 120° = tan (180° - 60°) = - tan 60° = - $\sqrt{3}$.
- (iv) Here $\theta = 150^{\circ}$
 - $\therefore \text{ Slope} = \tan \theta = \tan 150^\circ = \tan (180^\circ 30^\circ)$

$$= -\tan 30^{\circ} = -\frac{1}{\sqrt{3}}$$

Ex.2 Find the slope of the line passing through the points :

(i)
$$(1, 6)$$
 and $(-4, 2)$ (ii) $(5, 9)$ and $(2, 9)$

Sol. (i) Let
$$A = (1, 6)$$
 and $B = (-4, 2)$

$$\therefore \qquad \text{Slope of AB} = \frac{2-6}{-4-1} = \frac{-4}{-5} = \frac{4}{5}$$
$$\left(\text{Using slope} = \frac{y_2 - y_1}{x_2 - x_1} \right)$$
$$(ii) \text{ Let} \qquad \text{A} = (5, 9), \text{ B} = (2, 9)$$
$$\therefore \qquad \text{Slope of AB} = \frac{9-9}{2-5} = \frac{0}{-3} = 0$$

2. Point - Slope form :

 $y-y_1 = m (x - x_1)$ is the equation of a straight line whose slope is m & which passes through the point (x_1, y_1) .

Solved Examples

- **Ex.3** Find the equation of a line passing through (2, -3) and inclined at an angle of 135° with the positive direction of x-axis.
- Sol. Here, m = slope of the line = tan 135° = tan $(90^\circ + 45^\circ) = -\cot 45^\circ = -1$, $x_1 = 2$, $y_1 = -3$ So, the equation of the line is $y - y_1 = m (x - x_1)$ i.e. y - (-3) = -1 (x - 2) or y + 3 = -x + 2 or x + y + 1 = 0
- 3. Slope form :- y = mx + c

Where 'm' is the slope of the line and 'c' is the length of the intercept made by it on y axis

for general form ax + by + c = 0



Solved Examples

- **Ex.4** Find the equation of a line with slope –1 and cutting off an intercept of 4 units on negative direction of y-axis.
- Sol. Here m = -1 and c = -4. So, the equation of the line is y = mx + ci.e. y = -x - 4 or x + y + 4 = 0
- **Ex.5** Equation of a line which is passing through (3, -4) and making an angle of 45° with x axis is
- [1] x y 7 = 0[3] x - y + 7 = 0Sol. y - (-4) = tan 45°(x-3) $\Rightarrow x - y - 7 = 0$ Ans.[1] [2] x + y + 7 = 0[4] x + y - 7 = 0 $\Rightarrow y + 4 = x - 3$ Ans.[1]

4. Two point form : The equation of a line passing through two given points (x_1, y_1) and $(x_2 - y_2)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

Solved Examples

Ex.6 Equation of a line passing through (3, -4) and (4, 3) is [1] y = 7x + 25 [2] y = 7x - 25[3] y = -7x - 25 [4] y = -7x + 25

Sol. $y + 4 = \frac{3+4}{4-3} (x-3)$ or y = 7x - 25 **Ans.[2]**

- **Ex.7** Find the equation of the line joining the points (-1, 3) and (4, -2)
- Sol. Here the two points are $(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (4, -2)$.

So, the equation of the line in two-point form is

$$y-3 = \frac{3-(-2)}{-1-4} (x+1) \quad \Rightarrow \quad y-3 = -x-1$$

$$\Rightarrow x+y-2 = 0$$

5. Determinant form : Equation of line passing through

$$(x_1, y_1)$$
 and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Solved Examples

Ex.8 Find the equation of line passing through (2, 4) & (-1, 3).

Sol. $\begin{vmatrix} x & y & 1 \\ 2 & 4 & 1 \\ -1 & 3 & 1 \end{vmatrix} = 0 \implies x - 3y + 10 = 0$

6. Intercept form: $\frac{x}{a} + \frac{y}{b} = 1$

It is the equation of straight line which cuts off intercepts a and b on the axes of x and y respectively.

- (a) length of the intercept of the line between coordinate axes $AB = \sqrt{a^2 + b^2}$
- (b) Area of $\triangle OAB = \frac{ab}{2}$
- (c) Coordinates of the points P which divides AB in the ratio $m_1 : m_2$ are $\left(\frac{m_2 a}{m_1 + m_2}, \frac{m_1 b}{m_1 + m_2}\right)$

Ex.9 Equation of a line which makes intercepts 3 and 4 on x axis and y axis respectively is

[1]
$$4x + 3y = 12$$

[3] $6x + y = 12$
[4] $4x - 3y = 12$

- **Sol.** $\frac{x}{3} + \frac{y}{4} = 1 \Rightarrow 4x + 3y = 12$ **Ans.[1]**
- **Ex.10** Find the equation of the line which passes through the point (3, 4) and the sum of its intercepts on the axes is 14.
- Sol. Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ (i) This passes through (3, 4), therefore $\frac{3}{a} + \frac{4}{b} = 1$...(ii) It is given that $a + b = 14 \implies b = 14 - a$. Putting b = 14 - a in (ii), we get $\frac{3}{a} + \frac{4}{14 - a} = 1$ $\Rightarrow a^2 - 13a + 42 = 0$ $\Rightarrow (a - 7) (a - 6) = 0 \Rightarrow a = 7, 6$ For a = 7, b = 14 - 7 = 7 and for a = 6, b = 14 - 6 = 8. Putting the values of a and b in (i), we get the

Putting the values of a and b in (1), we get the equations of the lines

$$\frac{x}{7} + \frac{y}{7} = 1$$
 and $\frac{x}{6} + \frac{y}{8} = 1$ or
x + y = 7 and 4x + 3y = 24

7. Perpendicular/Normal form :

 $x\cos \alpha + y\sin \alpha = p$ (where $p > 0, 0 \le \alpha < 2\pi$) is the equation of the straight line where the length of the perpendicular from the origin O on the line is p and this perpendicular makes an angle α with positive x-axis.

Solved Examples

Ex.11 Find the equation of the line which is at a distance 3 from the origin and the perpendicular from the origin to the line makes an angle of 30° with the positive direction of the x-axis.

Sol. Here p = 3, $\alpha = 30^{\circ}$

 \therefore Equation of the line in the normal form is

$$x \cos 30^{\circ} + y \sin 30^{\circ} = 3 \text{ or } x \frac{\sqrt{3}}{2} + \frac{y}{2} = 3 \text{ or}$$

 $\sqrt{3} x + y = 6$

Ex.12 Equation of a line on which length of pependicular from origin is 4 and inclination of this perpendicular is 60° with the positive direction of x axis is

[1]
$$x + y\sqrt{3} = 8$$

[3] $y - x\sqrt{3} = 8$
[3] $y - x\sqrt{3} = 8$
Sol. $x \cos 60^\circ + y \sin 60^\circ = 4$
 $\frac{x}{2} + \frac{y\sqrt{3}}{2} = 4 \implies x + y\sqrt{3} = 8$
Ans.[1]

8. Parametric form

 $\frac{\mathbf{x} - \mathbf{x}_1}{\cos \theta} = \frac{\mathbf{y} - \mathbf{y}_1}{\sin \theta} = \mathbf{r}$

It is the equation of a straight line passes through a given point A (x₁, y₁) and makes an angle θ with x axis. The coordinates (x, y) of any point P on this line are (x₁ + r cos θ , y₁ + r sin θ). The distance of this point P from the given point A is $\sqrt{(x_1 + r \cos \theta - x_1)^2 + (y_1 + r \sin \theta - y_1)^2} = r$

