TRIGONOMETRIC EQUATIONS

TRIGONOMETRIC EQUATION :

An equation involving one or more trigonometric ratios of an unknown angle is called a trigonometric equation.

SOLUTION OF TRIGONOMETRIC EQUATION :

A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.

e.g. if $\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots\ldots\ldots\ldots$

Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature) and can be classified as :

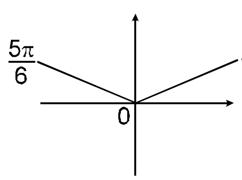
(i) Principal solution (ii) General solution.

Principal solutions :

The solutions of a trigonometric equation which lie in the interval $[0, 2\pi)$ are called Principal solutions.

e.g. Find the Principal solutions of the equation $\sin x = \frac{1}{2}$.

Solution **:** $\because \sin x = \frac{1}{2}$



$\because$ there exists two values

i.e. $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ which lie in $[0, 2\pi)$ and whose sine is $\frac{1}{2}$

$\therefore$ Principal solutions of the equation $\sin x = \frac{1}{2}$ are $\frac{\pi}{6}, \frac{5\pi}{6}$

General Solution :

The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called General solution.

General solution of some standard trigonometric equations are given below.

GENERAL SOLUTION OF SOME STANDARD TRIGONOMETRIC EQUATIONS :

(i) If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.

(ii) If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

(iii) If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

(iv) If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$, $n \in I$.

(v) If $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$, $n \in I$.

(vi) If $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$, $n \in I$.

[**Note:** α is called the principal angle]

SOME IMPORTANT DEDUCTIONS :

(i) $\sin\theta = 0 \Rightarrow \theta = n\pi,\ n \in I$

(ii) $\sin\theta = 1 \Rightarrow \theta = (4n + 1)\frac{\pi}{2},\ n \in I$

(iii) $\sin\theta = -1 \Rightarrow \theta = (4n - 1)\frac{\pi}{2},\ n \in I$

(iv) $\cos\theta = 0 \Rightarrow \theta = (2n + 1)\frac{\pi}{2},\ n \in I$

(v) $\cos\theta = 1 \Rightarrow \theta = 2n\pi,\ n \in I$

(vi) $\cos\theta = -1 \Rightarrow \theta = (2n + 1)\pi,\ n \in I$

(vii) $\tan\theta = 0 \Rightarrow \theta = n\pi,\ n \in I$

Solved Examples

Ex.31 Solve $\sin\theta = \frac{\sqrt{3}}{2}$.

Sol. $\because \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \sin\theta = \sin\frac{\pi}{3}$

$\therefore \theta = n\pi + (-1)^n \frac{\pi}{3},\ n \in I$

Ex.32 The general solution of $\cos\theta = \frac{1}{2}$ is –

(A) $2n\pi \pm \frac{\pi}{6};\ n \in I$ (B) $n\pi \pm \frac{\pi}{6};\ n \in I$

(C) $2n\pi \pm \frac{\pi}{3};\ n \in I$ (D) $n\pi \pm \frac{\pi}{3};\ n \in I$

Sol. If $\cos\theta = \frac{1}{2}$, or $\cos\theta = \cos\left(\frac{\pi}{3}\right)$

$\theta = 2n\pi \pm \frac{\pi}{3}$; $n \in I$ **Ans.[C]**

Ex.33 Solve : $\sec 2\theta = -\frac{2}{\sqrt{3}}$

Sol. $\because \sec 2\theta = -\frac{2}{\sqrt{3}}$

$\Rightarrow \cos 2\theta = -\frac{\sqrt{3}}{2} \Rightarrow \cos 2\theta = \cos\frac{5\pi}{6}$

$\Rightarrow 2\theta = 2n\pi \pm \frac{5\pi}{6}, n \in I \Rightarrow \theta = n\pi \pm \frac{5\pi}{12}, n \in I$

Ex.34 Solve $\tan\theta = 2$

Sol. $\because \tan\theta = 2$ ....(i)

Let $2 = \tan\alpha \Rightarrow \tan\theta = \tan\alpha$

$\Rightarrow \theta = n\pi + \alpha$, where $\alpha = \tan^{-1}(2)$, $n \in I$

Ex.35 Solve $\cos^2\theta = \frac{1}{2}$

Sol. $\because \cos^2\theta = \frac{1}{2} \Rightarrow \cos^2\theta = \left(\frac{1}{\sqrt{2}}\right)^2$

$\Rightarrow \cos^2\theta = \cos^2\frac{\pi}{4} \Rightarrow \theta = n\pi \pm \frac{\pi}{4}, n \in I$

Ex.36 Solve $4\tan^2\theta = 3\sec^2\theta$

Sol. $\because 4\tan^2\theta = 3\sec^2\theta$ ....(i)

For equation (i) to be defined $\theta \neq (2n+1)\frac{\pi}{2}, n \in I$

$\because$ equation (i) can be written as:

$\frac{4\sin^2\theta}{\cos^2\theta} = \frac{3}{\cos^2\theta}$ $\because \theta \neq (2n+1)\frac{\pi}{2}, n \in I$

$\therefore \cos^2\theta \neq 0$

$\Rightarrow 4\sin^2\theta = 3 \Rightarrow \sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2$

$\Rightarrow \sin^2\theta = \sin^2\frac{\pi}{3} \Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in I$

Ex.37 If $\sin\theta + \sin 3\theta + \sin 5\theta = 0$, then the general value of θ is –

(A) $\frac{n\pi}{6}, \frac{m\pi}{12}$; m, n $\in I$

(B) $\frac{n\pi}{3}, m\pi \pm \frac{\pi}{3}$; m, n $\in I$

(C) $\frac{n\pi}{3}, m\pi \pm \frac{\pi}{6}$; m, n $\in I$

(D) None of these

Sol. If $(\sin 5\theta + \sin\theta) + \sin 3\theta = 0$

or, $2\sin 3\theta \cos 2\theta + \sin 3\theta = 0$

or, $\sin 3\theta (2\cos 2\theta + 1) = 0$

Case I

$\sin 3\theta = 0 \Rightarrow 3\theta = n\pi$; $n \in I$

$\Rightarrow \theta = \frac{n\pi}{3}$; $n \in I$

Case II

$2\cos 2\theta + 1 = 0 \Rightarrow \cos 2\theta = -\frac{1}{2}$

$\Rightarrow \cos 2\theta = \cos\frac{2\pi}{3} \Rightarrow \theta = m\pi \pm \frac{\pi}{3}$; $m \in I$

So the general solution of the given equation is $\theta = \frac{n\pi}{3}$ and $\theta = m\pi \pm \frac{\pi}{3}$

where m, n $\in I$ **Ans.[B]**

Ex.38 If $2\cos^2\theta + 3\sin\theta = 0$, then general value of θ is –

(A) $n\pi + (-1)^n\frac{\pi}{6}$; $n \in I$

(B) $2n\pi \pm \frac{\pi}{6}$; $n \in I$

(C) $n\pi + (-1)^{n+1}\frac{\pi}{6}$; $n \in I$

(D) None of these

Sol. If $2\cos^2\theta + 3\sin\theta = 0$

$\Rightarrow 2(1 - \sin^2\theta) + 3\sin\theta = 0$

$\Rightarrow 2\sin^2\theta - 3\sin\theta - 2 = 0$

$\Rightarrow 2\sin^2\theta - 4\sin\theta + \sin\theta - 2 = 0$

$\Rightarrow 2\sin\theta(\sin\theta - 2) + (\sin\theta - 2) = 0$

$\Rightarrow (\sin\theta - 2)(2\sin\theta + 1) = 0$

Case I

If $\sin\theta - 2 = 0$ $\sin\theta = 2$

Which is not possible because $-1 \le \sin\theta \le 1$

Case II

If $2\sin\theta + 1 = 0$

$\Rightarrow \sin\theta = -\frac{1}{2}$ or, $\sin\theta = \sin\left(\frac{-\pi}{6}\right)$

$\Rightarrow \theta = n\pi + (-1)^n\left(\frac{-\pi}{6}\right)$; $n \in I$

$\Rightarrow \theta = n\pi + (-1)^{n+1}\left(\frac{\pi}{6}\right)$; $n \in I$ **Ans.[C]**

Ex.39 If $\cos 3x = -1$, where $0^\circ \le x \le 360^\circ$, then $x =$

(A) $60^\circ, 180^\circ, 300^\circ$ (B) 180°

(C) $60^\circ, 180^\circ$ (D) $180^\circ, 300^\circ$

Sol. If $\cos 3x = -1 = \cos(2n+1)\pi$

or, $3x = (2n+1)\pi$ $x = (2n+1)\frac{\pi}{3}$

i.e., $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ **Ans.[A]**

Ex.40 If $\sin 3\theta = \sin\theta$, then the general value of θ is -

(A) $2n\pi, (2n+1)\frac{\pi}{3}$ (B) $n\pi, (2n+1)\frac{\pi}{4}$

(C) $n\pi, (2n+1)\frac{\pi}{3}$ (D) None of these

Sol. $\sin 3\theta = \sin\theta$ or, $3\theta = m\pi + (-1)^m\theta$

For (m) even i.e. $m = 2n$,

then $\theta = \frac{2n\pi}{2} = n\pi$ and for (m) odd

i.e. $m = (2n+1)$ or, $\theta = (2n+1)\frac{\pi}{4}$ **Ans.[B]**

TYPES OF TRIGONOMETRIC EQUATIONS :

Type-1

Trigonometric equations which can be solved by use of factorization.

