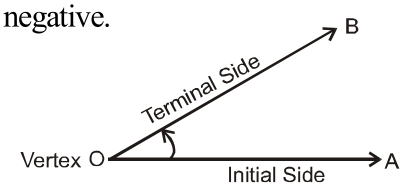
Trigonometric Ratios

TRIGONOMETRY

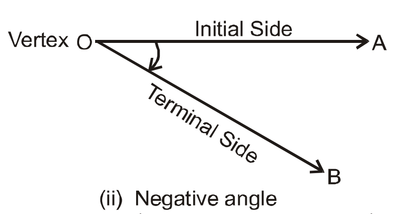
The word 'trigonometry' is derived from the Greek words 'trigon' and ' metron' and it means 'measuring the sides and angles of a triangle'.

ANGLE :

Angle is a measure of rotation of a given ray about its initial point. The original ray is called the initial side and the final position of the ray after rotation is called the terminal side of the angle. The point of rotation is called the vertex. If the direction of rotation is anticlockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is negative.



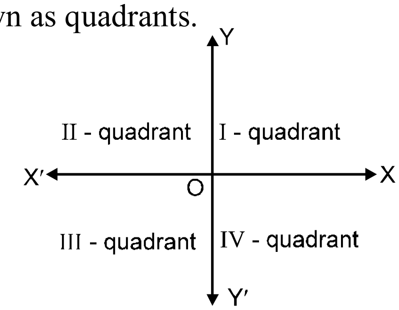
(i) Positive angle
(anticlockwise measurement)



(ii) Negative angle
(clockwise measurement)

Quadrant :

Let XOX′ and YOY′ be two lines at right angles in the plane of the paper. These lines divide the plane of the paper into four equal parts which are known as quadrants.



The lines XOX′ and YOY′ are known as x-axis and y-axis respectively. These two lines taken together are known as the coordinate axes. The regions XOY, YOX′, X′OY′ and Y′OX are known as the first, the second, the third and the fourth quadrant respectively.

SYSTEMS FOR MEASUREMENT OF ANGLES :

An angle can be measured in the following systems.

1. **Sexagesimal System (British System) :**
   The principal unit in this system is degree ($^\circ$). One right angle is divided into 90 equal parts and each part is called one degree (1°).

One degree is divided into 60 equal parts and each part is called one minute. Minute is denoted by (1'). One minute is equally divided into 60 equal parts and each part is called one second (1").

i. e. $\frac{1}{360}$ of a complete circular turn is called a degree (°), $\frac{1}{60}$ of a degree is called a minute ($'$) and $\frac{1}{60}$ of aminute is called a second ($''$).

One right angle = 90°, $1° = 60'$, $1' = 60''$

Solved Examples

Ex.1 30^o 30' is equal to -

(A) $\left(\frac{41}{2}\right)^o$ (B) 61^o

(C) $\left(\frac{61}{2}\right)^o$ (D) None of these

Sol. We know that, 30' = $\left(\frac{1}{2}\right)^o$

$30^o + \left(\frac{1}{2}\right)^o = \left(\frac{61}{2}\right)^o$

Ans. [C]

2. Centesimal System (French System) :

The principal unit in system is grade and is denoted by (^{g}). One right angle is divided into 100 equal parts, called grades, and each grade is subdivided into 100 minutes, and each minutes into 100 seconds.

i.e. $\frac{1}{400}$ of a complete circular turn is called a grade (^{g}), $\frac{1}{100}$ of a grade is called a minute (`) and $\frac{1}{100}$ of a minute is called a second (``).

$\therefore$ One right angle = 100^g; 1^g = 100` ; 1` = 100``

Note :

The minutes and seconds in the Sexagesimal system are different with the minutes and seconds respectively in the Centesimal System. Symbols in both systems are also different.

Solved Examples

Ex.2 50' is equal to -

(A) 1^g (B) $\left(\frac{1}{2}\right)^g$

(C) $\left(\frac{1}{4}\right)^g$ (D) None of these

Sol. 100' is equal to 1^g

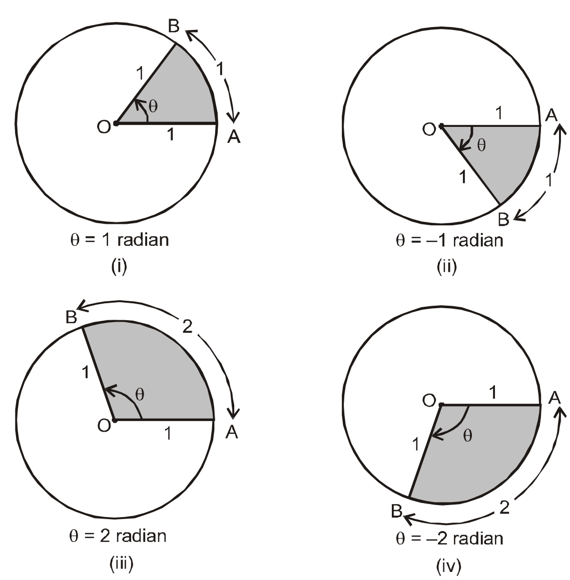
50' is equal to $\left(\frac{1}{100}\times 50\right)^g = \left(\frac{1}{2}\right)^g$ **Ans.[B]**

3. Circular System (Radian Measurement)

The angle subtended by an arc of a circle whose length is equal to the radius of the circle at the centre of the circle is called a radian. In this system the unit of measurement is radian (^{c})

As the circumference of a circle of radius 1 unit is 2π, therefore one complete revolution of the initial side subtends an angle of 2π radian.

More generally, in a circle of radius r, an arc of length r will subtend an angle of 1 radian. It is well-known that equal arcs of a circle subtend equal angle at the centre. Since in a circle of radius r, an arc of length r subtends an angle whose measure is 1 radian, an arc of length ℓ will subtend an angle whose measure is $\frac{\ell}{r}$ radian. Thus, if in a circle of radius r, arc of length ℓ subtends an angle θ radian at the centre, we have $\theta = \frac{\ell}{r}$ or $\ell = r\theta$.



θ = 1 radian (i)

θ = –1 radian (ii)

θ = 2 radian (iii)

θ = –2 radian (iv)

Some Important Conversion :

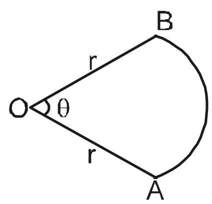
π Radian = 180^o	One radian = $\left(\frac{180}{\pi}\right)^o$
$\frac{\pi}{6}$ Radian = 30^o	$\frac{\pi}{4}$ Radian = 45^o
$\frac{\pi}{3}$ Radian = 60^o	$\frac{\pi}{2}$ Radian = 90^o
$\frac{2\pi}{3}$ Radian = 120^o	$\frac{3\pi}{4}$ Radian = 135^o
$\frac{5\pi}{6}$ Radian = 150^o	$\frac{7\pi}{6}$ Radian = 210^o
$\frac{5\pi}{4}$ Radian = 225^o	$\frac{5\pi}{3}$ Radian = 300^o

Note :

If no symbol is mentioned while showing measurement of angle, then it is considered to be measured in radians.

e.g. $\theta = 15$ implies 15 radian

Area of circular sector :



Area = $\frac{1}{2}r^2\theta$ sq. units

RELATION BETWEEN RADIAN, DEGREE AND GRADE :

$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$$

Where,

D = angle in degree

G = angle in grade

R = angle in radian

Solved Examples

Ex.3 340^o is equal to -

(A) $\left(\frac{\pi}{9}\right)^C$ (B) $\left(\frac{17\pi}{9}\right)^C$

(C) $\left(\frac{17\pi}{6}\right)^C$ (D) $\left(\frac{16\pi}{9}\right)^C$

Sol. We know , $180^o = \pi^C$

$$340^o = \left(\frac{\pi}{180}\times 340\right)^C = \left(\frac{17\pi}{9}\right)^C$$ **Ans.[B]**

Ex.4 Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15^o.