Solved Examples

Ex.13 Equation of a line which passes through point A(2,3) and makes an angle of 45° with x axis. If this line meet the line x + y + 1 = 0 at point P then distance AP is

$$[1] \ _{2\sqrt{3}} \ \ [2] \ _{3\sqrt{2}} \qquad [3] \ _{5\sqrt{2}} \ \ [4] \ _{2\sqrt{5}}$$

Sol. Here $x_1 = 2$, $y_1 = 3$ and $\theta = 45^{\circ}$

hence
$$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r$$

from first two parts $\Rightarrow x-2=y-3$
 $\Rightarrow x-y+1=0$
Co-ordinate of point P on this line is $\left(2+\frac{r}{\sqrt{2}},3+\frac{r}{\sqrt{2}}\right)$
if this point is on line $x+y+1=0$ then
 $\left(2+\frac{r}{\sqrt{2}}\right)+\left(3+\frac{r}{\sqrt{2}}\right)+1=0$
 $\Rightarrow r=-3\sqrt{2}$; $|r|=3\sqrt{2}$ Ans.[2]

- **Ex.14** Find the equation of the line through the point A(2, 3) and making an angle of 45° with the x-axis. Also determine the length of intercept on it between A and the line x + y + 1 = 0
- **Sol.** The equation of a line through A and making an angle of 45° with the x-axis is

$$\frac{x-2}{\cos 45^{\circ}} = \frac{y-3}{\sin 45^{\circ}} \text{ or } \frac{x-2}{\frac{1}{\sqrt{2}}} = \frac{y-3}{\frac{1}{\sqrt{2}}} \text{ or }$$

Suppose this line meets the line x + y + 1 = 0 at P such that AP = r. Then the co-ordinates of P are given by

$$\frac{x-2}{\cos 45^{\circ}} = \frac{y-3}{\sin 45^{\circ}} = r$$

$$\Rightarrow x = 2 + r \cos 45^{\circ}, y = 3 + r \sin 45^{\circ}$$

$$\Rightarrow x = 2 + \frac{1}{\sqrt{2}}, y = 3 + \frac{1}{\sqrt{2}}$$

Thus, the co-ordinates of P are $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$ Since P lies on x + y + 1 = 0,

so $2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} + 1 = 0$ $\Rightarrow \sqrt{2} r = -6 \Rightarrow r = -3\sqrt{2}$ $\Rightarrow \text{ length } AP = |r| = 3\sqrt{2}$

Thus, the length of the intercept = $3\sqrt{2}$.

REDUCTION OF GENERAL FORM OF EQUATION INTO STANDARD FORM

General form of equation ax + by + c = 0 then its

1. Slope intercept form is

$$y = -\frac{ax}{b} - \frac{c}{b}$$
, here slope $m = -\frac{a}{b}$,
intercept $c = \frac{-c}{b}$

2. Intercept form is

$$\frac{x}{-c/a} + \frac{y}{-c/b} = 1$$
, here x intercept is $= -\frac{c}{a}$,
y intercept is $= -\frac{c}{b}$

3. Normal form is :- To change the general form of a line into normal form, first take c to right hand side and make it positive, then divide the whole equation

by
$$\sqrt{a^2 + b^2}$$
 like
 $-\frac{ax}{\sqrt{a^2 + b^2}} - \frac{by}{\sqrt{a^2 + b^2}} = \frac{c}{\sqrt{a^2 + b^2}}$,
here $\cos\alpha = -\frac{a}{\sqrt{a^2 + b^2}}$
 $\sin\alpha = -\frac{b}{\sqrt{a^2 + b^2}}$ and $p = \frac{c}{\sqrt{a^2 + b^2}}$

Solved Examples

Ex.15 Find slope, x-intercept & y-intercept of the line 2x - 3y + 5 = 0.

Sol. Here,
$$a = 2, b = -3, c = 5$$

 $\therefore \text{ slope} = -\frac{a}{b} = \frac{2}{3}$
x-intercept $= -\frac{c}{a} = -\frac{5}{2}$ y-intercept $= \frac{5}{3}$

ANGLE BETWEEN TWO LINES

1. The angle θ between two lines whose slopes are m₁ and m₂ is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

- (i) If the lines are parallel, then $m_1 = m_2$
- (ii) If the lines are perpendicular, then $m_1 m_2 = -1$
- 2. If the given equation of straight line are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, then the angle between the lines ' θ ' is given by

$$\tan\theta = \frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}$$

(i) If the lines are parallel, then

 $a_{2}b_{1} - a_{1}b_{2} = 0 \text{ or } \frac{a_{1}}{a_{2}} = \frac{b_{1}}{b_{2}}$

(ii) If the lines are perpendicular, then $a_1a_2 + b_1b_2 = 0$

(iii) Coincident :-
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

(iv) In particulars, angle between lines $a_1 x + b_1 y + c_1$ = 0 and $b_2 y + c_2 = 0$ is $\tan^{-1} \left| \frac{a_1}{b_1} \right|$ and angle between lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x$ $+ c_2 = 0$ is $\tan^{-1} \left| \frac{b_1}{a_1} \right|$ (v) Angle between lines

(v) Angle between lines

$$x \cos \alpha + y \sin \alpha = p_1$$
 and $x \cos \beta + y \sin \beta = p_2$
 $= |\alpha - \beta|$

- **Ex.16** The acute angle between two lines is $\pi/4$ and slope of one of them is 1/2. Find the slope of the other line.
- Sol. If θ be the acute angle between the lines with slopes

$$m_{1} \text{ and } m_{2}, \text{ then } \tan \theta = \left| \frac{m_{1} - m_{2}}{1 + m_{1}m_{2}} \right|$$
Let $\theta = \frac{\pi}{4}$ and $m_{1} = \frac{1}{2}$

$$\therefore \tan \frac{\pi}{4} = \left| \frac{\frac{1}{2} - m_{2}}{1 + \frac{1}{2}m_{2}} \right|$$

$$\Rightarrow 1 = \left| \frac{1 - 2m_{2}}{2 + m_{2}} \right|$$

$$\Rightarrow \frac{1 - 2m_{2}}{2 + m_{2}} = +1 \text{ or } -1$$

Ex.17 Find the equation of the straight line which passes through the origin and making angle 60° with the line

$$\mathbf{x} + \sqrt{\mathbf{3}} \mathbf{y} + 3\sqrt{\mathbf{3}} = 0.$$

Sol. Given line is $x + \sqrt{3} y + 3\sqrt{3} = 0$.

⇒
$$y = \left(-\frac{1}{\sqrt{3}}\right) x - 3$$

∴ Slope of (1) = $-\frac{1}{\sqrt{3}}$.

Let slope of the required line be m. Also between these lines is given to be 60° .

$$\Rightarrow \tan 60^{\circ} = \left| \frac{m - \left(-\frac{1}{\sqrt{3}}\right)}{1 + m \left(-\frac{1}{\sqrt{3}}\right)} \right|$$

$$\Rightarrow \sqrt{3} = \left| \frac{\sqrt{3m} + 1}{\sqrt{3} - m} \right| \qquad \Rightarrow \quad \frac{\sqrt{3m} + 1}{\sqrt{3} - m} = \pm \sqrt{3}$$

$$\frac{\sqrt{3m} + 1}{\sqrt{3} - m} = \sqrt{3} \qquad \Rightarrow \quad \sqrt{3} m + 1 = 3 - \sqrt{3} m$$

$$\Rightarrow m = \frac{1}{\sqrt{3}}$$

Using y = mx + c, the equation of the required line is

$$y = \frac{1}{\sqrt{3}} x + 0$$

i.e. $x - \sqrt{3} y = 0$.
(:: This passes through origin, so $c = 0$)

 $\frac{\sqrt{3}m+1}{\sqrt{3}-m} = -\sqrt{3} \implies \sqrt{3}m+1 = -3 + \sqrt{3}m$ \Rightarrow m is not defined

 \therefore The slope of the required line is not defined. Thus, the required line is a vertical line. This line is to pass through the origin.