Solved Examples

Ex.41 Solve $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$.

Sol. $\because$ $(2\sin x - \cos x)(1+\cos x) = \sin^2 x$

$\Rightarrow (2\sin x - \cos x)(1+\cos x) - \sin^2 x = 0$

$\Rightarrow (2\sin x - \cos x)(1+\cos x) - (1-\cos x)(1+\cos x) = 0$

$\Rightarrow (1+\cos x)(2\sin x - 1) = 0$

$\Rightarrow 1 + \cos x = 0$ or $2\sin x - 1 = 0$

$\Rightarrow \cos x = -1$ or $\sin x = \frac{1}{2}$

$\Rightarrow x = (2n+1)\pi, n \in I$ or $\sin x = \sin\frac{\pi}{6}$

$\Rightarrow x = n\pi + (-1)^n\frac{\pi}{6}, n \in I$

$\therefore$ Solution of given equation is

$(2n+1)\pi, n \in I$ or $n\pi + (-1)^n\frac{\pi}{6}, n \in I$

Type - 2

Trigonometric equations which can be solved by reducing them in quadratic equations.

Solved Examples

Ex.42 Solve $2\cos^2 x + 4\cos x = 3\sin^2 x$

Sol. $\because$ $2\cos^2 x + 4\cos x - 3\sin^2 x = 0$

$\Rightarrow 2\cos^2 x + 4\cos x - 3(1 - \cos^2 x) = 0$

$\Rightarrow 5\cos^2 x + 4\cos x - 3 = 0$

$\Rightarrow \left\{\cos x - \left(\frac{-2+\sqrt{19}}{5}\right)\right\}\left\{\cos x - \left(\frac{-2-\sqrt{19}}{5}\right)\right\} = 0$...(ii)

$\because$ $\cos x \in [-1, 1]\ \forall\ x \in R$

$\therefore$ $\cos x \ne \frac{-2-\sqrt{19}}{5}$

$\therefore$ equation (ii) will be true if $\cos x = \frac{-2+\sqrt{19}}{5}$

$\Rightarrow \cos x = \cos\alpha$, where $\cos\alpha = \frac{-2+\sqrt{19}}{5}$

$\Rightarrow x = 2n\pi \pm \alpha$ where $\alpha = \cos^{-1}\left(\frac{-2+\sqrt{19}}{5}\right), n \in I$

Type-3

Trigonometric equations which can be solved by transforming a sum or difference of trigonometric ratios into their product.

Solved Examples

Ex43. Solve $\cos 3x + \sin 2x - \sin 4x = 0$

Sol. $\cos 3x + \sin 2x - \sin 4x = 0$

$\Rightarrow \cos 3x + 2\cos 3x.\sin(-x) = 0$

$\Rightarrow \cos 3x - 2\cos 3x.\sin x = 0$

$\Rightarrow \cos 3x(1 - 2\sin x) = 0$

$\Rightarrow \cos 3x = 0$ or $1 - 2\sin x = 0$

$\Rightarrow 3x = (2n+1)\frac{\pi}{2}, n \in I$ or $\sin x = \frac{1}{2}$

$\Rightarrow x = (2n+1)\frac{\pi}{6}, n \in I$ or $x = n\pi + (-1)^n\frac{\pi}{6}, n \in I$

$\therefore$ solution of given equation is

$(2n+1)\frac{\pi}{6}, n \in I$ or $n\pi + (-1)^n\frac{\pi}{6}, n \in I$

Type-4

Trigonometric equations which can be solved by transforming a product of trigonometric ratios into their sum or difference.

Solved Examples

Ex.44 Solve sin5x.cos3x = sin6x.cos2x

Sol. $\because$ sin5x.cos3x = sin6x.cos2x

$\Rightarrow$ 2sin5x.cos3x = 2sin6x.cos2x

$\Rightarrow$ sin8x + sin2x = sin8x + sin4x

$\Rightarrow$ sin4x – sin2x = 0

$\Rightarrow$ 2sin2x.cos2x – sin2x = 0

$\Rightarrow$ sin2x (2cos2x – 1) = 0

$\Rightarrow$ sin2x = 0 or 2cos2x – 1 = 0

$\Rightarrow 2x = n\pi, n \in I$ or $\cos 2x = \frac{1}{2}$

$\Rightarrow x = \frac{n\pi}{2}, n \in I$ or $2x = 2n\pi \pm \frac{\pi}{3}, n \in I$

$\Rightarrow x = n\pi \pm \frac{\pi}{6}, n \in I$

$\therefore$ Solution of given equation is

$\frac{n\pi}{2}, n \in I$ or $n\pi \pm \frac{\pi}{6}, n \in I$

Type 5.

GENERAL SOLUTION OF TRIGONOMETRICAL EQUATION :

acos θ + bsin θ = c

Consider a trigonometrical equation acosθ + bsinθ = c,

where a, b, c $\in$ R and $|c| \le \sqrt{a^2 + b^2}$

To solve this type of equation, first we reduce them in the form cosθ = cosα or sinθ = sinα.

Algorithm to solve equation of the form

acosθ + bsinθ = c

Step I Obtain the equation acosθ + bsinθ = c

Step II Put a = r cosα and b = rsinα,

where $r = \sqrt{a^2 + b^2}$ and $\tan\alpha = \frac{b}{a}$ i.e.

$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

Step III Using the substitution in step - II, the equation reduces r cos(θ – α) = c

$\Rightarrow \cos(\theta - \alpha) = \frac{c}{r} \Rightarrow \cos(\theta - \alpha) = \cos\beta$ (say)

Step IV Solve the equation obtained in step III by using the formula.

General solution of Trigonometrical equation acosθ + bsinθ = c

Solved Examples

Ex.45 General solution of equation

$\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$ is –

(A) $n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$; n $\in$ I

(B) $2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$; n $\in$ I

(C) $2n\pi \pm \frac{\pi}{4} - \frac{\pi}{6}$; n $\in$ I

(D) None of these

Sol. $\sqrt{3}\cos\theta + \sin\theta = \sqrt{2}$ ....(1)

this is the form of a cosθ + bsinθ = c

where $a = \sqrt{3}$, b = 1 and $c = \sqrt{2}$

Let a = r cosα, and b = rsinα

i.e., $\sqrt{3}$ = rcosα and 1 = rsinα

then r = 2 and $\tan\alpha = \frac{1}{\sqrt{3}}$, so $\alpha = \frac{\pi}{6}$

Substituting $a = \sqrt{3}$ = rcosα

and b = 1 = r sin α in the equation (1)

so, r [cosα cosθ + sinθ sinα] = $\sqrt{2}$

or, r cos(θ – α) = $\sqrt{2}$

or, $2\cos\left(\theta - \frac{\pi}{6}\right) = \sqrt{2}$

or, $\cos\left(\theta - \frac{\pi}{6}\right) = \frac{1}{\sqrt{2}}$

or, $\cos\left(\theta - \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{4}\right)$

or, $\theta - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}$; n $\in$ I

$\theta = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$; n $\in$ I **Ans.[B]**

Ex.46 The number of solutions of the equation $5\sec\theta - 13 = 12\tan\theta$ in $[0, 2\pi]$ is

(A) 2 (B) 1

(C) 4 (D) 0

Sol. $5\sec\theta - 13 = 12\tan\theta$

or, $13\cos\theta + 12\sin\theta = 5$

or, $\frac{13}{\sqrt{13^2+12^2}}\cos\theta + \frac{12}{\sqrt{13^2+12^2}}\sin\theta$

$= \frac{5}{\sqrt{13^2+12^2}}$ or, $\cos(\theta-\alpha) = \frac{5}{\sqrt{313}}$,

where $\cos\alpha = \frac{13}{\sqrt{313}}$

$\therefore \quad \theta = 2n\pi \pm \cos^{-1}\frac{5}{\sqrt{313}} + \alpha$

$= 2n\pi \pm \cos^{-1}\frac{5}{\sqrt{313}} + \cos^{-1}\frac{13}{\sqrt{313}}$

As $\cos^{-1}\frac{5}{\sqrt{313}} > \cos^{-1}\frac{13}{\sqrt{313}}$,

then $\theta \in [0, 2\pi]$, when n = 0 (0ne value, taking positive sign) and when n = 1 (One value, taking negative sign.) **Ans.[A]**

Ex.47 The general solution of

$\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$ is -

(A) $\theta = 2r\pi + \frac{\pi}{2}$, $r \in Z$

(B) $\theta = 2r\pi$, $r \in Z$

(C) $\theta = 2r\pi + \frac{\pi}{2}$ and $\theta = 2r\pi$, $r \in Z$

(D) None of these

Sol. We have, $\tan\left(\frac{\pi}{2}\sin\theta\right) = \cot\left(\frac{\pi}{2}\cos\theta\right)$

$\Rightarrow \tan\left(\frac{\pi}{2}\sin\theta\right) = \tan\left(\frac{\pi}{2} - \frac{\pi}{2}\cos\theta\right)$

$\Rightarrow \frac{\pi}{2}\sin\theta = r\pi + \frac{\pi}{2} - \frac{\pi}{2}\cos\theta$, $r \in Z$

$\Rightarrow \sin\theta + \cos\theta = (2r+1)$, $r \in Z$

$\Rightarrow \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{2r+1}{\sqrt{2}}$, $r \in Z$