Sol. Let s be the length of the arc subtending an angle θ at the centre of a circle of radius r.

then , $\theta = \frac{s}{r}$

Here, r = 5 cm, and $\theta = 15^o = \left(15\times\frac{\pi}{180}\right)^C$

$\theta = \left(\frac{\pi}{12}\right)^C$ $\theta = \frac{s}{r} \Rightarrow \frac{\pi}{12} = \frac{s}{5}$

$s = \frac{5\pi}{12}$ cm. **Ans. [C]**

TRIGONOMETRICAL RATIOS OR FUNCTIONS

In the right angled triangle OMP , we have base (OM) = x, perpendicular (PM) = y and hypotenuse (OP) = r, then we define the following trigonometric ratios which are known as trigonometric function.

$\sin\theta = \frac{P}{H} = \frac{y}{r}$ $\cos\theta = \frac{B}{H} = \frac{x}{r}$

$\tan\theta = \frac{P}{B} = \frac{y}{x}$ $\cot\theta = \frac{B}{P} = \frac{x}{y}$

$\sec\theta = \frac{H}{B} = \frac{r}{x}$ $\text{cosec}\theta = \frac{H}{P} = \frac{r}{y}$

Note :

(1) It should be noted that $\sin\theta$ does not mean the product of sin and θ. The $\sin\theta$ is correctly read sin of angle θ.

(2) These functions depend only on the value of the angle θ and not on the position of the point P chosen on the terminal side of the angle θ.

Fundamental Trigonometrical Identities :

(a) $\sin\theta = \frac{1}{\text{cosec}\theta}$ (b) $\cos\theta = \frac{1}{\sec\theta}$

(c) $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

(d) $1 + \tan^2\theta = \sec^2\theta$ or, $\sec^2\theta - \tan^2\theta = 1$

$(\sec\theta - \tan\theta) = \frac{1}{(\sec\theta + \tan\theta)}$

(e) $\sin^2\theta + \cos^2\theta = 1$

(f) $1 + \cot^2\theta = \text{cosec}^2\theta$

$(\text{cosec}\theta - \cot\theta) = \frac{1}{\text{cosec}\theta + \cot\theta}$

Solved Examples

Ex.5 $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=$

(A) $\dfrac{1-\sin\theta}{\cos\theta}$ (B) $\dfrac{1-\cos\theta}{\sin\theta}$

(C) $\dfrac{1+\sin\theta}{\cos\theta}$ (D) $\dfrac{1+\cos\theta}{\sin\theta}$

Sol. $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\dfrac{(\tan\theta+\sec\theta)-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1}$

$[\because \sec^2\theta-\tan^2\theta=1]$

$=\dfrac{(\sec\theta+\tan\theta)\{1-(\sec\theta-\tan\theta)\}}{\tan\theta-\sec\theta+1}$

$=\dfrac{(\sec\theta+\tan\theta)(\tan\theta-\sec\theta+1)}{\tan\theta-\sec\theta+1}$

$=\sec\theta+\tan\theta=\dfrac{1}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}=\dfrac{1+\sin\theta}{\cos\theta}$ **Ans.[C]**

Ex.6 The value of the expression -

$1-\dfrac{\sin^2 y}{1+\cos y}+\dfrac{1+\cos y}{\sin y}-\dfrac{\sin y}{1-\cos y}$ is equal to-

(A) 0 (B) 1

(C) sin y (D) cos y

Sol. $1-\dfrac{\sin^2 y}{1+\cos y}+\dfrac{1+\cos y}{\sin y}-\dfrac{\sin y}{1-\cos y}$

$=\dfrac{1+\cos y-\sin^2 y}{1+\cos y}+\dfrac{1-\cos^2 y-\sin^2 y}{\sin y\,(1-\cos y)}$

$=\dfrac{\cos y+\cos^2 y}{1+\cos y}+0=\cos y$ **Ans.[D]**

Ex.7 If cosec θ – sin θ = m and sec θ – cos θ = n then $(m^2n)^{2/3}+(n^2m)^{2/3}$ equals to -

(A) 0 (B) 1

(C) –1 (D) 2

Sol. cosec θ – sin θ = m

$m=\dfrac{1}{\sin\theta}-\sin\theta=\dfrac{\cos^2\theta}{\sin\theta}$ ...(i)

$n=\dfrac{1}{\cos\theta}-\cos\theta=\dfrac{\sin^2\theta}{\cos\theta}$ ...(ii)

$m\times n=\dfrac{\cos^2\theta}{\sin\theta}\cdot\dfrac{\sin^2\theta}{\cos\theta}=\sin\theta\cos\theta$

from (i) and (ii)

from (i) $\cos^2\theta=m\,.\,\sin\theta$

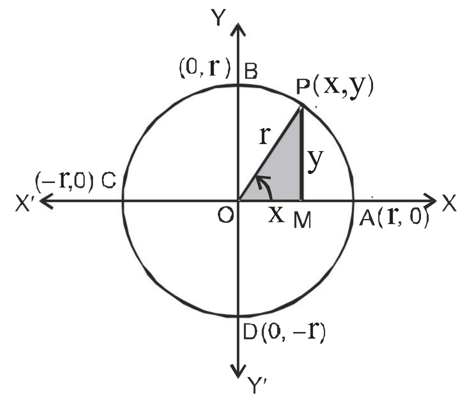
or $\cos^3\theta=m\sin\theta\cos\theta=m\,.\,(mn)=m^2n$

Similarly $\sin^3\theta=n^2m$

since $\sin^2\theta+\cos^2\theta=1$

$(n^2m)^{2/3}+(m^2n)^{2/3}=1$ **Ans.[B]**

SIGN OF THE TRIGONOMETRIC FUNCTIONS :



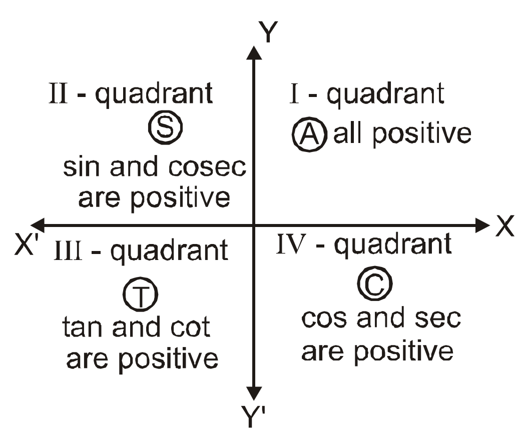
(i) If θ is in the first quadrant then P(x, y) lies in the first quadrant. Therefore x > 0, y > 0 and hence the values of all the trigonometric functions are positive.