 \therefore The equation of the required line is x = 0

Ex.18 The angle between the lines
$$2x - y + 1 = 0$$
 and $x + 5 = 0$ is given by

- [1] $\tan^{-1}\frac{1}{2}$ [2] $\tan^{-1}1$ [3] $\tan^{-1}2$ [4] none of these $\tan^{-1}\frac{|-1|}{|-1|}$ (-1) (1/2) (-5) (-5)
- **Sol.** $\tan^{-1} \left| \frac{-1}{2} \right| = \tan^{-1} (1/2)$ (use form (iv)) **Ans.[1]**
- **Ex.19** The angle between the lines 2x y + 1 = 0 and 2y + 1 = 0 is given by

[1]
$$\tan^{-1}\frac{1}{2}$$
 [2] $\tan^{-1} 1$
[3] $\tan^{-1} 2$ [4] none of these

Sol.
$$\tan^{-1} \left| \frac{2}{-1} \right| = \tan^{-1} 2$$
 (use form (iv)) **Ans.[3]**

Ex.20 Angle between y = x + 6 and

$$y = \sqrt{3} \times +7 \text{ is here } m_1 = 1 \quad m_2 = \sqrt{3}$$

[1] 75 [2] 45 [3] 15 [4] 30
$$d_1 \tan 0 = \frac{|1 - \sqrt{3}|}{|1 - \sqrt{3}|} = 10 = \tan^{-1} \frac{1 - \sqrt{3}}{3} = 15^\circ \text{ Ans } 10^\circ \text{ fm}$$

Sol.
$$\tan \theta = \left| \frac{1 - \sqrt{3}}{1 + \sqrt{3}} \right| \Rightarrow \theta = \tan^{-1} \frac{1 - \sqrt{3}}{1 + \sqrt{3}} = 15^{\circ}$$
 Ans.[3]

Ex.21 If A(1,2), B(-1, 3) and C (3, -5) be vertices of a triangle then find $\angle B$

[1]
$$\tan^{-1}\frac{1}{2}$$
 [2] $\tan^{-1}1$
[3] $\tan^{-1}2$ [4] $\tan^{-1}\frac{3}{4}$

Sol. Slope of $AB = m_1 = -1/2$

Slope of BC =
$$m_2 = -2$$

 $\therefore \angle B = \tan^{-1} \left| \frac{-1/2 + 2}{1 + (-1/2)(-2)} \right| = \tan^{-1} (3/4)$ Ans.[4]

Ex.22 If $x + 4y - 5 = 0$ and $4x + ky + 7 = 0$ are two
perpendicular lines then k is
(1) 3 (2) 4 (3) -1 (4) -4
Sol. $m_1 = -\frac{1}{4}$ $m_2 = -\frac{4}{k}$
Two lines are perpendicular if $m_1 m_2 = -1$
$\Rightarrow \left(-\frac{1}{4}\right) \times \left(-\frac{4}{k}\right) = -1 \Rightarrow k = -1 \text{ Ans.[3]}$
Ex.23 If $7x + 3y + 9 = 0$ and $y = kx + 7$ are two parallel lines then k is
$(1) 3/7 (2) -7/3 \qquad (3) 3 \qquad (4) 7$
Sol. $m_1 = -\frac{7}{3}$ $m_2 = k$
Two lines are parallel if $m_1 = m_2$; $k = -7/3$ Ans.[2]

Parallel Lines :

- (i) When two straight lines are parallel their slopes are equal. Thus any line parallel to y = mx + c is of the type y = mx + d, where 'd' is a
- parameter. (ii) Two lines ax + by + c = 0 and a'x + b'y + c' = 0are parallel if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$.

Thus any line parallel to ax + by + c = 0 is of the type ax + by + k = 0, where k is a parameter.

(iii) The distance between two parallel lines with equations $ax + by + c_1 = 0$ &

$$ax + by + c_2 = 0$$
 is $= \left| \frac{c_1 - c_2}{\sqrt{a^2 + b^2}} \right|$.

Note that coefficients of x & y in both the equations must be same.

(iv) The area of the parallelogram $= \frac{p_1 p_2}{\sin \theta}$, where $p_1 \& p_2$ are distances between two pairs of opposite sides $\& \theta$ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1 x + c_1$, $y = m_1 x + c_2$ and $y = m_2 x + d_1$, $y = m_2 x + d_2$ is given



Solved Examples

Ex.24 Find the equation of the straight line that has y-intercept 4 and is parallel to the straight line 2x - 3y = 7.

Sol. Given line is 2x - 3y = 7

(1)
$$\Rightarrow$$
 $3y = 2x - 7 \Rightarrow$ $y = \frac{2}{3}x - \frac{7}{3}$

 \therefore Slope of (1) is 2/3

The required line is parallel to (1), so its slope is also 2/3, y-intercept of required line = 4

 \therefore By using y = mx + c form, the equation of the required line is

$$y = \frac{2}{3}x + 4$$
 or $2x - 3y + 12 = 0$

- **Ex.25** Two sides of a square lie on the lines x + y = 1and x + y + 2 = 0. What is its area?
- **Sol.** Clearly the length of the side of the square is equal to the distance between the parallel lines

$$x + y - 1 = 0$$
(i) and
 $x + y + 2 = 0$ (ii)

Putting x = 0 in (i), we get y = 1. So (0, 1) is a point on line (i).

Now, Distance between the parallel lines

= length of the
$$\perp$$
 from (0, 1) to x + y + 2 = 0

$$=\frac{|0+1+2|}{\sqrt{1^2+1^2}}=\frac{3}{\sqrt{2}}\frac{3}{\sqrt{2}}$$

Thus, the length of the side of the square is $\frac{3}{\sqrt{2}}$ and

hence its area =
$$\left(\frac{3}{\sqrt{2}}\right)^2 = \frac{9}{2}$$

Ex.26 Find the area of the parallelogram whose sides are x + 2y + 3 = 0, 3x + 4y - 5 = 0,

$$2x + 4y + 5 = 0$$
 and $3x + 4y - 10 = 0$



Here,	$c_1 = -\frac{3}{2},$	$c_2 = -\frac{5}{4},$	
$d_1 = \frac{10}{4},$	$d_2 = \frac{5}{4}, m_1 = -$	$\frac{1}{2}, m_2 = -$	$\frac{3}{4}$
∴ Area=	$= \frac{\left(\frac{-\frac{3}{2}+\frac{5}{4}}{-\frac{1}{2}+\frac{3}{4}}\right)\left(\frac{10}{4}-\frac{1}{2}+\frac{3}{4}\right)}{\left(-\frac{1}{2}+\frac{3}{4}\right)}$	$\frac{-\frac{5}{4}}{2} = \frac{5}{4}$	sq. units

Ex.27 The distance between the lines 3x + 4y - 5 = 0and 3x + 4y + 7 = 0

Sol. = $\frac{|-5-7|}{\sqrt{9+16}} = \frac{12}{5}$

Ex.28 Equation of a line which passes through (4,6) and parallel to 3x - 7y + 2 = 0 is

(1) 3x - 7y + 30 = 0(2) 3x + 7y + 30 = 0(3) 3x - 7y - 30 = 0(4) 3x + 7y - 30 = 0

Sol. Let the equation is 3x - 7y + k = 0 this line passes through (4, 6) Hence $3(4) - 7(6) + k = 0 \Rightarrow k = 30$

the requiired equation is 3x - 7y + 30 = 0Ans.[1]

Perpendicular Lines:

(i) When two lines of slopes $m_1 \& m_2$ are at right angles, the product of their slopes is -1, i.e. $m_1 m_2 = -1$. Thus any line perpendicular to y = mx + c is of the form

 $y = -\frac{1}{m}x + d$, where 'd' is any parameter.