$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \frac{2r+1}{\sqrt{2}}$, $r \in Z$

$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$

$\Rightarrow \theta - \frac{\pi}{4} = 2r\pi \pm \frac{\pi}{4}$, $r \in Z$

$\Rightarrow \theta = 2r\pi \pm \frac{\pi}{4} + \frac{\pi}{4}$, $r \in Z$

$\Rightarrow \theta = 2r\pi,\ 2r\pi + \frac{\pi}{2}$, $r \in Z$

But $\theta = 2r\pi + \frac{\pi}{2}$, $r \in Z$ gives extraneous roots as it does not satisfy the given equation. Therefore $\theta = 2r\pi$, $r \in Z$ **Ans.[B]**

Ex.48 Let n be positive integer such that

$\sin\frac{\pi}{2n} + \cos\frac{\pi}{2n} = \frac{\sqrt{n}}{2}$. Then -

(A) $6 \le n \le 8$ (B) $4 < n \le 8$

(C) $6 \le n \le 8$ (D) $4 < n < 8$

Sol. $\sin\frac{\pi}{2n} + \cos\frac{\pi}{2n} = \sqrt{2}\sin\left(\frac{\pi}{2n} + \frac{\pi}{4}\right)$

or, $\sin\left(\frac{\pi}{2n} + \frac{\pi}{4}\right) = \frac{\sqrt{n}}{2\sqrt{2}}$

since $\frac{\pi}{4} < \frac{\pi}{2n} + \frac{\pi}{4} < \frac{3\pi}{4}$ for $n > 1$

or, $\frac{1}{\sqrt{2}} < \frac{\sqrt{n}}{2\sqrt{2}} \le 1$ or, $2 < \sqrt{n} \le 2\sqrt{2}$

or, $4 < n \le 8$.

If n = 1, L.H.S. = 1, R.H.S. = 1/2

Similarly for n = 8, $\sin\left(\frac{\pi}{16} + \frac{\pi}{4}\right) \ne 1$

$\therefore$ $4 < n < 8$ **Ans.[D]**

SOLUTIONS IN THE CASE OF TWO EQUATIONS ARE GIVEN

Two equations are given and we have to find the values of variable θ which may satisfy both the given equations, like $\cos\theta = \cos\alpha$ and $\sin\theta = \sin\alpha$ so the common solution is $\theta = 2n\pi + \alpha$, $n \in I$

Similarly, $\sin\theta = \sin\alpha$ and $\tan\theta = \tan\alpha$

so the common solution is, $\theta = 2n\pi + \alpha$, $n \in I$

Rule : Find the common values of θ between 0 and 2π and then add $2\pi n$ to this common value.

Solutions in the case of two equations are given

Solved Examples

Ex.49 The most general value of θ satisfying the equations $\cos\theta = \frac{1}{\sqrt{2}}$ and $\tan\theta = -1$ is –

(A) $n\pi + \frac{7\pi}{4}$; $n \in I$

(B) $n\pi + (-1)^n \frac{7\pi}{4}$;$n \in I$

(C) $2n\pi + \frac{7\pi}{4}$; $n \in I$

(D) None of these

Sol. $\cos\theta = \frac{1}{\sqrt{2}} = \cos\left(\frac{\pi}{4}\right)$

$\theta = 2n\pi \pm \frac{\pi}{4}$; $n \in I$

Put $n = 1$, $\theta = \frac{9\pi}{4}, \frac{7\pi}{4}$

$\tan\theta = -1 = \tan\left(\frac{-\pi}{4}\right)$

$\theta = n\pi - \frac{\pi}{4}$; $n \in I$

putn $= 1$, $\theta = \frac{3\pi}{4}$

put n $= 2$, $\theta = \frac{7\pi}{4}$

The common value which satisfies both these equation is $\left(\frac{7\pi}{4}\right)$

Hence the general value is $2n\pi + \frac{7\pi}{4}$

Ans.[C]

INTRODUCTION

In any triangle ABC, the side BC, opposite to the angle A is denoted by a ; the side CA and AB, opposite to the angles B and C respectively are denoted by b and c respectively. Semiperimeter of the triangle is denoted by s and its area by Δ or S. In this chapter, we shall discuss various relations between the sides a , b, c and the angles A,B,C of Δ ABC.

SINE RULE

The sides of a triangle (any type of triangle) are proportional to the sines of the angle opposite to them in triangle ABC, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

Note :-

(1) The above rule can also be written as $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$

(2) The sine rule is very useful tool to express sides of a triangle in terms of sines of angle and vice-versa in the following manner.

$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$ (Let)

$\Rightarrow n = k \sin A, b = k \sin B, c = k \sin C$

similarly, $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=\lambda$ (Let)

$\Rightarrow \sin A = \lambda a, \sin B = \lambda b, \sin C = \lambda c$

Solved Examples

Ex.1 In a triangle ABC, if a = 3, b = 4 and sin A = $\frac{3}{4}$, then $\angle B =$

[1] 60° [2] 90° [3] 45° [4] 30°

Sol. [2] We have,

$\frac{\sin A}{a}=\frac{\sin B}{b}$ or $\sin B = \frac{b}{a}\sin A$

since, a = 3, b = 4, sin A = $\frac{3}{4}$,

we get, $\sin B = \frac{4}{3}\times\frac{3}{4}=1$ $\therefore \angle B = 90°$

Ex.2 If A = 75°, B = 45°, then $b + c\sqrt{2} =$

[1] a [2] a + b + c

[3] 2a [4] $\frac{1}{2}$ (a + b + c)

Sol. [3] C = 180° – 120° = 60°

Use sine rule $\frac{a}{\sin 75°}=\frac{b}{\sin 45°}=\frac{c}{\sin 60°}=k$

$\Rightarrow (b + c\sqrt{2}) = k(\sin 45° + \sqrt{2}\sin 60°)$

$= k\frac{\sqrt{3}+1}{\sqrt{2}} = 2k\frac{\sqrt{3}+1}{2\sqrt{2}} = 2k\sin 75° = 2k\sin A = 2a$

Ex.3 Angles of a triangle are in 4 : 1 : 1 ratio. the ratio between its greatest side and perimeter is

[1] $\frac{3}{2+\sqrt{3}}$ [2] $\frac{\sqrt{3}}{2+\sqrt{3}}$

[3] $\frac{\sqrt{3}}{2-\sqrt{3}}$ [4] $\frac{1}{2+\sqrt{3}}$

Sol. [2] Angles are in ratio 4 : 1 : 1.

$\Rightarrow$ angles are 120°, 30°, 30°.

If asides opposite to these angles are a, b, c respectively, then a will be the greatest side. Now from sine formula

$\frac{a}{\sin 120°}=\frac{b}{\sin 30°}=\frac{c}{\sin 30°}$

$\Rightarrow \frac{a}{\sqrt{3}/2}=\frac{b}{1/2}=\frac{c}{1/2} \Rightarrow \frac{a}{\sqrt{3}}=\frac{b}{1}=\frac{c}{1}=k$ (say)

then $a = \sqrt{3}k$, perimeter $= (2+\sqrt{3})k$

$\therefore$ required ratio $= \frac{\sqrt{3}k}{(2+\sqrt{3})k} = \frac{\sqrt{3}}{2+\sqrt{3}}$

Ex.4 In any ΔABC, prove that $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$.

Sol. Since $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (let)

$\Rightarrow$ a = k sinA, b = k sinB and c = k sinC

$\therefore$ L.H.S. $= \frac{a+b}{c} = \frac{k(\sin A + \sin B)}{k \sin C}$

$$= \frac{\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$= \frac{\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}\cos\frac{C}{2}} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}} = \text{R.H.S.}$$

Hence L.H.S. = R.H.S. **Proved**

Ex.5 In any ΔABC, prove that

$(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$

Sol. Since a = k sinA, b = k sinB and c = k sinC

$\therefore$ $(b^2 - c^2) \cot A = k^2 (\sin^2 B - \sin^2 C) \cot A = k^2 \sin (B + C) \sin (B - C) \cot A$

$\therefore$ $= k^2 \sin A \sin (B - C) \frac{\cos A}{\sin A}$

$= - k^2 \sin (B - C) \cos (B + C)$

$(\because \cos A = - \cos (B + C))$

$= -\frac{k^2}{2} [2\sin (B - C) \cos (B + C)]$

$= -\frac{k^2}{2} [\sin 2B - \sin 2C]$ ....(i)

Similarly $(c^2 - a^2) \cot B = -\frac{k^2}{2}$

$[\sin 2C - \sin 2A]$ ....(ii)

and $(a^2 - b^2) \cot C = -\frac{k^2}{2} [\sin 2A - \sin 2B]$ ....(iii)

adding equations (i), (ii) and (iii), we get

$(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$

Hence Proved

COSINE FORMULAE

In any ΔABC, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$

$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

Note : In particular

$\angle A = 60° \Rightarrow b^2 + c^2 - a^2 = bc$

$\angle B = 60° \Rightarrow c^2 + a^2 - b^2 = ca$

$\angle C = 60° \Rightarrow a^2 + b^2 - c^2 = ab$

Solved Examples

Ex.6 In a ΔABC, a = 2cm, b = 3cm, c = 4 cm then cosA equal to -

[1] $\frac{8}{7}$ [2] $\frac{7}{8}$ [3] $\frac{1}{8}$ [4] $\frac{1}{7}$

Sol. By the cosine rule,

$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$\cos A = \frac{3^2 + 4^2 - 2^2}{2(3)(4)}$

$\cos A = \frac{21}{24}$

$\cos A = \frac{7}{8}$ **Ans. [2]**

Ex.7 In a triangle ABC, if B = 30° and c = $\sqrt{3}$ b, then A can be equal to-

[1] 45° [2] 60° [3] 90° [4] 120°

Sol. [3] We have $\cos B = \frac{c^2 + a^2 - b^2}{2ca} \Rightarrow \frac{\sqrt{3}}{2}$

$= \frac{3b^2 + a^2 - b^2}{2 \times \sqrt{3} b \times a}$

$\Rightarrow a^2 - 3ab + 2b^2 = 0 \Rightarrow (a - 2b)(a - b) = 0$

$\Rightarrow$ Either $a = b \Rightarrow A = 30°$

$\Rightarrow a = 2b \Rightarrow a^2 = 4b^2 = b^2 + c^2 \Rightarrow A = 30°$

or $a = 2b \Rightarrow a^2 = 4b^2 = b^2 + c^2 \Rightarrow A = 90°$.