(ii) If θ is in the II quadrant then P(x, y) lies in the II quadrant. Therefore x < 0, y > 0 and hence the values sin, cosec are positive and the remaining are negative.

(iii) If θ is in the III quadrant then P(x, y) lies in the III quadrant. Therefore x < 0, y < 0 and hence the values of tan, cot are positive and the remaining are negative.

(iv) If θ is in the IV quadrant then P(x, y) lies in the IV quadrant. Therefore x > 0, y < 0 and hence the values of cos, sec are positive and the remaining are negative.

To be Remember :



Values of trigonometric functions of certain popular angles are shown in the following table :

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	$\sqrt{\frac{0}{4}}=0$	$\sqrt{\frac{1}{4}}=\frac{1}{2}$	$\sqrt{\frac{2}{4}}=\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$	$\sqrt{\frac{4}{4}}=1$
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	N.D.

N.D. implies not defined

The values of cosec x, sec x and cot x are the reciprocal of the values of sin x, cosx and tan x, respectively.

Solved Examples

Ex.8 The values of $\sin\theta$ and $\tan\theta$ if $\cos\theta = -\frac{12}{13}$ and θ lies in the third quadrant is-

(A) $-\frac{5}{13}$ and $\frac{5}{12}$ (B) $\frac{5}{12}$ and $-\frac{5}{13}$

(C) $-\frac{12}{13}$ and $-\frac{5}{13}$ (D) None of these

Sol. We have $\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \sin\theta = \pm\sqrt{1-\cos^2\theta}$

In the third quadrant $\sin\theta$ is negative, therefore

$\sin\theta = -\sqrt{1-\cos^2\theta}$

$\Rightarrow \sin\theta = -\sqrt{1-\left(-\frac{12}{13}\right)^2} = -\frac{5}{13}$

then, $\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = -\frac{5}{13}\times\frac{13}{-12} = \frac{5}{12}$

Ans.[A]

Ex.9 If $\sec\theta = \sqrt{2}$, and $\frac{3\pi}{2} < \theta < 2\pi$. Then the value of $\frac{1+\tan\theta+\text{cosec}\theta}{1+\cot\theta-\text{cosec}\theta}$ is-

(A) – 1 (B) $\pm\frac{1}{\sqrt{2}}$

(C) $-\sqrt{2}$ (D) 1

Sol. If $\sec\theta = \sqrt{2}$ or, $\cos\theta = \frac{1}{\sqrt{2}}$,

$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

But θ lies in the fourth quadrant in which $\sin\theta$ is negative.

$\sin\theta = -\frac{1}{\sqrt{2}}$, $\text{cosec}\ \theta = -\sqrt{2}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = -\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{1}$

$\Rightarrow \tan\theta = -1 \quad \Rightarrow \cot\theta = -1$

then, $\frac{1+\tan\theta+\text{cosec}\theta}{1+\cot\theta-\text{cosec}\theta} = \frac{1-1-\sqrt{2}}{1-1+\sqrt{2}} \Rightarrow -1$

Ans. [A]

TRIGONOMETRIC RATIOS OF ALLIED ANGLES :

If θ is any angle, then $-\theta$, $\frac{\pi}{2}\pm\theta$, $\pi\pm\theta$, $\frac{3\pi}{2}\pm\theta$, $2\pi\pm\theta$ etc. are called allied angles.

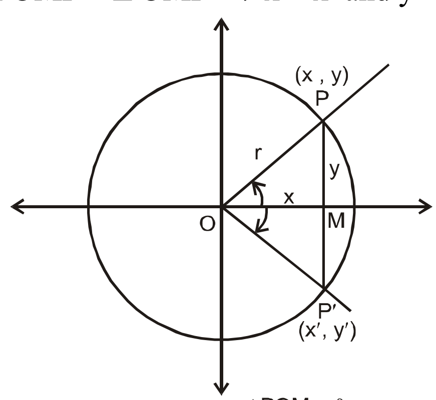
* **Trigonometric Ratios of** $(-\theta)$:

Let θ be an angle in the standard position in the I quadrant. Let its terminal side cuts the circle with centre ‘O’ and radius r in P (x, y).

Let P′ (x′, y′) be the point of intersection of the terminal side of the angle $-\theta$ in the standard position with the circle.

Now $\angle MOP = \angle MOP'$ (numerically) and P & P′ have the same projection M in the x - axis

$\therefore\ \Delta OMP \equiv \Delta OMP' \Rightarrow x = x'$ and $y = -y'$



$\angle POM = \theta$

$\angle MOP' = -\theta$

$\therefore\ \sin(-\theta) = \frac{y'}{r} = \frac{-y}{r} = -\sin\theta$.

$\cos(-\theta) = \frac{x'}{r} = \frac{x}{r} = \cos\theta$.

$\tan(-\theta) = \frac{y'}{x'} = -\frac{y}{x} = -\tan\theta.$

$\cot(-\theta) = \frac{x'}{y'} = \frac{x}{-y} = -\cot\theta.$

$\sec(-\theta) = \frac{r}{x'} = \frac{r}{x} = \sec\theta.$

$\text{cosec}(-\theta) = \frac{r}{y'} = \frac{r}{-y} = -\text{cosec}\,\theta.$

Similarly if θ is in the other quadrants then the above results can also be proved.

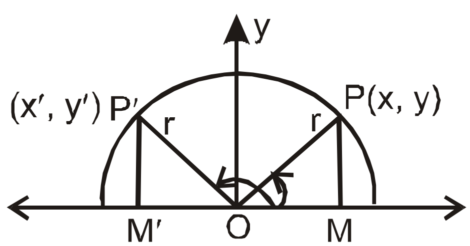
* **Trigonometric Ratios of π – θ**

Let θ be an angle in the standard position in the I quadrant. Let its terminal side cuts the circle with centre 'O' and radius r at P (x, y). Let P′ (x′, y′) be the point of intersection of the terminal side of the angle π – θ with the circle. Let M and M′ be the projections of P and P′ respectively in the x-axis.

Since Δ OM′P′ ≡ Δ OMP, x′ = – x, y′ = y

$\therefore \sin(\pi-\theta) = \frac{y'}{r} = \frac{y}{r} = \sin\theta.$

$\cos(\pi-\theta) = \frac{x'}{r} = -\frac{x}{r} = -\cos\theta.$



∠MOP = θ
∠MOP′ = π – θ

$\tan(\pi-\theta) = \frac{y'}{x'} = \frac{y}{-x} = -\tan\theta.$

$\cot(\pi-\theta) = \frac{x'}{y'} = -\frac{x}{y} = -\cot\theta.$

$\sec(\pi-\theta) = \frac{r}{x'} = \frac{r}{-x} = -\sec\theta.$

$\text{cosec}(\pi-\theta) = \frac{r}{y'} = \frac{r}{y} = \text{cosec}\,\theta.$

* **Trigonometric Ratios of $\left(\frac{\pi}{2}-\theta\right)$:**

Similarly we can easily prove the following results.

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$, $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$,

$\text{cosec}\left(\frac{\pi}{2}-\theta\right) = \sec\theta$, $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$,

$\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$, $\sec\left(\frac{\pi}{2}-\theta\right) = \text{cosec}\,\theta$

* **Trigonometric Ratios of $\left(\frac{\pi}{2}+\theta\right)$**

Similarly we can easily prove the following results.