(ii) Two lines ax + by + c = 0 and a'x + b'y + c' = 0 are perpendicular if aa' + bb' = 0. Thus any line perpendicular to ax + by + c = 0 is of the form bx - ay + k = 0, where 'k' is any parameter.

Solved Examples

Ex.29 Find the equation of the straight line that passes through the point (3, 4) and perpendicular to the line 3x + 2y + 5 = 0 Sol. The equation of a line perpendicular to 3x + 2y + 5 = 0 is

 $2x - 3y + \lambda = 0 \qquad \qquad \dots \dots \dots (i)$

This passes through the point (3, 4)

 $\therefore 3 \times 2 - 3 \times 4 + \lambda = 0 \Rightarrow \lambda = 6$

Putting $\lambda = 6$ in (i), we get 2x - 3y + 6 = 0, which is the required equation.

<u>Aliter</u>

The slope of the given line is -3/2. Since the required line is perpendicular to the given line. So, the slope of the required line is 2/3. As it passes through

(3, 4). So, its equation is $y - 4 = \frac{2}{3} (x - 3)$ or 2x - 3y + 6 = 0

POSITION OF A POINT RELATIVE TO A LINE

- (i) The point (x_1, y_1) lies on the line ax + by + c = 0 if, $ax_1 + by_1 + c = 0$
- (ii) If $P(x_1, y_1)$ and $Q(x_2, y_2)$ do not lie on the line as + by +c = 0 then they are on the same side of the line, if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign and they lie on the opposite side of line if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the opposite sign.
- (iii) (x_1, y_1) is on origin or non origin sides of the line ax + by + c = 0 if $ax_1 + by_1 + c$ and c are of the same or opposite signs.

Solved Examples

Ex.30 Points (3,4) and (-9,6) lie the line 7x + 5y - 9 = 0

- [1] on same side
- [2] on opposite side
- [3] on the line
- [4] nothing can be predicted

Sol. Putting both the points in the given equation

$$\Rightarrow$$
 7 × 3 + 5 × 4 - 9 = 32 and 7 × (-9) + 5(6) - 9
= -42

as both are of opposite sign so they lie opposite side of the given line

Ans.[2]

- **Ex.31** Show that (1, 4) and (0, -3) lie on the opposite sides of the line x + 3y + 7 = 0.
- Sol. At (1, 4), the value of x + 3y + 7 = 1 + 3(4) + 7= 20 > 0.

At (0, -3), the value of x + 3y + 7 = 0 + 3(-3) + 7= -2 < 0

:. The points (1, 4) and (0, -3) are on the opposite sides of the given line.

The ratio in which a given line divides the line segment joining two points :

Let the given line ax + by + c = 0 divide the line segment joining $A(x_1, y_1) \& B(x_2, y_2)$ in the ratio

m : n, then $\frac{m}{n} = -\frac{ax_1+by_1+c}{ax_2+by_2+c}$. If A & B are on the

same side of the given line then m/n is negative but if A & B are on opposite sides of the given line, then m/n is positive

Solved Examples

Ex.32 Find the ratio in which the line joining the points A (1, 2) and B(-3, 4) is divided by the line x + y - 5 = 0.

Sol. Let the line x + y = 5 divides AB in the ratio k : 1 at P

 $\therefore \text{ co-ordinate of P are } \left(\frac{-3k+1}{k+1}, \frac{4k+2}{k+1}\right)$ Since P lies on x + y - 5 = 0

$$\therefore \ \frac{-3k+1}{k+1} + \frac{4k+2}{k+1} - 5 = 0 \implies k = -\frac{1}{2}$$

 \therefore Required ratio is 1 : 2 extremally.

<u>Aliter</u> Let the ratio is m : n

$$\therefore \ \frac{m}{n} = -\frac{(1 \times 1 + 1 \times 2 - 5)}{1 \times (-3) + 1 \times 4 - 5} = -\frac{1}{2}$$

 \therefore ratio is 1 : 2 externally.

EQUATION OF A LINE MAKING A GIVEN ANGLE WITH ANOTHER LINE

The equation of a line passing through (x_1, y_1) and making an angle α with the line y = mx + c is given by



Solved Examples

Ex.33 One vertex of an equilateral triangle is (2,3) and the equation of line opposite to the vertex is x+y=2, then equation of remaining two sides are

(1)
$$y - 3 = \pm 2 (x - 2)$$

(2)
$$y - 3 = (\sqrt{3} \pm 1) (x - 2)$$

(3)
$$y - 3 = (2 \pm \sqrt{3}) (x - 2)$$

(4) None of these

Sol. Since the two sides make an angle of 60° each with side x + y = 2. Therefore equation of these sides will be

$$y - 3 = \frac{-1 \pm \tan 60^{\circ}}{1 \mp (-1) \tan 60^{\circ}} (x - 2) = \frac{-1 \pm \sqrt{3}}{1 \pm \sqrt{3}} (x - 2)$$

$$\Rightarrow y - 3 = (2 \pm \sqrt{3}) (x - 2) \quad \text{Ans.[3]}$$

CONCURRENCY OF THREE LINES

Three lines	a ₁ x	+b	₁ y+	$c_1 =$	0
	a ₂ x	+b	₂ y+	$c_{2} =$	0
and	a ₃ x	+b	₃ y+	$c_3 =$	0
	a ₁	b_1	c ₁		
are concurrent iff	a ₂	b_2	c_2	= 0	
	a_3	b_3	c_3		

Note :

If P = 0, Q = 0, R = 0 are equations of three lines, then $P + Q + R = 0 \Rightarrow$ the lines are concurrent. But its converse may not be true i.e., if the three lines are concurrent then it is not necessary that P + Q + R = 0.

Ex.34 If the lines x + y = 1, 2x - y = 0 and $x + 2y + \lambda = 0$ are concurrent then find λ .

Sol. Using condition of concurrency, we have

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow 1(-\lambda) - 1(2\lambda) - (4+1) = 0$$

$$\Rightarrow -3\lambda - 5 = 0 \Rightarrow \lambda = -5/3. \text{ Ans.[1]}$$

Length of perpendicular from a point on a line :

The length of perpendicular from $P(x_1, y_1)$ on ax +



Solved Examples

Ex.35 Find the distance between the line 12x - 5y + 9 = 0 and the point (2, 1)

Sol. The required distance =
$$\left| \frac{12 \times 2 - 5 \times 1 + 9}{\sqrt{12^2 + (-5)^2}} \right|$$

= $\frac{|24 - 5 + 9|}{13} = \frac{28}{13}$

- **Ex.36** Find all points on x + y = 4 that lie at a unit distance from the line 4x + 3y 10 = 0.
- Sol. Note that the co-ordinates of an arbitrary point on x + y = 4 can be obtained by putting x = t (or y = t) and then obtaining y (or x) from the equation of the line, where t is a parameter.