Ex.8 In a triangle ABC if a = 13, b = 8 and c = 7, then find sin A.

Sol. $\because$ $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{64+49-169}{2.8.7}$

$\Rightarrow \cos A = -\dfrac{1}{2} \Rightarrow A = \dfrac{2\pi}{3}$

$\therefore$ $\sin A = \sin\dfrac{2\pi}{3} = \dfrac{\sqrt{3}}{2}$ **Ans.**

Ex.9 In a ΔABC, prove that a (b cos C – c cos B) $= b^2 - c^2$

Sol. Since $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$ & $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$

$\therefore$ L.H.S. $= a\left\{b\left(\dfrac{a^2+b^2-c^2}{2ab}\right)-c\left(\dfrac{a^2+c^2-b^2}{2ac}\right)\right\}$

$= \dfrac{a^2+b^2-c^2}{2} - \dfrac{(a^2+c^2-b^2)}{2} = (b^2-c^2) =$ R.H.S.

Hence L.H.S. = R.H.S. **Proved**

Ex.10 If in a ΔABC, $\angle A = 60°$, then find the value of

$\left(1+\dfrac{a}{c}+\dfrac{b}{c}\right)\left(1+\dfrac{c}{b}-\dfrac{a}{b}\right)$.

Sol. $\left(1+\dfrac{a}{c}+\dfrac{b}{c}\right)\left(1+\dfrac{c}{b}-\dfrac{a}{b}\right) = \left(\dfrac{c+a+b}{c}\right)\left(\dfrac{b+c-a}{b}\right)$

$= \dfrac{(b+c)^2-a^2}{bc} = \dfrac{(b^2+c^2-a^2)+2bc}{bc}$

$= \dfrac{b^2+c^2-a^2}{bc} + 2 = 2\left(\dfrac{b^2+c^2-a^2}{2bc}\right)+2$

$= 2\cos A + 2 = 3$ $\{\because \angle A = 60°\}$

$\therefore$ $\left(1+\dfrac{a}{c}+\dfrac{b}{c}\right)\left(1+\dfrac{c}{b}-\dfrac{a}{b}\right) = 3$

PROJECTION FORMULA

In any ΔABC

(i) a = b cosC + c cosB

(ii) b = c cosA + a cosC

(iii) c = a cosB + b cosA

Solved Examples

Ex.11 In a triangle ABC,

prove that a(b cosC – c cosB) $= b^2 - c^2$

Sol. $\because$ L.H.S. = a (b cosC – c cosB)

= b (a cosC) – c (a cosB) ....(i)

$\because$ From **projection rule**, we know that

b = a cosC + c cosA $\Rightarrow$ a cosC = b – c cosA

& c = a cosB + b cosA $\Rightarrow$ a cosB = c – b cosA

Put values of a cosC and a cosB in equation (i), we get

L.H.S. = b (b – c cos A) – c(c – b cos A)

$= b^2 -$ bc cos A $- c^2 +$ bc cos A

$= b^2 - c^2$ = R.H.S.

Hence L.H.S. = R.H.S. **Proved**

Note: We have also proved a (b cosC – c cosB) $= b^2 - c^2$ by using **cosine – rule** in solved ***Example.**

Solved Examples

Ex.12 In a ΔABC, prove that (b + c) cos A + (c + a) cos B + (a + b) cos C = a + b + c.

Sol. $\because$ L.H.S. = (b + c) cos A + (c + a) cos B + (a + b) cos C

= b cos A + c cos A + c cos B + a cos B + a cos C + b cos C

= (b cos A + a cos B) + (c cos A + a cos C) + (c cos B + b cos C)

= a + b + c = R.H.S.

Hence L.H.S. = R.H.S. **Proved**

Ex.13 In any ΔABC $2\left[a\sin^2\left(\dfrac{C}{2}\right)+c\sin^2\left(\dfrac{A}{2}\right)\right]$ equals

[1] a + c – b [2] a – c + b

[3] a + c + b [4] none of these

Sol. $2\left[a\sin^2\left(\dfrac{C}{2}\right)+c\sin^2\left(\dfrac{A}{2}\right)\right]$

$\Rightarrow$ [a (1 – cos C) + c (1– cos A)]

$\Rightarrow$ a + c – (a cos C + c cos A)

$\Rightarrow$ a + c – b [By projection formulae] **Ans. [1]**

Ex.14 Solve :

$b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}$ in term of k where k is perimeter of the ΔABC.

[1] $\frac{k}{2}$ [2] $\frac{k}{4}$

[3] k [4] none of these

Sol. [1] Here,

$b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}$

$\Rightarrow \frac{b}{2}(1+\cos C)+\frac{c}{2}(1+\cos B)$

$\Rightarrow \frac{b+c}{2}+\frac{1}{2}(b\cos C+c\cos B)$

[using projection formula]

$\Rightarrow \frac{b+c}{2}+\frac{1}{2}a \Rightarrow \frac{a+b+c}{2}$

$\therefore\ b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}=\frac{k}{2}$

[where k = a + b + c, given]

NAPIER'S ANALOGY - TANGENT RULE

In any ΔABC

(i) $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$

(ii) $\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}$

(iii) $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$

Solved Examples

Ex.15 Find the unknown elements of the ΔABC in which $a=\sqrt{3}+1$, $b=\sqrt{3}-1$, $C=60°$.

Sol. $\because\ a=\sqrt{3}+1,\ b=\sqrt{3}-1,\ C=60°$

$\because\ A+B+C=180°$

$\therefore\ A+B=120°$ ....(i)

$\because$ From law of tangent, we know that $\tan\left(\frac{A-B}{2}\right)$

$=\frac{a-b}{a+b}\cot\frac{C}{2}$

$=\frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\cot 30°=\frac{2}{2\sqrt{3}}\cot 30°$

$\Rightarrow \tan\left(\frac{A-B}{2}\right)=1$

$\therefore\ \frac{A-B}{2}=\frac{\pi}{4}=45°$

$\Rightarrow\ A-B=90°$ ....(ii)

From equation (i) and (ii), we get

$A=105°$ and $B=15°$

Now,

$\because$ From **sine-rule,** we know that $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

$\therefore\ c=\frac{a\sin C}{\sin A}=\frac{(\sqrt{3}+1)\sin 60°}{\sin 105°}=\frac{(\sqrt{3}+1)\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$

$\because\ \sin 105°=\frac{\sqrt{3}+1}{2\sqrt{2}}$

$\Rightarrow c=\sqrt{6}$

$\therefore\ c=\sqrt{6}$, $A=105°$, $B=15°$ Ans.

Ex.16 In a ΔABC, $b=\sqrt{3}+1$, $c=\sqrt{3}-1$, $\angle A=60°$ then the value of $\tan\left(\frac{B-C}{2}\right)$ is

[1] 2 [2] 1/2 [3] 1 [4] 3

Sol. $\tan\left(\frac{B-C}{2}\right)=\left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$

putting the value of b, c and $\angle A$

$\tan\left(\frac{B-C}{2}\right)=\frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\cot(30°)$

$\Rightarrow \tan\left(\frac{B-C}{2}\right)=1$ **Ans. [3]**

Ex.17 If $\tan\left(\frac{B-C}{2}\right)=x\cot\left(\frac{A}{2}\right)$, find the value of x.

[1] $\frac{c-a}{c+a}$ [2] $\frac{a-b}{a+b}$ [3] $\frac{b-c}{b+c}$ [4] None

Sol. By the formulae

$\tan\left(\frac{B-C}{2}\right)=\frac{b-c}{b+c}\cot\left(\frac{A}{2}\right)$ **Ans. [3]**

TRIGONOMETRIC FUNCTIONS OF HALF ANGLES :

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{bc}}$, $\sin\frac{B}{2} = \sqrt{\frac{(s-c)\,(s-a)}{ca}}$,

$\sin\frac{C}{2} = \sqrt{\frac{(s-a)\,(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s\,(s-a)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{s\,(s-b)}{ca}}$, $\cos\frac{C}{2}$

$= \sqrt{\frac{s\,(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)\,(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)} = \frac{(s-b)(s-c)}{\Delta}$,

where $s = \frac{a+b+c}{2}$ is semi perimeter and Δ is the area of triangle.