$\sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$, $\tan\left(\frac{\pi}{2}+\theta\right) = -\cot\theta$,

$\text{cosec}\left(\frac{\pi}{2}+\theta\right) = \sec\theta$, $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$,

$\cot\left(\frac{\pi}{2}+\theta\right) = -\tan\theta$, $\sec\left(\frac{\pi}{2}+\theta\right) = -\text{cosec}\,\theta$

* **Trigonometric Ratios of (π + θ)**

Similarly we can easily prove the following results.

$\sin(\pi+\theta) = -\sin\theta$, $\tan(\pi+\theta) = \tan\theta$,

$\text{cosec}(\pi+\theta) = -\text{cosec}\,\theta$, $\cos(\pi+\theta) = -\cos\theta$,

$\cot(\pi+\theta) = \cot\theta$, $\sec(\pi+\theta) = -\sec\theta$

* **Trigonometric Ratios of $\left(\frac{3\pi}{2}-\theta\right)$**

Similarly we can easily prove the following results.

$\sin\left(\frac{3\pi}{2}-\theta\right) = -\cos\theta$, $\tan\left(\frac{3\pi}{2}-\theta\right) = \cot\theta$,

$\text{cosec}\left(\frac{3\pi}{2}-\theta\right) = -\sec\theta$ $\cos\left(\frac{3\pi}{2}-\theta\right) = -\sin\theta$,

$\cot\left(\frac{3\pi}{2}-\theta\right) = \tan\theta$, $\sec\left(\frac{3\pi}{2}-\theta\right) = -\text{cosec}\,\theta$,

* **Trigonometric Ratios of $\left(\frac{3\pi}{2}+\theta\right)$**

Similarly we can easily prove the following results.

$\sin\left(\frac{3\pi}{2}+\theta\right) = -\cos\theta$, $\cos\left(\frac{3\pi}{2}+\theta\right) = \sin\theta$,

$\tan\left(\frac{3\pi}{2}+\theta\right) = -\cot\theta$, $\cot\left(\frac{3\pi}{2}+\theta\right) = -\tan\theta$,

$\sec\left(\frac{3\pi}{2}+\theta\right) = \text{cosec}\,\theta$, $\text{cosec}\left(\frac{3\pi}{2}+\theta\right) = -\sec\theta$

Think, and fill up the blank blocks in following table.

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1									
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0									
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$N.D.$									

Solved Examples

Ex.10 $\sin 315^\circ =$

(A) $\frac{1}{\sqrt{2}}$ (B) $-\frac{1}{\sqrt{2}}$

(C) $\frac{1}{2}$ (D) None of these

Sol. $\sin 315^\circ = \sin(270^\circ + 45^\circ)$

$= -\cos 45^\circ \quad [\because \sin(270^\circ + \theta) = -\cos\theta]$

$= -\frac{1}{\sqrt{2}}$ **Ans.[B]**

Ex.11 $\cos 510^\circ \cos 330^\circ + \sin 390^\circ \cos 120^\circ =$

(A) 2 (B) – 1

(C) 0 (D) $\frac{1}{\sqrt{2}}$

Sol. $\cos 510^\circ \cos 330^\circ + \sin 390^\circ \cos 120^\circ$

$= \cos(360^\circ + 150^\circ)\cos(360^\circ - 30^\circ) + \sin(360^\circ + 30^\circ)\cos(90^\circ + 30^\circ)$

$= \cos 150^\circ \cos 30^\circ + \sin 30^\circ(-\sin 30^\circ)$

$= \cos(180^\circ - 30^\circ)\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{4}$

$= -\cos 30^\circ\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{4}$

$= -\frac{3}{4} - \frac{1}{4} = -1$ **Ans.[B]**

Ex.12 $\frac{\operatorname{cosec}(2\pi+\theta).\cos(2\pi+\theta)\tan(\pi/2+\theta)}{\sec(\pi/2+\theta).\cos\theta.\cot(\pi+\theta)} =$

(A) 2 (B) – 1

(C) 4 (D) 1

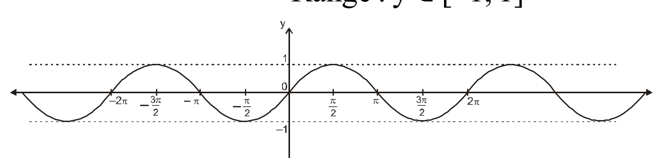
Sol. $\frac{\operatorname{cosec}(2\pi+\theta).\cos(2\pi+\theta)\tan(\pi/2+\theta)}{\sec(\pi/2+\theta).\cos\theta.\cot(\pi+\theta)}$

$= \frac{\operatorname{cosec}\theta\,.\cos\theta\,(-\cot\theta)}{(-\operatorname{cosec}\theta).\cos\theta.\cot\theta} = 1$ **Ans.[D]**

TRIGONOMETRIC FUNCTIONS:

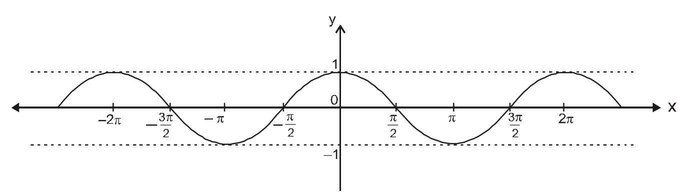
(i) y = sin x Domain : $x \in R$

Range : $y \in [-1, 1]$



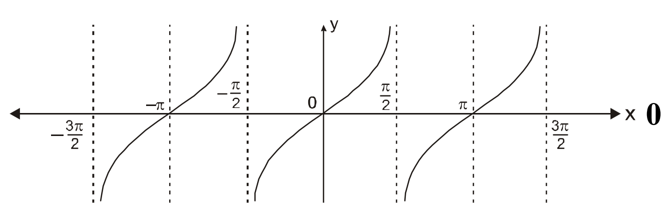
(ii) y = cos x Domain : $x \in R$

Range : $y \in [-1, 1]$



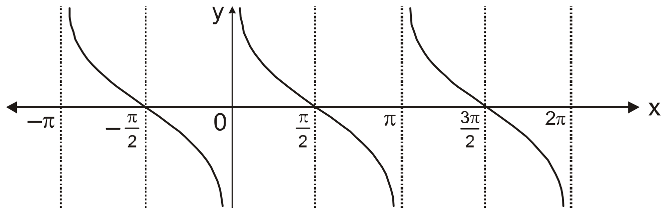
(iii) y = tan x Domai : $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}, n \in I$

Range : $y \in R$



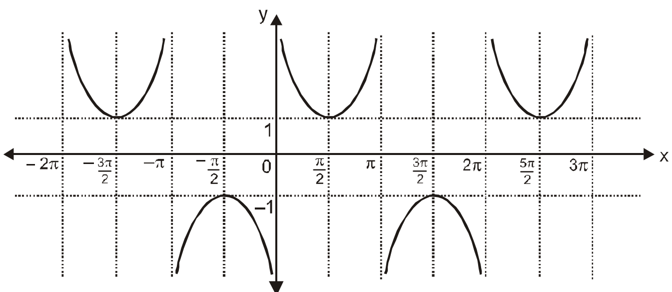
(iv) y = cot x Domain : $x \in R - \{n\pi\},\ n \in I$