Putting x = t in the equation x + y = 4 of the given line, we obtain y = 4 - t. So, co-ordinates of an arbitrary point on the given line are P(t, 4 - t). Let P(t, 4-t) be the required point. Then, distance of P from the line 4x + 3y - 10 = 0 is unity i.e.

$$\Rightarrow \left| \frac{4t + 3(4 - t) - 10}{\sqrt{4^2 + 3^2}} \right| = 1 \Rightarrow |t + 2| = 5$$

$$\Rightarrow t + 2 = \pm 5 \Rightarrow t = -7 \text{ or } t = 3$$

Hence, required points are (-7, 11) and (3, 1)

Self practice problem :

Find the length of the altitudes from the vertices of the triangle with vertices :(-1, 1), (5, 2) and (3, -1).

Answer:
$$\frac{16}{\sqrt{13}}$$
, $\frac{8}{\sqrt{5}}$, $\frac{16}{\sqrt{37}}$

Reflection of a point about a line :

(i) Foot of the perpendicular from a point (x_1, y_1) on the line ax + by + c = 0 is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

(ii) The image of a point (x_1, y_1) about the line ax + by + c = 0 is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -2\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

Solved Examples

Ex.37 Find the foot of perpendicular of the line drawn from P (-3, 5) on the line x - y + 2 = 0.

Sol. Slope of PM = -1

$$P(-3, 5)$$

$$M = x - y + 2 = 0$$

: Equation of PM is

$$-y - 2 = 0$$
(i)

solving equation (i) with x - y + 2 = 0,

we get co-ordinates of M(0, 2)

<u>Aliter</u>

X +

Here,
$$\frac{x+3}{1} = \frac{y-5}{-1} = -\frac{(1 \times (-3) + (-1) \times 5 + 2)}{(1)^2 + (-1)^2}$$

$$\Rightarrow \frac{x+3}{1} = \frac{y-5}{-1} = 3$$

$$\Rightarrow x+3=3 \qquad \Rightarrow \qquad x=0 \quad \text{and}$$

$$y-5=-3 \qquad \Rightarrow \qquad y=2$$

$$\therefore \text{ M is } (0, 2)$$

Ex.38 Find the image of the point P(-1, 2) in the line Self practice problems : mirror 2x - 3y + 4 = 0.

Sol. Let image of P is Q.



 \therefore PM = MQ & PQ \perp AB

Let Q is (h, k)

$$\therefore \text{ M is}\left(\frac{h-1}{2},\frac{k+2}{2}\right)$$

It lies on 2x - 3y + 4 = 0.

$$\therefore 2 \left(\frac{h-1}{2}\right) - 3 \left(\frac{k+2}{2}\right) + 4 = 0.$$

or $2h - 3k = 0$ (i)

slope of PQ =
$$\frac{k-2}{h+1}$$

$$PQ \perp AB$$

$$\therefore \frac{k-2}{h+1} \times \frac{2}{3} = -1.$$

$$\Rightarrow 3h + 2k - 1 = 0. \qquad \dots \dots \dots (ii)$$

soving (i) & (ii), we get $h = \frac{3}{13}, k = \frac{2}{13}$

$$\therefore \text{ Image of P}(-1, 2) \text{ is } Q\left(\frac{3}{13}, \frac{2}{13}\right)$$

<u>Aliter</u>

The image of P(-1, 2) about the line 2x - 3y + 4 = 0

is
$$\frac{x+1}{2} = \frac{y-2}{-3} = -2 \frac{[2(-1)-3(2)+4]}{2^2 + (-3)^2}$$

 $\frac{x+1}{2} = \frac{y-2}{-3} = \frac{8}{13}$
⇒ $13x + 13 = 16$ ⇒ $x = \frac{3}{13}$ &
 $13y - 26 = -24$ ⇒ $y = \frac{2}{13}$
 \therefore image is $\left(\frac{3}{13}, \frac{2}{13}\right)$

- Find the foot of perpendicular of the line drawn from 1. (-2, -3) on the line 3x - 2y - 1 = 0.
- Find the image of the point (1, 2) in y-axis. 2.

Answers (27)
$$\left(\frac{-23}{13}, \frac{-41}{13}\right)$$
 (28) $(-1, 2)$

BISECTORS OF THE ANGLES BETWEEN TWO LINES

Bisectors of the two angles between two lines $a_x + b_y$ $b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are given by



First write the equation of the lines so that the constant terms are positive. Then

- (a) If $a_1a_2 + b_1b_2 > 0$ then on taking positive sign in the above bisectors equation we shall get the obtuse angle bisector and on taking negative sign we shall get the acute angle bisector.
- (b) If $a_1a_2 + b_1b_2 < 0$ then positive sign give the acute angle bisector and negative sign gives the obtuse angle bisector.
- (c) On taking positive sign we shall get equation of the bisector of the angle which contains the origin and negative sign gives the equation of the bisector which does not contain origin.

- **Ex.39** Find the equation of the bisector of the acute angle between the lines 3x - 4y + 7 = 0 and 12x + 5y - 2= 0
 - $[1] 11x 3y + 9 = 0 \qquad [2] 9x 11y + 9 = 0$ $[3] 11x - 3y - 9 = 0 \qquad [4] 11x + 3y - 9 = 0$

Sol. The given equation can be written as

3x - 4y + 7 = 0 and -12x - 5y + 2 = 0

Here $a_1a_2 + b_1b_2 = 3(-12) - 4(-5) < 0$, so positive sign gives the acute angle bisector which is

$$\frac{3x - 4y + 7}{5} = \frac{-12x - 5y + 2}{13}$$

$$\Rightarrow 99x - 27y + 81 = 0 \Rightarrow 11x - 3y + 9 = 0$$

Ans.[1]

- **Note :** This is also the bisector of the angle in which origin lies (since c_1 , c_2 are positive and it has been obtained by taking positive sign)
- **Ex.40** Find the equations of the bisectors of the angle between the straight lines

3x - 4y + 7 = 0 and 12x - 5y - 8 = 0.

Sol. The equations of the bisectors of the angles between

$$3x - 4y + 7 = 0 \text{ and } 12 x - 5y - 8 = 0 \text{ are}$$
$$\frac{3x - 4y + 7}{\sqrt{3^2 + (-4)^2}} = \pm \frac{12x - 5y - 8}{\sqrt{12^2 + (-5)^2}}$$
or
$$\frac{3x - 4y + 7}{5} = \pm \frac{12x - 5y - 8}{13}$$

or $39x - 52y + 91 = \pm (60 x - 25 y - 40)$

Taking the positive sign, we get 21 x + 27 y - 131 = 0 as one bisector

Taking the negative sign, we get $99 \times 77 \times 51 = 0$ as the other bisector.