(iv) $\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

Solved Examples

Ex.18 In a triangle ABC if $\frac{s-a}{11} = \frac{s-b}{12} = \frac{s-c}{13}$, then $\tan^2(A/2) =$

[1] $\frac{143}{432}$ [2] $\frac{13}{33}$ [3] $\frac{11}{39}$ [4] $\frac{12}{37}$

Sol. [2] $\frac{s-a}{11} = \frac{s-b}{12} = \frac{s-c}{13} = \frac{3s-(a+b+c)}{11+12+13} = \frac{s}{36}$

Now $\tan^2\left(\frac{A}{2}\right) = \frac{(s-b)(s-c)}{s(s-a)} = \frac{12\times 13}{36\times 11} = \frac{13}{33}$

Ex.19 In a ΔABC, if a = 13, b = 14 and c = 15, then the value of $\sin\left(\frac{A}{2}\right)$ is

[1] $\frac{3}{5}$ [2] $\frac{1}{\sqrt{5}}$ [3] $\frac{7}{\sqrt{65}}$ [4] 6

Sol. We know that, 2s = a + b + c

2s = 42

s = 21

s – a = 8, s – b = 7, and s – c = 6

$\sin\left(\frac{A}{2}\right) = \sqrt{\frac{(s-b)(s-c)}{bc}}$

$= \sqrt{\frac{7\times 6}{14\times 15}} = \frac{1}{\sqrt{5}}$ **Ans. [2]**

Ex.20 In a triangle ABC, if $\cos\frac{A}{2} = \sqrt{\frac{b+c}{2c}}$, then

[1] $a^2 + b^2 = c^2$ [2] $b^2 + c^2 = a^2$
[3] $c^2 + a^2 = b^2$ [4] $b - c = c - a$

Sol. $\cos\frac{A}{2} = \sqrt{\frac{b+c}{2c}} \Rightarrow \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{b+c}{2c}}$

$\Rightarrow 2s(s-a) = b^2 + bc$

$\Rightarrow (a+b+c)(b+c-a) = 2b^2 + 2bc$

$\Rightarrow a^2 + b^2 = c^2$ **Ans. [1]**

Ex.21 In a ΔABC, the sides a, b and c are in A.P. Then $\left(\tan\frac{A}{2} + \tan\frac{C}{2}\right) : \cot\frac{B}{2}$ is equal to

[1] 3 : 2 [2] 1 : 2
[3] 3 : 4 [4] 2 : 3

Sol. $\left(\tan\frac{A}{2} + \tan\frac{C}{2}\right) : \cot\frac{B}{2}$

$\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\right] : \sqrt{\frac{s(s-b)}{(s-c)(s-a)}}$

$= \frac{(s-c)+(s-c)}{\sqrt{s}} : \sqrt{s}$

$= 2s - (a+c) : s$

$\Rightarrow b : \frac{a+b+c}{2}$

$\Rightarrow 2b : a + b + c = 2b : 3b$

[$\because$ a, b, c are in A.P. $\therefore$ 2b = a + c]

= 2 : 3 **Ans. [4]**

Ex.22 In a ΔABC if a, b, c are in A.P., then find the value of $\tan\frac{A}{2}.\tan\frac{C}{2}$

Sol. Since $\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}$ and $\tan\frac{C}{2} = \frac{\Delta}{s(s-c)}$

$\therefore \tan\frac{A}{2}.\tan\frac{C}{2} = \frac{\Delta^2}{s^2(s-a)(s-c)}$

$\because \Delta^2 = s(s-a)(s-b)(s-c)$

$\therefore \tan\frac{A}{2}.\tan\frac{C}{2} = \frac{s-b}{s} = 1 - \frac{b}{s}$ ....(i)

$\because$ it is given that a, b, c are in A.P. $\Rightarrow 2b = a + c$

$\because s = \frac{a+b+c}{2} = \frac{3b}{2}$

$\therefore \frac{b}{s} = \frac{2}{3}$ put in equation (i), we get

$\therefore \tan\frac{A}{2}.\tan\frac{C}{2} = 1 - \frac{2}{3}$

$\Rightarrow \tan\frac{A}{2}.\tan\frac{C}{2} = \frac{1}{3}$ **Ans.**

Ex.23 In any ΔABC, prove that $(a + b + c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right) = 2c\cot\frac{C}{2}$.

Sol. $\because$ L.H.S. $= (a+b+c)\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)$

$\because \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ and $\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$\therefore$ L.H.S. $= (a+b+c)$

$\left[\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\right]$

$= 2s\sqrt{\frac{s-c}{s}}\left[\sqrt{\frac{s-b}{s-a}} + \sqrt{\frac{s-a}{s-b}}\right]$

$= 2\sqrt{s(s-c)}\left[\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\right]$

$\because 2s = a+b+c \quad \therefore \quad 2s-b-a = c$

$= 2\sqrt{s(s-c)}\left[\frac{c}{\sqrt{(s-a)(s-b)}}\right] = 2c\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$\because \cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = 2c\cot\frac{C}{2}$ = R.H.S.

Hence L.H.S. = R.H.S. **Proved**

AREA OF A TRIANGLE

If Δ be the area of a triangle ABC, then

(i) $\Delta = \frac{1}{2}bc\ \sin A = \frac{1}{2}ca\ \sin B = \frac{1}{2}ab\sin C$

(ii) $\Delta = \frac{1}{2}\frac{a^2\sin B\sin C}{\sin(B+C)} = \frac{1}{2}\frac{b^2\sin C\sin A}{\sin(C+A)} = \frac{1}{2}\frac{c^2\sin A\sin B}{\sin(A+B)}$

(iii) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ (Hero's formula)

Form above results, we obtain following values of sin A, sin B, sin C

(iv) $\sin A = \frac{2\Delta}{bc} = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

(v) $\sin B = \frac{2\Delta}{ca} = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}$

(vi) $\sin C = \frac{2\Delta}{ab} = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$

Further with the help of (iv) , (v) (vi) we obtain

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\Delta}{abc}}$$

Solved Examples

Ex.24 Find the area of a triangle ABC in which $\angle A = 60^o$, b = 4 cm and c = $\sqrt{3}$ cm

[1] 3 sq. cm [2] 5 sq. cm
[3] 8 sq. cm [4] none

Sol. The area of triangle ABC is given by

$\Delta = \frac{1}{2}$ bc sin A $= \frac{1}{2} \times 4\sqrt{3} \times \sin 60^o$

$= 2\sqrt{3}\times\frac{\sqrt{3}}{2}$ = 3 sq. cm **Ans. [1]**

Ex.25 In any triangle ABC, if a = $\sqrt{2}$ cm, b = $\sqrt{3}$ cm and c = $\sqrt{5}$ cm, show that its area is $\frac{1}{2}\sqrt{6}$ sq. cm.

Sol. We know that, $\cos C = \frac{a^2+b^2-c^2}{2ab} = 0$

then, $\angle C = \frac{\pi}{2}$

so, $A = \frac{1}{2}$ ab sin C [$\because$ sin c = 1]

$\Delta = \frac{1}{2}\times(\sqrt{2})(\sqrt{3})(1)$

$\Delta = \frac{\sqrt{6}}{2}$ sq. cm

Ex.26 In a ΔABC, the sides are in the ratio 4 : 5 : 6. The ratio of the circumradius and the inradius is-

[1] 8 : 7 [2] 3 : 2 [3] 7 : 3 [4] 16 : 7

Sol. [4] Here a = 4k, b = 5k, c = 6k

$\therefore\ s = \frac{15k}{2}$

$\therefore\ \Delta = \sqrt{\frac{15k}{2}\left(\frac{15k}{2}-4k\right)\left(\frac{15k}{2}-5k\right)\left(\frac{15k}{2}-6k\right)} = \frac{15\sqrt{7}}{4}k^2$

But $R = \frac{abc}{4\Delta} = \frac{4k.5k.6k}{15\sqrt{7}\,k^2} = \frac{8}{\sqrt{7}}k$ and

$r = \frac{\Delta}{s} = \frac{15\sqrt{7}}{4}k^2.\frac{2}{15k} = \frac{\sqrt{7}}{2}k$

$\therefore\ \frac{R}{r} = \frac{\frac{8k}{\sqrt{7}}}{\frac{\sqrt{7}\,k}{2}} = \frac{16}{7} = 16:7$

Ex.27 In a ΔABC if b sinC(b cosC + c cosB) = 42, then find the area of the ΔABC.

Sol. $\because$ b sinC (b cosC + c cosB) = 42 ....(i) given

$\because$ From **projection rule**, we know that

a = b cosC + c cosB put in (i), we get

ab sinC = 42 ....(ii)

$\because\ \Delta = \frac{1}{2}$ ab sinC

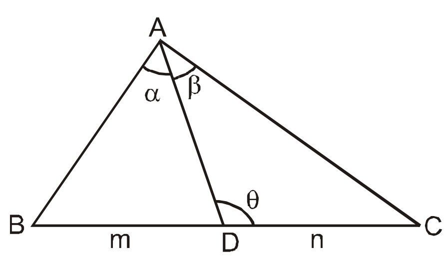
$\therefore$ from equation (ii), we get

$\therefore\ \Delta = 21$ sq. unit

m - n Rule

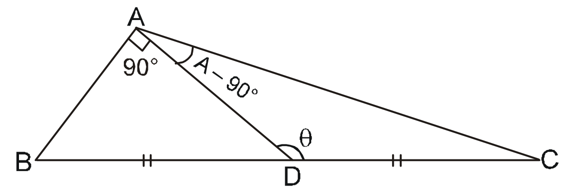
In any triangle ABC if D be any point on the base BC, such that BD : DC :: m : n and if $\angle BAD = \alpha$, $\angle DAC = \beta$, $\angle CDA = \theta$, then

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$
$$= n\cot B - m\cot C$$



Ex.28 If the median AD of a triangle ABC is perpendicular to AB, prove that tan A + 2tan B = 0.