Range : $y \in R$



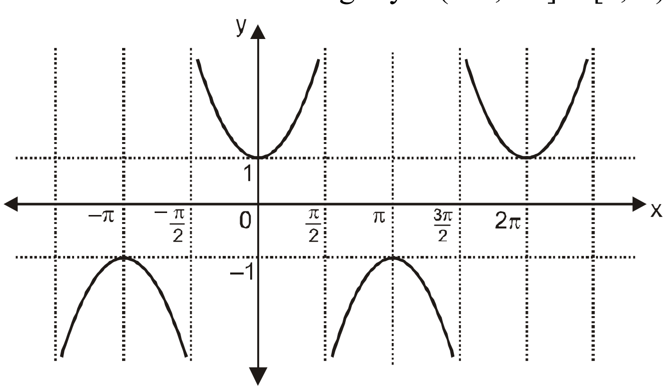
(v) y = cosec x Domain : $x \in R - \{n\pi\},\ n \in I$

Range : $y \in (-\infty, -1] \cup [1, \infty)$



(vi) y = sec x Domain : $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}, n \in I$

Range : $y \in (-\infty, -1] \cup [1, \infty)$



TRIGONOMETRIC FUNCTIONS OF SUM OR DIFFERENCE OF TWO ANGLES :

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B).\sin(A-B)$

(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B).\cos(A-B)$

(e) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(f) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

(g) $\sin(A+B+C) = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C$

(h) $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$

(i) $\tan(A+B+C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.$$

(j) $\tan(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n) = \dfrac{S_1 - S_3 + S_5 - ......}{1 - S_2 + S_4 - ....}$

where S_i denotes sum of product of tangent of angles taken i at a time

Solved Examples

Ex.13 Prove that

(i) $\sin(45^o + A)\cos(45^o - B) + \cos(45^o + A)\sin(45^o - B) = \cos(A - B)$

(ii) $\tan\left(\frac{\pi}{4} + \theta\right)\tan\left(\frac{3\pi}{4} + \theta\right) = -1$

Sol. (i) Clearly $\sin(45^o + A)\cos(45^o - B) + \cos(45^o + A)\sin(45^o - B)$

$= \sin(45^o + A + 45^o - B) = \sin(90^o + A - B)$

$= \cos(A - B)$

(ii) $\tan\left(\frac{\pi}{4} + \theta\right) \times \tan\left(\frac{3\pi}{4} + \theta\right)$

$= \dfrac{1 + \tan\theta}{1 - \tan\theta} \times \dfrac{-1 + \tan\theta}{1 + \tan\theta} = -1$

FORMULAE FOR PRODUCT INTO SUM OR DIFFERENCE CONVERSION

We know that,

$\sin A \cos B + \cos A \sin B = \sin(A + B)$ ......(i)

$\sin A \cos B - \cos A \sin B = \sin(A - B)$ ......(ii)

$\cos A \cos B - \sin A \sin B = \cos(A + B)$ .....(iii)

$\cos A \cos B + \sin A \sin B = \cos(A - B)$ .....(iv)

Adding (i) and (ii),

2 sin A cos B = sin (A + B) + sin (A – B)

Subtracting (ii) from (i),

2 cos A sin B = sin (A + B) – sin (A – B)

Adding (iii) and (iv),

2 cosA cos B = cos (A + B) + cos (A – B)

Subtraction (iii) from (iv).

2 sin A sin B = cos (A – B) – cos (A + B)

Formulae :

(a) 2 sin A cos B = sin (A + B) + sin (A – B)

(b) 2 cos A sin B = sin (A + B) – sin (A – B)

(c) 2 cos A cos B = cos (A + B) + cos (A – B)

(d) 2 sin A sin B = cos (A – B) – cos (A + B)

FORMULAE FOR SUM OR DIFFE RENCE INTO PRODUCT CONVERSION

We know that,

sin (A + B) + sin(A – B) = 2 sin A cos B ......(i)

Let A + B = C and A – B = D

then A = $\frac{C+D}{2}$ and B = $\frac{C-D}{2}$

Substituting in (i),

(a) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right).\cos\left(\frac{C-D}{2}\right)$

similarly other formula are,

(b) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right).\sin\left(\frac{C-D}{2}\right)$

(c) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right).\cos\left(\frac{C-D}{2}\right)$

* (d) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right).\sin\left(\frac{D-C}{2}\right)$

Formulae for sum or difference into product conversion

Solved Examples

Ex.14 Prove that sin 5A + sin 3A = 2sin 4A cos A

Sol. L.H.S. sin 5A + sin 3A = 2sin 4A cos A = R.H.S.

[$\because \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$]

Ex.15 Find the value of 2 sin 3θ cos θ – sin 4θ – sin 2θ

Sol. 2 sin 3θ cos θ – sin 4θ – sin 2θ = 2 sin 3θ cos θ – [2 sin 3θ cos θ] = 0

Ex.16 Prove that

(i) $\frac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta} = \tan 2\theta$

(ii) $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4\cos 2\theta\cos 4\theta$

Sol. (i) $\frac{2\sin 8\theta\cos\theta - 2\sin 6\theta\cos 3\theta}{2\cos 2\theta\cos\theta - 2\sin 3\theta\sin 4\theta}$

$= \frac{\sin 9\theta + \sin 7\theta - \sin 9\theta - \sin 3\theta}{\cos 3\theta + \cos\theta - \cos\theta + \cos 7\theta}$

$= \frac{2\sin 2\theta\cos 5\theta}{2\cos 5\theta\cos 2\theta} = \tan 2\theta$

(ii) $\frac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = \frac{\sin 5\theta\cos 3\theta + \sin 3\theta\cos 5\theta}{\sin 5\theta\cos 3\theta - \sin 3\theta\cos 5\theta}$

$= \frac{\sin 8\theta}{\sin 2\theta} = 4\cos 2\theta\cos 4\theta$

Ex.17 $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 =$

(A) $4\sin^2\left(\frac{\alpha+\beta}{2}\right)$ (B) $4\cos^2\left(\frac{\alpha+\beta}{2}\right)$

(C) $4\sin^2\left(\frac{\alpha-\beta}{2}\right)$ (D) $4\cos^2\left(\frac{\alpha-\beta}{2}\right)$

Sol. $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$

$= \left[2\cos\left(\frac{\alpha+\beta}{2}\right).\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2 + \left[2\sin\left(\frac{\alpha+\beta}{2}\right).\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2$

$= 4\cos^2\left(\frac{\alpha+\beta}{2}\right).\cos^2\left(\frac{\alpha-\beta}{2}\right)$

$+ 4\sin^2\left(\frac{\alpha+\beta}{2}\right).\cos^2\left(\frac{\alpha-\beta}{2}\right)$

$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right).\left[\cos^2\left(\frac{\alpha+\beta}{2}\right) + \sin^2\left(\frac{\alpha+\beta}{2}\right)\right]$

$= 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$ **Ans.[D]**

MULTIPLE AND SUB-MULTIPLE ANGLES :

(i) $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(iii) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

(iv) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

(v) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(vi) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(vii) $\sin\theta/2 = \sqrt{\dfrac{1-\cos\theta}{2}}$