Ex.41 For the straight lines 4x + 3y - 6 = 0 and 5x + 12y + 9 = 0, find the equation of the

- (i) bisector of the obtuse angle between them;
- (ii) bisector of the acute angle between them;

Sol. (i) The equations of the given straight lines are

4x -	+ 3y –	6 = 0)	(1)
_		~		(-)

5x + 12y + 9 = 0(2)

The equation of the bisectors of the angles between lines (1) and (2) are

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm \frac{5x+12y+9}{\sqrt{5^2+12^2}} \text{ or}$$
$$\frac{4x+3y-6}{5} = \pm \frac{5x+12y+9}{13}$$

Taking the positive sign, we have $\frac{4x+3y-6}{5}$

$$= \frac{5x + 12y + 9}{13}$$

or $52x + 39y - 78 = 25x + 60y + 45$
or $27x - 21y - 123 = 0$
or $9x - 7y - 41 = 0$

Taking the negative sign, we have $\frac{4x+3y-6}{5}$

$$= -\frac{5x + 12y + 9}{13}$$

or $52x + 39y - 78 = -25x - 60y - 45$
or $77x + 99y - 33 = 0$ or $7x + 9y - 3 = 0$
Hence the equation of the bisectors are
 $9x - 7y - 41 = 0$ (3)
and $7x + 9y - 3 = 0$ (4)
Now slope of line $(1) = -\frac{4}{3}$ and slope of the bisector
 $(3) = \frac{9}{7}$.
If θ be the acute angle between the line (1) and the

If θ be the acute angle between the line (1) and the bisector (3), then

$$\tan \theta = \left| \frac{\frac{9}{7} + \frac{4}{3}}{1 + \frac{9}{7} \left(-\frac{4}{3} \right)} \right| = \left| \frac{27 + 28}{21 - 36} \right| = \left| \frac{55}{-15} \right| = \frac{11}{3} > 1$$

$$\therefore \ \theta > 45^{\circ}$$

Hence 9x - 7y - 41 = 0 is the bisector of the obtuse angle between the given lines (1) and (2)

(ii) Since 9x - 7y - 41 is the bisector of the obtuse angle between the given lines, therefore the other bisector 7x + 9y - 3 = 0 will be the bisector of the acute angle between the given lines.

2nd Method :

Writing the equation of the lines so that constants become positive we have

-4x - 3y + 6 = 0(1) and 5x + 12y + 9 = 0(2) Here $a_1 = -4, a_2 = 5, b_1 = -3, b_2 = 12$ Now $a_1a_2 + b_1b_2 = -20 - 36 = -56 < 0$

 \therefore origin does not lie in the obtuse angle between lines (1) and (2) and hence equation of the bisector of the obtuse angle between lines (1) and (2) will be

$$\frac{-4x - 3y + 6}{\sqrt{(-4)^2 + (-3)^2}} = -\frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}}$$

or $13(-4x - 3y + 6) = -5(5x + 12y + 9)$
or $27x - 21y - 123 = 0$ or $9x - 7y - 41 = 0$
and the equation of the bisector of the acute angle
will be (origin lies in the acute angle)

$$\frac{-4x - 3y + 6}{\sqrt{(-4)^2 + (-3)^2}} = \frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}}$$

or 77x + 99y - 33 = 0 or 7x + 9y - 3 = 0

To discriminate between the bisector of the angle containing a point :

To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin. Rewrite the equations, ax + by + c = 0 & a'x + b'y + c' = 0 such that the constant terms c, c'

are positive. Then ;
$$\frac{ax + by + c}{\sqrt{a^2 + b^2}}$$

$$= + \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$
 gives the equation of the

bisector of the angle containing the origin

$$\& \frac{ax + by + c}{\sqrt{a^2 + b^2}} = -\frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}} \text{ gives the}$$

equation of the bisector of the angle not containing the origin. In general equation of the bisector which contains the point ($\alpha \beta$) is

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}} \text{ or }$$

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = -\frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$
 according as

 $a \alpha + b \beta + c$ and $a' \alpha + b' \beta + c'$ having same sign or otherwise.

Solved Examples

Ex.42 For the straight lines 4x + 3y - 6 = 0 and 5x + 12y + 9 = 0, find the equation of the bisector of the angle which contains the origin.

Sol. For point O(0, 0), 4x + 3y - 6 = -6 < 0 and 5x + 12y + 9 = 9 > 0

Hence for point O(0, 0) 4x + 3y - 6 and 5x + 12y + 9 are of opposite signs.

Hence equation of the bisector of the angle between the given lines containing the origin will be

$$\frac{4x + 3y - 6}{\sqrt{(4)^2 + (3)^2}} = -\frac{5x + 12y + 9}{\sqrt{5^2 + 12^2}}$$

or $\frac{4x + 3y - 6}{5} = -\frac{5x + 12y + 9}{13}$
or $52x + 39y - 78 = -25x - 60y - 45.$
or $77x + 99y - 33 = 0$ or $7x + 9y - 3 = 0$

LINES PASSING THROUGH THE POINT OF INTERSECTION OF TWO LINES

If equation of two lines $L_1 = a_1x + b_1y + c_1 = 0$ and $L_2 = a_2x + b_2y + c_2 = 0$ then the equation of the lines passing through the point of intersection of these lines is $L_1 + \lambda L_2 = 0$ or $(a_1x + b_1y + c = 0) + \lambda (a_2x + b_2y + c_2 = 0) = 0$, value of λ is obtained with the help of the additional infomation given in the problem.

Solved Examples

Ex.43 Equation of a line passing through the point of intersection of x + y - 3 = 0 and 2x - y + 1 = 0 and a point (2, -3) is [1] 4x + y - 5 = 0 [2] 4x - y - 5 = 0

$$\begin{bmatrix} 3 \end{bmatrix} 4x + 5y + 5 = 0 \qquad \begin{bmatrix} 4 \end{bmatrix} 4x + 2y - 5 = 0$$

Sol. $(x + y - 3) + \lambda (2x - y + 1) = 0$ as this line also passes through (2, -3)Hence $(2 - 3 - 3) + \lambda (2 \times 2 - (-3) + 1) = 0$ $-4 + 8\lambda = 0$; $\lambda = 1/2$ equation is (x + y - 3) + 1/2 (2x - y + 1) = 04x + y - 5 = 0Ans.[1]

- **Ex.44** Find the equation of the straight line which passes through the point (2, -3) and the point of intersection of the lines x + y + 4 = 0 and 3x y 8 = 0.
- Sol. Any line through the intersection of the lines x + y + 4 = 0 and 3x y 8 = 0 has the equation

 $(x + y + 4) + \lambda (3x - y - 8) = 0$ (i)

This will pass through (2, -3) if

 $(2-3+4) + \lambda (6+3-8) = 0 \text{ or } 3 + \lambda = 0$ $\Rightarrow \lambda = -3.$

Putting the value of λ in (i), the required line is (x + y + 4) + (-3) (3x - y - 8) = 0 or -8x + 4y + 28 = 0 or 2x - y - 7 = 0

<u>Aliter</u>

Solving the equations x+y+4=0 and 3x-y-8=0by cross-multiplication, we get x = 1, y = -5

So the two lines intersect at the point (1, -5). Hence the required line passes through (2, -3) and

(1,-5) and so its equation is $y+3 = \frac{-5+3}{1-2} (x-2)$ or 2x - y - 7 = 0

- **Ex.45** Obtain the equations of the lines passing through the intersection of lines 4x-3y-1=0 and 2x-5y+3=0 and equally inclined to the axes.
- **Sol.** The equation of any line through the intersection of the given lines is

$$(4x - 3y - 1) + \lambda (2x - 5y + 3) = 0$$

or x (2 \lambda + 4) - y (5\lambda + 3) + 3\lambda - 1 = 0(i)

Let m be the slope of this line. Then $m = \frac{2\lambda + 4}{5\lambda + 3}$

As the line is equally inclined with the axes, therefore $m = \tan 45^\circ$ or $m = \tan 135^\circ$

$$\Rightarrow m = \pm 1, \frac{2\lambda + 4}{5\lambda + 3} = \pm 1$$

$$\Rightarrow \lambda = -1 \text{ or } \frac{1}{3}, \text{ putting the values of } \lambda \text{ in (i)},$$

we get $2x + 2y - 4 = 0$ and $14x - 14y = 0$

i.e. x + y - 2 = 0 and x = y as the equations of the required lines.

A Pair of straight lines through origin:

- (i) A homogeneous equation of degree two, $ax^{2}+2hxy+by^{2}=0$ " always represents a pair of straight lines passing through the origin if :
 - (a) $h^2 > ab$ \Rightarrow lines are real & distinct.