Sol. From the figure, we see that $\theta = 90° + B$ (as θ is external angle of ΔABD)



Now if we apply **m-n rule** in ΔABC, we get

$(1 + 1) \cot (90° + B) = 1. \cot 90° - 1.\cot (A - 90°)$

$\Rightarrow\ -2 \tan B = \cot (90° - A)$

$\Rightarrow\ -2 \tan B = \tan A$

$\Rightarrow\ \tan A + 2 \tan B = 0$ **Hence proved.**

CIRCUMCIRCLE OF A TRIANGLE

A circle passing through the vertices of a triangle is called the circumcircle of the triangle.

The centre of the circumcircle is called the circum-centre of the triangle and it is the point of intersection of the perpendicular bisectors of the sides of the triangle.

The radius of the circumcircle is called the circum radius of the triangle and is usually denoted by R and is given by the following formulae

$$\boxed{R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}}$$

Where Δ is area of triangle and $s = \frac{a+b+c}{2}$

Solved Examples

Ex.29 The diameter of the circumcircle of a triangle with sides 5 cm, 6 cm and 7 cm is

[1] $\frac{3\sqrt{3}}{2}$ cm [2] $2\sqrt{6}$ cm

[3] $\frac{35}{48}$ cm [4] None of these

Sol. Radius of circumcircle is given by $R = \frac{abc}{4\Delta}$ and

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$

Here a = 5 cm, b = 6 cm, and c = 7 cm

$\therefore s = \frac{5+6+7}{2} = 9$

$\Delta = \sqrt{9(9-5)(9-6)(9-7)} = \sqrt{216} = 6\sqrt{6}$

$\Rightarrow R = \frac{5.6.7}{4.6.\sqrt{6}} = \frac{35}{4\sqrt{6}}$

Diameter $= 2R = \frac{35}{2\sqrt{6}}$ **Ans. [4]**

Ex.30 In a ΔABC, prove that $\sin A + \sin B + \sin C = \frac{s}{R}$

Sol. In a ΔABC, we know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ $= 2R$

$\therefore \sin A = \frac{a}{2R}, \sin B = \frac{b}{2R}$ and $\sin C = \frac{c}{2R}$.

$\therefore \sin A + \sin B + \sin C = \frac{a+b+c}{2R} = \frac{2s}{2R}$

$\because a + b + c = 2s$

$\Rightarrow \sin A + \sin B + \sin C = \frac{s}{R}$.

Ex.31 In a ΔABC if a = 13 cm, b = 14 cm and c = 15 cm, then find its circumradius.

Sol. $\because R = \frac{abc}{4\Delta}$ ....(i)

$\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\because s = \frac{a+b+c}{2} = 21$ cm

$\therefore \Delta = \sqrt{21\times 8\times 7\times 6} = \sqrt{7^2\times 4^2\times 3^2} \Rightarrow \Delta = 84\ \text{cm}^2$

$\therefore R = \frac{13\times 14\times 15}{4\times 84} = \frac{65}{8}$ cm

$\therefore R = \frac{65}{8}$ cm. Ans.

Ex.32 In a ΔABC, prove that

$s = 4R \cos\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}$.

Sol. In a ΔABC,

$\because \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}, \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ and

$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$ and $R = \frac{abc}{4\Delta}$

$\because$ R.H.S. $= 4R \cos\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}$.

$= \frac{abc}{\Delta} . s\sqrt{\frac{s(s-a)(s-b)(s-c)}{(abc)^2}} = s$

$\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$ = L.H.S.

Hence R.H.S = L.H.S. proved.

Ex.33 In a ΔABC, prove that

$\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{4R}{\Delta}$.

Sol. $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{4R}{\Delta}$

$\because$ L.H.S. $= \left(\frac{1}{s-a} + \frac{1}{s-b}\right) + \left(\frac{1}{s-c} - \frac{1}{s}\right)$

$= \frac{2s-a-b}{(s-a)(s-b)} + \frac{(s-s+c)}{s(s-c)}$ $\because 2s = a + b + c$

$= \frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)}$

$= c\left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]$

$= c\left[\frac{2s^2 - s(a+b+c) + ab}{\Delta^2}\right]$

$\therefore$ L.H.S. $= c\left[\frac{2s^2 - s(2s) + ab}{\Delta^2}\right] = \frac{abc}{\Delta^2} = \frac{4R\Delta}{\Delta^2}$

$= \frac{4R}{\Delta}$

$\because R = \frac{abc}{4\Delta} \Rightarrow abc = 4R\Delta$

$\therefore$ L.H.S. $= \frac{4R}{\Delta}$ = R.H.S.

INCIRCLE OF A TRIANGLE

The circle which can be inscribed within the triangle so as to touch all the three sides is called the incircle of the triangle.

The centre of the incircle is called the in centre of the triangle and it is the point of intersection of the internal bisectors of the angles of the triangle.

The radius of the incircle is called the inradius of the triangle and is usually denoted by r and is given by the following formula

In – Radius : The radius r of the inscribed circle of a triangle ABC is given by

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s - a)\tan\left(\frac{A}{2}\right)$, $r = (s - b)\tan\left(\frac{B}{2}\right)$ and $r = (s - c)\tan\left(\frac{C}{2}\right)$

(iii) $r = \frac{a\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{A}{2}\right)}$, $r = \frac{b\sin\left(\frac{A}{2}\right)\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{B}{2}\right)}$ and $r = \frac{c\sin\left(\frac{B}{2}\right)\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{C}{2}\right)}$

(iv) $r = 4R\sin\left(\frac{A}{2}\right).\sin\left(\frac{B}{2}\right).\sin\left(\frac{C}{2}\right)$

Solved Examples

Ex.34 The ratio of the circumradius and inradius of an equilateral triangle is

[1] 3 : 1 [2] 1 : 2 [3] 2 : $\sqrt{3}$ [4] 2 : 1

Sol. $\frac{r}{R} = \frac{a\cos A + b\cos B + c\cos C}{a+b+c}$

In equilateral triangle A = B = C = 60º

$= \frac{(a+b+c)\cos 60^\circ}{a+b+c} = \frac{1}{2}$ **Ans. [2]**

Ex.35 A ΔABC is right angle at B. Then the diameter of the inccircle of the triangle is

[1] 2(c + a – b) [2] c + a – 2b

[3] c + a – b [4] none of these

Sol. r = $r = \frac{\Delta}{s} = \frac{\left(\frac{1}{2}\right)ac}{\left(\frac{1}{2}\right)(a+b+c)} = \frac{ac}{(a+b+c)} = \frac{ac(c+a-b)}{(c+a)^2 - b^2}$

$= \frac{ac(c+a-b)}{c^2 + 2ca + a^2 - b^2}$ $= \frac{ac(c+a-b)}{2ca + b^2 - b^2}$

$= \frac{c+a-b}{2}$ $(\because a^2 + c^2 = b^2)$ **Ans. [4]**

Radius of The Ex- Circles

If r_1, r_2, r_3 are the radii of the ex-circles of ΔABC opposite to the vertex A, B, C respectively, then

(i) $r_1 = \frac{\Delta}{s-a}$; $r_2 = \frac{\Delta}{s-b}$; $r_3 = \frac{\Delta}{s-c}$

(ii) $r_1 = s\tan\frac{A}{2}$; $r_2 = s\tan\frac{B}{2}$; $r_3 = s\tan\frac{C}{2}$

(iii) $r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$ and so on

(iv) $r_1 = 4R\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$

Solved Examples

Ex.36 In a ΔABC, prove that

$r_1 + r_2 + r_3 - r = 4R = 2a\,\text{cosec}A$

Sol. $\because$ L.H.S $= r_1 + r_2 + r_3 - r$

$= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s}$

$= \Delta\left(\frac{1}{s-a} + \frac{1}{s-b}\right) + \Delta\left(\frac{1}{s-c} - \frac{1}{s}\right)$

$= \Delta\left[\left(\frac{s-b+s-a}{(s-a)(s-b)}\right) + \left(\frac{s-s+c}{s(s-c)}\right)\right]$

$= \Delta\left[\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)}\right]$

$= c\Delta\left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right]$

$= c\Delta\left[\frac{2s^2 - s(a+b+c) + ab}{\Delta^2}\right] = \frac{abc}{\Delta}$

$\because$ $a + b + c = 2s$

$\because$ $R = \frac{abc}{4\Delta}$ $\quad = 4R = 2a \text{ cosec } A$

$\because$ $\frac{a}{\sin A} = 2R$ $\quad = a \text{ cosec } A \quad = \text{R.H.S.}$

Hence L.H.S. = R.H.S. **proved**

Ex.37 If the area of a ΔABC is 96 sq. unit and the radius of the escribed circles are respectively 8, 12 and 24. Find the perimeter of ΔABC.