(viii) $\cos\theta/2 = \sqrt{\dfrac{1+\cos\theta}{2}}$

(ix) $\tan\theta/2 = \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$

Solved Examples

Ex.18 Prove that

(i) $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$

(ii) $\tan A + \cot A = 2\,\mathrm{cosec}\,2A$

(iii) $\dfrac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)} = \tan\dfrac{A}{2}\cot\dfrac{B}{2}$

Sol. (i) L.H.S. $\dfrac{\sin 2A}{1+\cos 2A} = \dfrac{2\sin A\cos A}{2\cos^2 A} = \tan A$

(ii) L.H.S. $\tan A + \cot A = \dfrac{1+\tan^2 A}{\tan A} = 2$

$\left(\dfrac{1+\tan^2 A}{2\tan A}\right) = \dfrac{2}{\sin 2A} = 2\,\mathrm{cosec}\,2A$

(iii) L.H.S. $\dfrac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)}$

$$= \dfrac{2\sin^2\frac{A}{2} + 2\sin\frac{A}{2}\sin\left(\frac{A}{2}+B\right)}{2\cos^2\frac{A}{2} - 2\cos\frac{A}{2}\cos\left(\frac{A}{2}+B\right)}$$

$$= \tan\frac{A}{2}\left[\dfrac{\sin\frac{A}{2}+\sin\left(\frac{A}{2}+B\right)}{\cos\frac{A}{2}-\cos\left(\frac{A}{2}+B\right)}\right]$$

$$= \tan\frac{A}{2}\left[\dfrac{2\sin\frac{A+B}{2}\cos\left(\frac{B}{2}\right)}{2\sin\frac{A+B}{2}\sin\left(\frac{B}{2}\right)}\right] = \tan\frac{A}{2}\cot\frac{B}{2}$$

Ex.19 $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} =$

(A) $\cot\left(\dfrac{\theta}{2}\right)$ (B) $\sin\left(\dfrac{\theta}{2}\right)$

(C) $\cos\left(\dfrac{\theta}{2}\right)$ (D) $\tan\left(\dfrac{\theta}{2}\right)$

Sol. $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}$

$= \dfrac{(1-\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta}$

$$= \dfrac{2\sin^2\left(\frac{\theta}{2}\right) + 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right) + 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$= \dfrac{2\sin\left(\frac{\theta}{2}\right)\left[2\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right]}{2\cos\left(\frac{\theta}{2}\right)\left[2\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right]} = \tan\left(\frac{\theta}{2}\right)$$ **Ans.[D]**

Ex.20 The value of $\left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\dfrac{5\pi}{8}\right)\left(1+\cos\dfrac{7\pi}{8}\right)$ is -

(A) $\dfrac{1}{2}$ (B) $\cos\dfrac{\pi}{8}$

(C) $\dfrac{1}{8}$ (D) $\dfrac{1+\sqrt{2}}{2\sqrt{2}}$

Sol. $\left(1+\cos\dfrac{3\pi}{8}\right)\left(1+\cos\left(\pi-\dfrac{3\pi}{8}\right)\right)\left(1+\cos\left(\pi-\dfrac{\pi}{8}\right)\right)$

$= \left(1+\cos\dfrac{\pi}{8}\right)\left(1+\cos\dfrac{3\pi}{8}\right)\left(1-\cos\dfrac{3\pi}{8}\right)\left(1-\cos\dfrac{\pi}{8}\right)$

$= \left(1-\cos^2\dfrac{\pi}{8}\right)\left(1-\cos^2\dfrac{3\pi}{8}\right)$

$= \dfrac{1}{4}\left(2-1-\cos\dfrac{\pi}{4}\right)\left(2-1-\cos\dfrac{3\pi}{4}\right)$

$= \dfrac{1}{4}\left(1-\cos\dfrac{\pi}{4}\right)\left(1-\cos\dfrac{3\pi}{4}\right)$

$= \dfrac{1}{4}\left(1-\dfrac{1}{\sqrt{2}}\right)\left(1+\dfrac{1}{\sqrt{2}}\right) = \dfrac{1}{4}\left(1-\dfrac{1}{2}\right) = \dfrac{1}{8}$

Ans.[C]

Ex.21 The value of sin 20° sin 40° sin 60° sin 80° is-

(A) $\frac{3}{8}$ (B) $\frac{1}{8}$

(C) $\frac{3}{16}$ (D) None of these

Sol. sin 20° sin 40° sin 60° sin 80°

$= \frac{\sqrt{3}}{2}\sin 20° \sin(60° - 20°)\sin(60° + 20°)$

$= \frac{\sqrt{3}}{2}\sin 20° (\sin^2 60° - \sin^2 20°)$

$= \frac{\sqrt{3}}{2}\sin 20° (\frac{3}{4} - \sin^2 20°)$

$= \frac{\sqrt{3}}{8}(3\sin 20° - 4\sin^3 20°)$

$= \frac{\sqrt{3}}{8}\sin 60°$

$= \frac{\sqrt{3}}{8} \cdot \frac{\sqrt{3}}{2} = \frac{3}{16}$ **Ans.[C]**

Alternate : By direct formula

$\sin\theta.\sin(60^o - \theta).\sin(60^o + \theta) = \frac{1}{4}\sin 3\theta$

$\Rightarrow \sin 60^o [\sin 20^o \sin(60^o - 20^o)$

$\sin(60^o + 20^o)]$

$= \sin 60^o \left[\frac{1}{4}\sin 60^o\right] = \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{16}$

Ex.22 $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$ equals to -

(A) 1/2 (B) 1/4 (C) 3/2 (D) 3/4

Sol. $= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$

$= \cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{\pi}{8}$

$= 2\left(\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8}\right)$

$= \frac{1}{2}\left[\left(2\cos^2\frac{\pi}{8}\right)^2 + \left(2\cos^2\frac{3\pi}{8}\right)^2\right]$

$= \frac{1}{2}\left[\left(1+\cos\frac{\pi}{4}\right)^2 + \left(1+\cos\frac{3\pi}{4}\right)^2\right]$

$= \frac{1}{2}\left[\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right] = \frac{1}{2}[2+1] = \frac{3}{2}$

Ans.[C]

CONDITIONAL TRIGONOMETRICAL IDENTITIES

We have certain trigonometric identities like, $\sin^2\theta + \cos^2\theta = 1$ and $1 + \tan^2\theta = \sec^2\theta$ etc.

Such identities are identities in the sense that they hold for all value of the angles which satisfy the given condition among them and they are called conditional identities.

If A, B, C denote the angle of a triangle ABC, then the relation A + B + C = π enables us to establish many important identities involving trigonometric ratios of these angles.

(I) If A + B + C = π, then A + B = π – C, B + C = π – A and C + A = π – B

(II) If A + B + C = π, then sin(A + B) = sin(π – C) = sinC

similarly, sin (B + C) = sin(π – A) = sin A and sin (C + A) = sin (π – B) = sin B

(III) If A + B + C = π, then cos(A + B) = cos(π – C) = – cos C

similarly, cos (B + C) = cos(π – A) = – cosA and cos (C + A) = cos (π – B) = – cos B

(IV) If A + B + C = π, then tan(A + B) = tan(π – C) = – tan C

similarly, tan (B + C) = tan(π – A) = – tanA and, tan (C + A) = tan (π – B) = – tan B

(V) If A + B + C = π, then $\frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$ and $\frac{B+C}{2} = \frac{\pi}{2} - \frac{A}{2}$ and $\frac{C+A}{2} = \frac{\pi}{2} - \frac{B}{2}$

$\sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\left(\frac{C}{2}\right)$

$\cos\left(\frac{A+B}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{C}{2}\right) = \sin\left(\frac{C}{2}\right)$

$\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\left(\frac{C}{2}\right)$

All problems on conditional identities are broadly divided into the following four types:

(I) Identities involving sines and cosines of the multiple or sub-multiples of the angles involved.