(b)
$$h^2 = ab \implies \text{lines are coincident}$$
.



(c) $h^2 < ab \implies$ lines are imaginary with real point of intersection i.e. (0, 0)

This equation is obtained by multiplying the two equations of lines $(m_1x - y)(m_2x - y) = 0$

$$\Rightarrow m_1 m_2 x^2 - (m_1 + m_2) xy + y^2 = 0$$

(ii) If $y = m_1 x \& y = m_2 x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$$m_1 + m_2 = -\frac{2h}{b} \& m_1 m_2 = \frac{a}{b}.$$

(iii) If θ is the acute angle between the pair of straight lines represented by,

$$ax^2 + 2hxy + by^2 = 0$$
, then $\tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b}$

- (iv) The condition that these lines are :
 - (a) at right angles to each other is a + b = 0. i.e. co-efficient of $x^2 + co$ -efficient of $y^2 = 0$.
 - (b) coincident is $h^2 = ab$.
 - (c) equally inclined to the axis of x is h=0.i.e. coeff. of xy=0.

Note that a homogeneous equation of degree n represents n straight lines passing through origin.

(v) The equation to the pair of straight lines bisecting the angles between the straight lines $ax^2 + 2hxy + by^2$

$$= 0$$
 is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$.

- **Ex.46** Show that the equation $6x^2 5xy + y^2 = 0$ represents a pair of distinct straight lines, each passing through the origin. Find the separate equations of these lines.
- Sol. The given equation is a homogeneous equation of second degree. So, it represents a pair of straight lines passing through the origin. Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we obtain a = 6, b = 1 and 2h = -5.

$$\therefore h^2 - ab = \frac{25}{4} - 6 = \frac{1}{4} > 0 \Rightarrow h^2 > ab$$

Hence, the given equation represents a pair of distinct lines passing through the origin.

Now,
$$6x^2 - 5xy + y^2 = 0$$

$$\Rightarrow \left(\frac{y}{x}\right)^2 - 5\left(\frac{y}{x}\right) + 6 = 0$$

$$\Rightarrow \left(\frac{y}{x}\right)^2 - 3\left(\frac{y}{x}\right) - 2\left(\frac{y}{x}\right) + 6 = 0$$

$$\Rightarrow \left(\frac{y}{x} - 3\right)\left(\frac{y}{x} - 2\right) = 0$$

$$\Rightarrow \frac{y}{x} - 3 = 0 \text{ or } \frac{y}{x} - 2 = 0 \Rightarrow y - 3x = 0 \text{ or } y - 2x = 0$$
So the given equation represents the straight line

So the given equation represents the straight lines y - 3x = 0 and y - 2x = 0.

- **Ex.47** Find the equations to the pair of lines through the origin which are perpendicular to the lines represented by $2x^2 7xy + 3y^2 = 0$.
- **Sol.** We have $2x^2 7xy + 3y^2 = 0$.

$$\Rightarrow 2x^2 - 6xy - xy + 3y^2 = 0$$

$$\Rightarrow 2x(x - 3y) - y(x - 3y) = 0$$

$$\Rightarrow (x - 3y) (2x - y) = 0$$

$$\Rightarrow$$
 x - 3y = 0 or 2x - y = 0

Thus the given equation represents the lines x - 3y = 0 and 2x - y = 0. The equations of the lines passing through the origin and perpendicular to the given lines

are
$$y - 0 = -3 (x - 0)$$
 and $y - 0 = -\frac{1}{2}(x - 0)$
[:: (Slope of $x - 3 y = 0$) is 1/3 and (Slope of $2x - y = 0$) is 2]
 $\Rightarrow y + 3x = 0$ and $2y + x = 0$

Ex.48 Find the angle between the pair of straight lines $4x^2 + 24xy + 11y^2 = 0$

Sol. Given equation is $4x^2 + 24xy + 11y^2 = 0$ Here a = coeff. of $x^2 = 4$, b = coeff. of $y^2 = 11$ and 2h = coeff. of xy = 24 \therefore h = 12

Now
$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| = \left| \frac{2\sqrt{144 - 44}}{4 + 11} \right| = \frac{4}{3}$$

Where θ is the acute angle between the lines.

- ∴ acute angle between the lines is $\tan^{-1}\left(\frac{4}{3}\right)$ and obtuse angle between them is $\pi - \tan^{-1}\left(\frac{4}{3}\right)$
- **Ex.49** Find the equation of the bisectors of the angle between the lines represented by $3x^2 - 5xy + 4y^2 = 0$
- Sol. Given equation is $3x^2 5xy + 4y^2 = 0$ (1) comparing it with the equation $ax^2 + 2hxy + by^2 = 0$ (2)

we have a = 3, 2h = -5; and b = 4

Now the equation of the bisectors of the angle

between the pair of lines (1) is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$

or
$$\frac{x^2 - y^2}{3 - 4} = \frac{xy}{-\frac{5}{2}}$$
; or $\frac{x^2 - y^2}{-1} = \frac{2xy}{-5}$

or $5x^2 - 2xy - 5y^2 = 0$

0

Ex.50 Lines represented by $2x^2 - 7xy + 3y^2 = 0$ are

[1] x + 2y = 0, x - 3y = 0 [2] x - 2y = 0, x + 3y = 0 [3] 2x - y = 0, 3x - y = 0 [4] 2x - y = 0, x - 3y = 0Sol. $2x^2 - 7xy + 3y^2 = 0 = (2x - y)(x - 3y)$ $\therefore 2x - y = 0, x - 3y = 0$ Ans.[4]

Ex.51 Angle between the lines $2x^2 - 7xy + 3y^2 = 0$ is

[1] 45 [2] 30 [3] 60 [4] 90 Sol. $\tan \theta = \frac{2\sqrt{(-7/2)^2 - 2 \times 3}}{2 + 3} \Rightarrow \tan \theta = 1 \Rightarrow \theta = 45^{\circ}$ Ans.[1]

General equation of second degree representing a pair of Straight lines :

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$$abc + 2fgh - af^{2} - bg^{2} - ch^{2} = 0$$
, i.e. if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Such an equation is obtained again by multiplying the two equation of lines

 $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$

(ii) The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only.

Solved Examples

- **Ex.52** Prove that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Find the co-ordinates of their point of intersection.
- **Sol.** Given equation is $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$

Writing the equation (1) as a quadratic equation in x we have

$$2x^{2} + (5y + 6) x + 3y^{2} + 7y + 4 = 0$$

$$\therefore x = \frac{-(5y+6) \pm \sqrt{(5y+6)^{2} - 4.2(3y^{2} + 7y + 4)}}{4}$$

$$= \frac{-(5y+6) \pm \sqrt{25y^{2} + 60y + 36 - 24y^{2} - 56y - 32}}{4}$$

$$= \frac{-(5y+6) \pm \sqrt{y^{2} + 4y + 4}}{4} = \frac{-(5y+6) \pm (y+2)}{4}$$

$$\therefore x = \frac{-5y-6+y+2}{4}, \frac{-5y-6-y-2}{4}$$

or $4x + 4y + 4 = 0$ and $4x + 6y + 8 = 0$
or $x + y + 1 = 0$ and $2x + 3y + 4 = 0$

Hence equation (1) represents a pair of straight lines whose equation are

$$x + y + 1 = 0$$
(1)

and 2x + 3y + 4 = 0(2)

Solving these two equations, the required point of intersection is (1, -2).