Sol. $\because$ $\Delta = 96$ sq. unit

$r_1 = 8, r_2 = 12$ and $r_3 = 24$

$\because$ $r_1 = \frac{\Delta}{s-a} \Rightarrow s - a = 12$ ....(i)

$\because$ $r_2 = \frac{\Delta}{s-b} \Rightarrow s - b = 8$ ....(ii)

$\because$ $r_3 = \frac{\Delta}{s-c} \Rightarrow s - c = 4$ ....(iii)

$\therefore$ adding equations (i), (ii) & (iii), we get

$3s - (a + b + c) = 24$

$s = 24$

$\therefore$ perimeter of $\Delta ABC = 2s = 48$ unit. Ans.

Ex.38 In a Δ ABC, if a = 18 cm and b = 24 cm and c = 30 cm then the value of r_1, r_2 and r_3 are

[1] 12 cm, 18 cm 36 cm [2] 12 cm, 8 cm, 30 cm

[3] 12 cm, 10 cm, 30 cm [4] 12 cm, 18 cm, 36 cm

Sol. a = 18 cm, b = 24 cm, c = 30 cm

$\therefore$ $2s = a + b + c = 72$ cm

$s = 36$ cm

But, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$\Delta = 216$ sq. units

then, $r_1 = \frac{\Delta}{s-a} = \frac{216}{18} = 12\text{cm}$

or, $r_2 = \frac{\Delta}{s-b} = \frac{216}{12} = 18\text{cm}$

or, $r_3 = \frac{\Delta}{s-c} = \frac{216}{6} = 36\text{cm}$

so, r_1, r_2, r_3 are 12 cm, 18 cm, and 36 cm

Ans. [4]

Ex.39 If the exradii of a triangle are in HP the corresponding sides are in

[1] A.P. [2] G.P.

[3] H.P. [4] None of these

Sol. $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}$ are in A.P. $\Rightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta}$ are in A.P.

$\Rightarrow$ $s - a, s - b, s - c$ are in A.P.

$\Rightarrow$ $-a, -b, -c$ are in A.P. **Ans. [1]**

Ex.40 Value of the $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$ is equal to-

[1] 1 [2] 2 [3] 3 [4] 0

Sol. [4]

$\frac{(b-c)}{r_1} + \frac{(c-a)}{r_2} + \frac{(a-b)}{r_3}$

$\Rightarrow (b-c)\left(\frac{s-a}{\Delta}\right) + (c-a)\left(\frac{s-b}{\Delta}\right) + (a-b).\left(\frac{s-c}{\Delta}\right)$

$\Rightarrow \frac{(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)}{\Delta}$

$= \frac{s(b-c+c-a+a-b)-[ab-ac+bc-ba+ac-bc]}{\Delta}$

$= \frac{0}{\Delta} = 0 = \textbf{R.H.S.}$

Thus, $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0$

Ex.41 Value of the $r \cot \frac{B}{2} \cot \frac{C}{2}$ is equal to-

[1] r_1 [2] r_2

[3] $2r_1$ [4] none of these

Sol. [1] $r \cot B/2. \cot C/2$

$\Rightarrow 4R \sin A/2.\sin B/2.\sin C/2. \frac{\cos B/2}{\sin B/2}.\frac{\cos C/2}{\sin C/2}$

[as $r = 4R \sin A/2 \sin B/2 \sin C/2$]

$\Rightarrow 4R. \sin A/2. \cos B/2. \cos C/2$

$\Rightarrow r_1$ = R.H.S. {as, $r_1 = 4R \sin A/2. \cos B/2. \cos C/2$}

$\therefore$ $r \cot B/2. \cot C/2 = r_1$

ORTHOCENTRE OF A TRIANGLE

The point of intersection of perpendiculars drawn from the vertices on the opposite sides of a triangle is called its orthocentre.

Let the perpendicular AD, BE and CF from the vertices A, B and C on the opposite sides BC, CA and AB of ABC, respectively, meet at O. Then O is the orthocentre of the ΔABC.

The triangle DEF is called the pedal triangle of the ΔABC.

The distances of the orthocentre from the vertices and the sides - If O is the orthocentre and DEF the pedal triangle of the ΔABC, where AD, BE, CF are the perpendiculars drawn from A, B,C on the opposite sides BC, CA, AB respectively, then

(i) OA = 2R cos A, OB = 2R cos B and OC = 2R cos C

(ii) OD = 2R cos B cos C, OE = 2R cos C cos A and OF = 2R cos A cos B

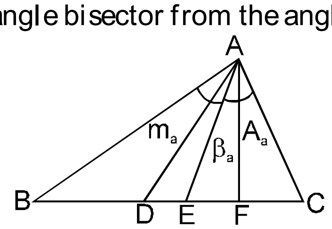
(iii) The circumradius of the pedal triangle $= \frac{R}{2}$

(iv) The area of pedal triangle = 2Δ cosA cos B cos C.

LENGTH OF ANGLE BISECTORS, MEDIANS & ALTITUDES

(i) Length of an angle bisector from the angle A = β_a

$= \dfrac{2bc\cos\frac{A}{2}}{b+c}$;



(ii) Length of median from the angle A = m_a

$= \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$ &

(iii) Length of altitude from the angle A $= A_a = \dfrac{2\Delta}{a}$

NOTE : $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

Solved Examples

Ex.42 AD is a median of the ΔABC. If AE and AF are medians of the triangles ABD and ADC respectively, and AD = m_1, AE = m_2, AF = m_3, then prove that $m_2^2 + m_3^2 - 2m_1^2 = \frac{a^2}{8}$.

Sol. $\because$ In ΔABC

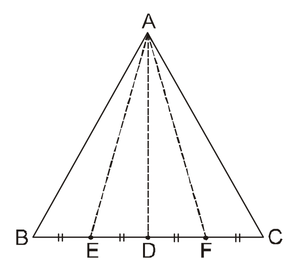
$AD^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2) = m_1^2$ ....(i)

$\because$ In ΔABD, $AE^2 = m_2^2 = \frac{1}{4}\left(2c^2 + 2AD^2 - \frac{a^2}{4}\right)$ ....(ii)

Similarly in ΔADC, $AF^2 = m_3^2 = \frac{1}{4}\left(2AD^2 + 2b^2 - \frac{a^2}{4}\right)$ ....(iii)

by adding equations (ii) and (iii), we get

$\because\ m_2^2 + m_3^2 = \frac{1}{4}\left(4AD^2 + 2b^2 + 2c^2 - \frac{a^2}{2}\right)$



$= AD^2 + \frac{1}{4}\left(2b^2 + 2c^2 - \frac{a^2}{2}\right) = AD^2 + \frac{1}{4}\left(2b^2 + 2c^2 - a^2 + \frac{a^2}{2}\right)$

$= AD^2 + \frac{1}{4}(2b^2 + 2c^2 - a^2) + \frac{a^2}{8}$

$= AD^2 + AD^2 + \frac{a^2}{8}$

$= 2AD^2 + \frac{a^2}{8} = 2m_1^2 + \frac{a^2}{8}$

$\because\ AD^2 = m_1^2$

$\therefore\ m_2^2 + m_3^2 - 2m_1^2 = \frac{a^2}{8}$ **Hence Proved**

THE DISTANCES OF THE SPECIAL POINTS FROM VERTICES AND SIDES OF TRIANGLE

(i) Circumcentre (O) : $OA = R$ and $O_a = R\cos A$

(ii) Incentre (I) : $IA = r\,\text{cosec}\,\frac{A}{2}$ and $I_a = r$

(iii) Excentre (I_1) : $I_1A = r_1\,\text{cosec}\,\frac{A}{2}$ and $I_{1a} = r_1$

(iv) Orthocentre (H) : $HA = 2R\cos A$ and $H_a = 2R\cos B\cos C$

(v) Centroid (G) : $GA = \frac{1}{3}\sqrt{2b^2+2c^2-a^2}$ and $G_a = \frac{2\Delta}{3a}$

Solved Examples

Ex.43 If x, y and z are respectively the distances of the vertices of the ΔABC from its orthocentre, then prove that (i) $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$ (ii) $x+y+z=2(R+r)$

Sol. $\because$ $x = 2R\cos A$, $y = 2R\cos B$, $z = 2R\cos C$ and and $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$

$\therefore\ \frac{a}{x}+\frac{b}{y}+\frac{c}{z} = \tan A+\tan B+\tan C$ ....(i)

& $\frac{abc}{xyz} = \tan A.\tan B.\tan C$ ....(ii)

$\because$ We know that in a ΔABC $\Sigma\tan A = \Pi\tan A$

$\therefore$ From equations (i) and (ii), we get $\frac{a}{x}+\frac{b}{y}+\frac{c}{z} = \frac{abc}{xyz}$

$\because\ x+y+z = 2R(\cos A+\cos B+\cos C)$

$\because$ in a ΔABC $\cos A+\cos B+\cos C$

$= 1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore\ x+y+z = 2R\left(1+4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\right)$

$= 2\left(R+4R\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\right)$

$\because\ r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore\ x+y+z = 2(R+r)$

SOME IMPORTANT RESULTS

(1) $\tan\frac{A}{2}\tan\frac{B}{2} = \frac{s-c}{s}$ $\quad\therefore\quad \cot\frac{A}{2}\cot\frac{B}{2} = \frac{s}{s-c}$

(2) $\tan\frac{A}{2}+\tan\frac{B}{2} = \frac{c}{s}\cot\frac{C}{2} = \frac{c}{\Delta}(s-c)$

(3) $\tan\frac{A}{2}-\tan\frac{B}{2} = \frac{a-b}{\Delta}(s-c)$

(4) $\cot\frac{A}{2}+\cot\frac{B}{2} = \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}\tan\frac{B}{2}} = \frac{c}{s-c}\cot\frac{C}{2}$

(5) Also note the following identities

(i) $\Sigma(p-q) = (p-q)+(q-r)+(r-p) = 0$

(ii) $\Sigma p(q-r) = p(q-r)+q(r-p)+r(p-q) = 0$

(iii) $\Sigma(p+a)(q-r)\ \Sigma p(q-r)+a\Sigma(q-r) = 0$

Solved Examples

Ex.44 In a triangle ABC if $\cot\frac{A}{2}\cot\frac{B}{2} = c$, $\cot\frac{B}{2}\cot\frac{C}{2} = a$ and $\cot\frac{C}{2}\cot\frac{A}{2} = b$, then $\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c} =$

[1] –1 [2] 0 [3] 1 [4] 2

Sol. [4] $\cot\frac{A}{2}\cot\frac{B}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}\times\frac{s(s-b)}{(s-c)(s-a)}} = c$

$\frac{s}{s-c} = c \Rightarrow \frac{1}{s-c} = \frac{c}{s}$ similarly

$\frac{1}{s-a} = \frac{a}{s}$ and $\frac{1}{s-b} = \frac{b}{s}$

so that $\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c} = \frac{a+b+c}{s} = \frac{2s}{s} = 2$

EXCENTRAL TRIANGLE

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.