(II) Identities involving squares of sines and cosines of the multiple or sub-multiples of the angles involved.

(III) Identities involving tangents and cotangents of the multiples or sub-multiples of the angles involved.

(IV) Identities involving cubes and higher powers of sines and cosines and some mixed identities.

1. **TYPE I : Identities involving sines and cosines of the multiple or sub-multiple of the angles involved.**

Working Method :

Step – 1 Use C & D formulae.

Step – 2 Use the given relation ($A + B + C = \pi$) in the expression obtained in step -1 such that a factor can be taken common after using multiple angles formulae in the remaining term.

Step – 3 Take the common factor outside.

Step – 4 Again use the given relation ($A+B+C=\pi$) within the bracket in such a manner so that we can apply C & D formulae.

Step – 5 Find the result according to the given options.

2. **TYPE II : Identities involving squares of sines and cosines of multiple or sub-multiples of the angles involved.**

Working Method :

Step – 1 Arrange the terms of the identity such that either $\sin^2A - \sin^2B = \sin(A+B).\sin(A-B)$ or $\cos^2A - \sin^2B = \cos(A+B).\cos(A-B)$ can be used.

Step – 2 Take the common factor outside.

Step – 3 Use the given relation ($A + B + C = \pi$) within the bracket in such a manner so that we can apply C & D formulae.

Step – 4 Find the result according to the given options.

3. **Type III : Identities for tan and cot of the angles**

Working Method :

Step – 1 Express the sum of the two angles in terms of third angle by using the given relation ($A + B + C = \pi$).

Step – 2 Taking tangent or cotangent of the angles of both the sides.

Step – 3 Use sum and difference formulae in the left hand side.

Step – 4 Use cross multiplication in the expression obtained in the step 3

Step – 5 Arrange the terms as per the result required.

Conditional trigonometrical identities

Solved Examples

Ex.23 If $A + B + C = \pi$, then

$\sin 2A + \sin 2B + \sin 2C =$

(A) $4\sin A \sin B \cos C$.

(B) $4\sin A \sin B \sin C$.

(C) $4\cos A \sin B \sin C$.

(D) None of these

Sol. $\sin 2A + \sin 2B + \sin 2C$

$= 2\sin\left(\frac{2A+2B}{2}\right).\cos\left(\frac{2A-2B}{2}\right) + \sin 2C$

$= 2\sin(A + B).\cos(A - B) + \sin 2C$

$= 2\sin(\pi - C).\cos(A - B) + \sin 2C$

$[\because A + B + C = \pi, A + B = \pi - C$

$\therefore \sin(A + B) = \sin(\pi - C) = \sin C]$

$= 2\sin C \cos(A - B) + 2\sin C \cos C$

$= 2\sin C[\cos(A - B) + \cos C]$

$= 2\sin C[\cos(A - B) - \cos(A + B)]$

$[\because \cos(A - B) - \cos(A + B) = 2\sin A.\sin B$, By C & D formula]

$= 2\sin C[2\sin A \sin B]$

$= 4\sin A \sin B \sin C$

Ans.[B]

Ex.24 If A + B + C = π, then tanA + tanB + tanC =

(A) cotA.tanB.tanC (B) tanA.cotB.tanC

(C) tanA.tanB.tanC (D) None of these

Sol. $A + B + C = \pi$ $A + B = \pi - C$

$\Rightarrow \tan(A + B) = \tan(\pi - C)$

$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

$\Rightarrow$ tanA + tanB = – tanC + tanA.tanB.tanC

tanA + tanB + tanC = tanA.tanB.tanC

Ans.[C]

Ex.25 If $A + B + C = \frac{3\pi}{2}$, then

cos 2A + cos 2B + cos 2C =

(A) 1 – 4 cos A cos B cos C

(B) 4 sin A sin B sin C

(C) 1 + 2 cos A cos B cos C

(D) 1 – 4 sin A sin B sin C

Sol. cos 2A + cos 2B + cos 2C

$= 2\cos(A + B)\cos(A - B) + \cos 2C$

$= 2\cos\left(\frac{3\pi}{2} - C\right)\cos(A - B) + \cos 2C$

$\because A + B + C = \frac{3\pi}{2}$

$= -2\sin C\cos(A - B) + 1 - 2\sin^2 C$

$= 1 - 2\sin C\,[\cos(A - B) + \sin C)$

$= 1 - 2\sin C\left[\cos(A-B) + \sin\left(\frac{3\pi}{2} - (A+B)\right)\right]$

$= 1 - 2\sin C\,[\cos(A - B) - \cos(A + B)]$

$= 1 - 4\sin A\sin B\sin C$ **Ans.[D]**

Ex.26 In any triangle ABC, sin A – cos B = cos C, then angle B is -

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

Sol. We have, sin A – cos B = cos C

sin A = cos B + cos C

$2\sin\frac{A}{2}\cos\frac{A}{2} = 2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$

$2\sin\frac{A}{2}\cos\frac{A}{2} = 2\cos\left(\frac{\pi - A}{2}\right)\cos\left(\frac{B-C}{2}\right)$

$\because A + B + C = \pi$

$2\sin\frac{A}{2}\cos\frac{A}{2} = 2\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$

$\cos\frac{A}{2} = \cos\frac{B-C}{2}$

or A = B – C

But A + B + C = π

Therefore $2B = \pi \Rightarrow B = \pi/2$

Ans.[A]

THE GREATEST AND LEAST VALUE OF THE EXPRESSION [a sinθ + b cosθ]

Let $a = r\cos\alpha$ ...(1)

and $b = r\sin\alpha$ ...(2)

Squaring and adding (1) and (2)

then $a^2 + b^2 = r^2$ or, $r = \sqrt{a^2 + b^2}$

$\therefore$ $a\sin\theta + b\cos\theta$

$= r(\sin\theta\cos\alpha + \cos\theta\sin\alpha) = r\sin(\theta + \alpha)$

But $-1 \le \sin\theta \le 1$

so $-1 \le \sin(\theta + \alpha) \le 1$

then $-r \le r\sin(\theta + \alpha) \le r$

hence,

$-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$ then the greatest and least values of $a\sin\theta + b\cos\theta$ are respectively $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$

The greatest and least value of the expression [a sinθ + b cosθ]

Solved Examples

Ex.27 The maximum value of 3 sinθ + 4 cosθ is-

(A) 2 (B) 3

(C) 4 (D) 5

Sol. $-\sqrt{25} \le 3\sin\theta + 4\cos\theta \le \sqrt{25}$

[By the standard results]

or, $-5 \le 3\sin\theta + 4\cos\theta \le 5$

so the maximum value is 5.