EQUATION OF LINES JOINING THE INTERSECTION POINTS OF A LINE & A CURVE TO THE ORIGIN

Let lx + my + n = 0(1) and the second degree curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (2) then their joint equation is

$$ax^{2} + 2hxy + by^{2} + 2gx \left(\frac{\ell x + my}{-n}\right) + 2fy$$
$$\left(\frac{\ell x + my}{-n}\right) + c \left(\frac{\ell x + my}{-n}\right)^{2} = 0$$

i.e. making the equation (2) homogenous using equation (1)

Solved Examples

Ex.53 Equation of pair of straight lines which are formed
by joining the origin and the points of intersection of
a circle
$$x^2 + y^2 = a^2$$
 and a line $y = mx + c$ is
[1] $x^2 (c^2 + a^2m^2) + y (c^2 + a^2) - 2ma^2xy = 0$
[2] $x^2 (c^2 - a^2m^2) - y (c^2 - a^2) + 2ma^2xy = 0$
[3] $x^2 (c^2 - a^2m^2) + y (c^2 - a^2) - 2ma^2xy = 0$
[4] $x^2 (c^2 - a^2m^2) + y (c^2 + a^2) - 2ma^2xy = 0$
Sol. $x^2 + y^2 = a^2 \left(\frac{y - mx}{c}\right)^2$
 $\Rightarrow x^2 (c^2 - a^2m^2) + y (c^2 - a^2) - 2ma^2xy = 0$
Ans.[3]

Ex.54 For what value of k, the equation $kx^2 - 10xy + 12y^2 + 5x - 16y - 3=0$ represents a pair of straight lines

Sol. Here
$$a = k, b = 12, c = -3$$
,
 $h = -5, g = 5/2, f = -8$
 $\therefore \Delta = k(12) (-3) + 2 (-8) (5/2) (-5) -k(-8)^2 - 12(5/2)^2 - (-3) (-5)^2 = 0$
 $\Rightarrow k = 2$ Ans.[1]

Ex.55 The angle and equation of bisectors of angle between the lines represented by the equation $2x^2 - 7xy + 3y^2 = 0$ are [1] 45° , $7x^2 - 2xy - 7y^2 = 0$ [2] 60° , $7x^2 - 2xy - 7y^2 = 0$ [3] 120° , $7x^2 - 2xy - 7y^2 = 0$ [4] 45° , $7x^2 + 2xy + 7y^2 = 0$ Sol. Since $2x^2 - 7xy + 3y^2 = (2x - y) (x - 3y)$ \therefore given lines are 2x - y = 0 and x - 3y = 0since a = 2, h = -7/2, b = 3Therefore angle between lines given by $\tan \theta = \frac{2\sqrt{(-7/2)^2 - 6}}{2 + 3} = 1 \implies \theta = 45^\circ$

Again equations of bisectors are given by

$$\frac{x^2 - y^2}{2 - 3} = \frac{xy}{-7/2} \implies 7x^2 - 2xy - 7y^2 = 0$$
Ans.[1]

Ex.56 Prove that the angle between the lines joining the origin to the points of intersection of the straight line y = 3x + 2 with the curve $x^2 + 2xy + 3y^2 + 4x + 8y$ - 11 = 0 is $\tan^{-1} \frac{2\sqrt{2}}{2}$.

Sol. Equation of the given curve is $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ and equation of the given straight line is

$$y-3x=2; \qquad \therefore \frac{y-3x}{2}=1$$

Making equation (1) homogeneous equation of the second degree in x any y with the help of (1), we

have
$$x^2 + 2xy + 3y^2 + 4x\left(\frac{y-3x}{2}\right) + 8y\left(\frac{y-3x}{2}\right)$$

 $-11\left(\frac{y-3x}{2}\right)^2 = 0$

or
$$x^{2} + 2xy + 3y^{2} + \frac{1}{2} (4xy + 8y^{2} - 12x^{2} - 24xy)$$

 $-\frac{11}{4} (y^{2} - 6xy + 9x^{2}) = 0$
or $4x^{2} + 8xy + 12y^{2} + 2(8y^{2} - 12x^{2} - 20xy) - 11$
 $(y^{2} - 6xy + 9x^{2}) = 0$
or $-119x^{2} + 34xy + 17y^{2} = 0$
or $119x^{2} - 34xy - 17y^{2} = 0$

or $7x^2 - 2xy - y^2 = 0$

This is the equation of the lines joining the origin to the points of intersection of (1) and (2).

Comparing equation (3) with the equation $ax^2 + 2hxy + by^2 = 0$

we have a = 7, b = -1 and 2h = -2 i.e. h = -1

If θ be the acute angle between pair of lines (3), then

$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| = \left| \frac{2\sqrt{1 + 7}}{7 - 1} \right| = \frac{2\sqrt{8}}{6} = \frac{2\sqrt{2}}{3}$$
$$\therefore \quad \theta = \tan^{-1} \frac{2\sqrt{2}}{3}.$$

TRANSFORMATION OF AXES

1. Parallel transformation

Let origin O(0,0) be shifted to a point (a,b) by moving the x-axis and y-axis, parallel to themselves. If the co-ordinate of point P with reference to old axis are (x_1, y_1) then co-ordinate of this point with respect to new axis will be $(x_1 - a, y_1 - b)$



2. Rotational transformation:

Let OX and OY be the old axes and OX' and OY' be the new axes obtained by rotating the old axes OX and OY through an angle θ , again, if coordinates of any point P(x,y) with reference to new axes will be (x',y'), then

 $x' = x \cos \theta + y \sin \theta$ $y' = -x \sin \theta + y \cos \theta$ $x = x' \cos \theta - y' \sin \theta$ $y = x' \sin \theta + y' \cos \theta$

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

$$\begin{array}{ccc} x\downarrow & y\downarrow \\ x'\rightarrow & \cos\theta & \sin\theta \\ y'\rightarrow & -\sin\theta & \cos\theta \end{array}$$

3 Reflection (image) of a point:

Let (x, y) be any point, then its image with respect to

(i)
$$x - axis \Rightarrow (x, -y)$$

(ii)
$$y - axis \Rightarrow (-x, y)$$

(iii) origin
$$\Rightarrow$$
 (-x,-y)

 $(iv) \text{ line } y = x \Rightarrow (y, x)$

Solved Examples

Ex.57 If axes are transformed from origin to the point (-2,1) then new co-ordinates of (4,-5) is
[1] (6,-6) [2] (-6,6)
[3] (4,-4) [4] (6,6)

Sol. [4-(-2),-5-1] = (6,-6) **Ans.[1]**

Ex.58 If the new coordinates of a point after the rotation of axes in the negative direction by an angle of $\frac{\pi}{3}$ are (4,2) then coordinates with respect to old axes are

$$[1] (2+\sqrt{3},-2\sqrt{3}+1)$$

$$[2] (2-\sqrt{3},-2\sqrt{3}-1)$$

$$[3] (2+\sqrt{3}, 2\sqrt{3}+1)$$

$$[4] (2+\sqrt{3},-2\sqrt{3}-1)$$
Sol.
$$\begin{bmatrix} x & y \\ 4 & \cos(-60^{\circ}) & \sin(-60^{\circ}) \\ 2 & -\sin(-60^{\circ}) & \cos(-60^{\circ}) \end{bmatrix}$$

$$\therefore x = x'\cos\alpha - y'\sin\alpha$$

$$= 4\cos(-60) - 2\sin(-60) = 2 + \sqrt{3}$$

$$y = x'\sin\alpha + y'\cos\alpha$$

$$= 4\sin(-60) + 2\cos(-60) = -2\sqrt{3} + 1$$
Hence the coordinates are $(2+\sqrt{3},-2\sqrt{3}+1)$
Ans.[1]