(i) ΔABC is the pedal triangle of the $\Delta I_1 I_2 I_3$.

(ii) Its angles are $\frac{\pi}{2}-\frac{A}{2}$, $\frac{\pi}{2}-\frac{B}{2}$ and $\frac{\pi}{2}-\frac{C}{2}$.

(iii) Its sides are $4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$.

(iv) $II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

(v) Incentre I of Δ ABC is the orthocentre of the excentral $\Delta I_1 I_2 I_3$.

DISTANCE BETWEEN SPECIAL POINTS

(i) Distance between circumcentre and orthocentre
$OH^2 = R^2(1 - 8\cos A\cos B\cos C)$

(ii) Distance between circumcentre and incentre
$OI^2 = R^2\left(1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right) = R^2 - 2Rr$

(iii) Distance between circumcentre and centroid
$OG^2 = R^2 - \frac{1}{9}(a^2+b^2+c^2)$

Solved Examples

Ex.45 If I is the incentre and I_1, I_2, I_3 are the centres of escribed circles of the ΔABC, prove that

(i) $II_1 . II_2 . II_3 = 16R^2r$

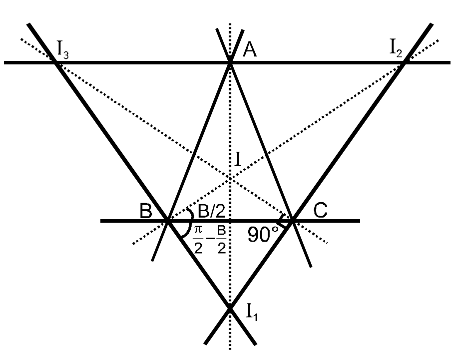
(ii) $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$

Sol. (i)

$\because$ We know that

$II_1 = a\sec\frac{A}{2}$, $II_2 = b\sec\frac{B}{2}$ and $II_3 = c\sec\frac{C}{2}$

$\because$ $I_1I_2 = c.\ \text{cosec}\frac{C}{2}$,



$I_2I_3 = a\ \text{cosec}\frac{A}{2}$ and $I_3I_1 = b\ \text{cosec}\frac{B}{2}$

$\because$ $II_1 . II_2 . II_3 = abc\sec\frac{A}{2}.\sec\frac{B}{2}.\sec\frac{C}{2}$ ......(i)

$\because$ $a = 2R\sin A$, $b = 2R\sin B$ and $c = 2R\sin C$

$\therefore$ equation (i) becomes

$\because$ $II_1 . II_2 . II_3 = (2R\sin A)(2R\sin B)(2R\sin C)$

$\sec\frac{A}{2}\sec\frac{B}{2}\sec\frac{C}{2}$

$$= 8R^3 . \frac{\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right)\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)}{\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}$$

$= 64R^3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\because r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\therefore$ $II_1 . II_2 . II_3 = 16R^2r$ **Hence Proved**

(ii) $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$

$\because$ $II_1^2 + I_2I_3^2 = a^2\sec^2\frac{A}{2} + a^2\text{cosec}^2\frac{A}{2}$

$$= \frac{a^2}{\sin^2\frac{A}{2}\cos^2\frac{A}{2}}$$

$\because$ $a = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2}$

$$\therefore\ II_1^2 + I_2I_3^2 = \frac{16R^2\sin^2\frac{A}{2}.\cos^2\frac{A}{2}}{\sin^2\frac{A}{2}.\cos^2\frac{A}{2}} = 16R^2$$

Similarly we can prove

$II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2 = 16R^2$

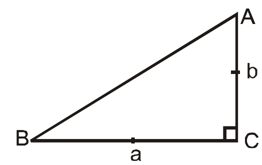
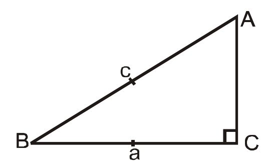
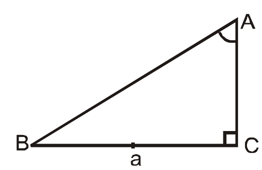
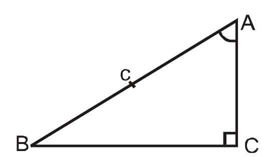
Hence $II_1^2 + I_2I_3^2 = II_2^2 + I_3I_1^2 = II_3^2 + I_1I_2^2$

SOLUTION OF TRIANGLES

Introduction – In a triangle, there are six elements viz. three sides and three angles. In plane geometry we have done that if three of the elements are given, at least one of which must be a side, then the other three elements can be uniquely determined. The procedure of determining unknown elements from the known elements is called solving a triangle.

SOLUTION OF A RIGHT ANGLED TRIANGLE

Let triangle ABC is right angled and $\angle C = 90°$. Then in different cases its solution is determined as shown in the following table.

	Given	To find	Formulae	Figure
(i)	(Two sides) a , b	A,B,C	tanA = a/b, B = 90° – A, C = a/sin A or tanB = b/a A = 90° – B C=b/sinB	
(ii)	(hypotenuse and one side) c, a	A,B, b	sin A = a / c B = 90° – A b = c cos A or b = a cot A B = 90° –A	
(iii)	(one side and one angle) a,A	B,b,c	b = a cot A $c = \frac{a}{\sin A}$ B = 90° –A	
(iv)	(hypotenuse and one angle) c, A	B, a, b	a = c sin A b = c cos A	

SOLUTION OF A GENERAL TRIANGLE

In different cases, solution of a general triangle is determined as follows :

Case I .

When three sides a ,b ,c are given :

In this case remaining elements i.e., angles A,B,C are determine by using following formulae

$\sin A = \frac{2\Delta}{bc}$, $\sin B = \frac{2\Delta}{ca}$. $\sin C = \frac{2\Delta}{ab}$

$\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}$, $\tan\frac{B}{2} = \frac{\Delta}{s(s-b)}$. $\tan\frac{C}{2} = \frac{\Delta}{s(s-c)}$

$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, similar results for $\tan\frac{B}{2} = \tan\frac{C}{2}$

$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, similar results for cos B, cos C

(use cosine formula when a, b, c are small numbers)

Case II.

When two sides say a , b and angle C between them are given :

In this case remaining elements A,B,c are determined by suing following formulae :

$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$, $\frac{A+B}{2}=90^\circ-\frac{C}{2}$

$c=\frac{a\sin C}{\sin A}$ or $c^2=a^2+b^2-2ab\cos C$

Case III

When two angles A,B, and one side a are given:

In this case remaining elements C, b, c are determined by using following formulae :

$C=180^\circ-(A+B)$ $b=\frac{a\sin B}{\sin A}$; $c=\frac{a\sin C}{\sin A}$

Note : - If angles A, B and side c be given, then we use following results

$C=180^\circ-(A+B)$ $b=\frac{c\sin B}{\sin C}$, $a=\frac{c\sin A}{\sin C}$

Case IV .

When two sides a , b and angle opposite to one side, say A, are given

In this case remaining elements B, C, c are determined by using following formulae :

$\sin B=\frac{b\sin A}{a}$ .....(1)

$C=180^\circ-(A+B)$ .....(2)

$c=\frac{a\sin C}{\sin A}$ .....(3)

while using above formulae, (1) may given following possibilities :

(i) When $A<90^\circ$ and $a<b\sin A$:

In this case $\sin B=\frac{b\sin A}{a}\Rightarrow\sin B>1$ which is not possible. hence no triangle will be possible

(ii) When $A<90^\circ$ and $a=b\sin A$:

In this case $\sin B=1\Rightarrow B=90^\circ\Rightarrow$ only one triangle is possible which is right angled at B.

(iii) When $A<90^\circ$ and $a>b\sin A$:

In this case $\sin B=\frac{b\sin A}{a}$ gives two such angles say B_1 , B_2 that $B_1+B_2=180^\circ$

(iv) when $A>90^\circ$

If $a\le b$, then B is also obtuse angle which is not possible.

If $a>b$, then $A>B$ and C will be an acute angle. So solution will exist.

Note : Above case (iv) is called ambiguous case.