Ans.[D]

MISCELLANEOUS POINTS

(A) Some useful Identities :

(a) $\tan(A+B+C) = \dfrac{\sum \tan A - \tan A \tan B \tan C}{1 - \sum \tan A . \tan B}$

(b) $\tan\theta = \cot\theta - 2\cot 2\theta$

(c) $\tan 3\theta = \tan\theta . \tan(60^\circ - \theta) . \tan(60^\circ + \theta)$

(d) $\tan(A+B) - \tan A - \tan B$

$= \tan A . \tan B . \tan(A+B)$

(e) $\sin\theta \sin(60^\circ - \theta)\sin(60^\circ + \theta) = \dfrac{1}{4}\sin 3\theta$

(f) $\cos\theta\cos(60^\circ - \theta)\cos(60^\circ + \theta) = \dfrac{1}{4}\cos 3\theta$

(B) Some useful series :

(a) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha + 2\beta) + ... +$ to n terms

$$= \frac{\sin\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\left[\sin\left(\frac{n\beta}{2}\right)\right]}{\sin\left(\frac{\beta}{2}\right)} \; ; \beta \neq 2n\pi$$

(b) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta)$ +..... + to n terms

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\left[\sin\left(\frac{n\beta}{2}\right)\right]}{\sin\left(\frac{\beta}{2}\right)} \; ; \beta \neq 2n\pi$$

Series

Solved Examples

Ex.28 $\cos\left(\frac{\pi}{14}\right) + \cos\left(\frac{3\pi}{14}\right) + \cos\left(\frac{5\pi}{14}\right) =$

(A) $\frac{1}{2}\tan\left(\frac{\pi}{14}\right)$ (B) $\frac{1}{2}\cos\left(\frac{\pi}{14}\right)$

(C) $\frac{1}{2}\cot\left(\frac{\pi}{14}\right)$ (D) None of these

Sol. Here $\alpha = \frac{\pi}{14}$, $\beta = \frac{2\pi}{14}$ and n = 3.

$$S = \frac{\cos\left[\frac{\pi}{14} + \left(\frac{3-1}{2}\right)\left(\frac{2\pi}{14}\right)\right]\sin\left(\frac{2\pi}{14}\times\frac{3}{2}\right)}{\sin\left(\frac{2\pi}{14}\times\frac{1}{2}\right)}$$

$$= \frac{2\cos\left(\frac{3\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)}{2\sin\left(\frac{\pi}{14}\right)} \qquad S = \frac{\sin\left(\frac{6\pi}{14}\right)}{2\sin\left(\frac{\pi}{14}\right)}$$

$$= \frac{\frac{1}{2}\sin\left(\frac{\pi}{2} - \frac{\pi}{14}\right)}{\sin\left(\frac{\pi}{14}\right)} = \frac{1}{2}\cot\left(\frac{\pi}{14}\right)$$ **Ans.[C]**

(C) Sine, cosine and tangent of some angle less than 90º.

	15º	18º	22½º	36º
sin	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}\sqrt{2-\sqrt{2}}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
cos	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{1}{2}\sqrt{2+\sqrt{2}}$	$\frac{\sqrt{5}+1}{4}$
tan	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\sqrt{2}-1$	$\sqrt{5-2\sqrt{5}}$

(D) Domain and Range of Trigonometrical Function

Trig. Function	Domain	Range
sin θ	R	[–1, 1]
cos θ	R	[–1, 1]
tan θ	R – {2n + 1) π/2, n ∈ z}	(∞, ∞) or R
cosec θ	R – {nπ, n ∈ z}	(–∞, –1] ∪ [1, ∞)
sec θ	R – ({2n + 1) π/2, n ∈ z}	(–∞, –1] ∪ [1, ∞)
cot θ	R – {nπ, n ∈ z}	(–∞, ∞) = R

(E) An Increasing Product series :

$p = \cos\alpha . \cos 2\alpha . \cos 2^2\alpha ..... \cos(2^{n-1}\alpha) =$

$$\begin{cases} \dfrac{\sin 2^n\alpha}{2^n \sin\alpha}, \text{if } \alpha \neq n\pi \\ 1, \text{ if } \alpha = 2k\pi \\ -1, \text{if } \alpha = (2k+1)\pi \end{cases}$$

(F) Continued sum of sine & cosine series :

(i) $\sin(\alpha) + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + ...$

$\sin[\alpha + (n-1)\beta]$

$$= \frac{\sin\left(\frac{n}{2}\times\text{difference}\right)\sin\left(\frac{1^{st}\angle + last\angle}{2}\right)}{\sin\left(\frac{\text{difference}}{2}\right)}$$

(ii) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ...$

$+ \cos[\alpha + (n-1)\beta]$

$$= \frac{\sin\left(\frac{n}{2}\times \text{difference}\right)}{\sin\left(\frac{\text{difference}}{2}\right)} \cos\left(\frac{1^{st}\angle + \text{last}\angle}{2}\right)$$

(F) Conversion 1 radian = 180º/π = 57º 17′ 45″

and $1^o = \frac{\pi}{180} = 0.01475$ radians (approximately)

(G) Basic right angled triangle are (pythogerian Triplets)

3, 4, 5 ; 5, 12, 13; 7, 24, 25; 8, 15, 17;9,40, 41;11, 60, 61; 12, 35, 37; 20, 21, 29 etc.

(H) Each interior angle of a regular polygon of n sides

$= \frac{n-2}{n} \times 180$ degrees

Solved Examples

Ex.29 tan 9° – tan27° – tan 63° + tan 81° is equals to -

(A) 0 (B) 1

(C) –1 (D) 4

Sol. tan 9° + tan 81° – (tan 27° + tan 63°)

(tan 9° + cot 9°) – (tan 27° + cot 27°)

$= \left(\frac{\sin 9^o}{\cos 9^o} + \frac{\cos 9^o}{\sin 9^o}\right) - \left(\frac{\sin 27^o}{\cos 27^o} + \frac{\cos 27^o}{\sin 27^o}\right)$

$= \frac{1}{\sin 9^o \cos 9^o} - \frac{1}{\cos 27^o \sin 27^o}$

$= \frac{2}{\sin 18} - \frac{2}{\sin 54^\circ} = \frac{2}{\sin 18^\circ} - \frac{2}{\sin 36^o}$

$= \frac{2\times 4}{\sqrt{5}-1} - \frac{2\times 4}{\sqrt{5}+1} = 8\left[\frac{\sqrt{5}+1-\sqrt{5}+1}{(\sqrt{5}-1)(\sqrt{5}+1)}\right]$

$= \frac{16}{4} = 4$ **Ans.[D]**

Ex.30 $\cos^3x.\ \sin 2x = \sum_{m=1}^{n} a_m \sin mx$ is an identity in x. Then -

(A) $a_3 = \frac{3}{8}$, $a_2 = 0$ (B) $n = 5$, $a_1 = \frac{1}{4}$

(C) $\Sigma a_m = \frac{3}{4}$ (D) All the above

Sol. $\cos^3x.\ \sin 2x = \frac{\cos 3x + 3\cos x}{4}.\sin 2x$

$= \frac{1}{8}(\sin 5x - \sin x) + \frac{3}{8}(\sin 3x + \sin x)$

$= \frac{1}{4}\sin x + \frac{3}{8}\sin 3x + \frac{1}{8}\sin 5x.$

$\therefore n = 5,\ a_1 = \frac{1}{4},\ a_2 = 0,\ a_3 = \frac{3}{8},$

$a_4 = 0,\ a_5 = \frac{1}{8}$ **Ans.[D]**