Algebra

Importance : Algebra based 2-3 questions are essentially asked in almost all competitive exams obviously this chapter should be given sufficient time and practice done.

Scope of questions : Questions based on different algebraic expressions, equations (e.g. quadratic or higher order, square root, cube root and inverse) or based on graphic representation of equations and the value of a variable is asked or an equation is required to be validated.

Way to success : Solution of questions of this chapter can be ensured by memorising the concerved formulae/rules and by regular practice.

Polynomials : An algebraic expression in which the variables involved have only non-negative integral powers is called a polynomial.

General Form : $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ is a polynomial in variable x, where $a_0, a_1, a_2, a_3 \ldots a_n$ are real numbers and n is non-negative integer.

Remainder Theorem : Let $f(x)$ be a polynomial of degree $n \geq 1$, and let a be any real number. When $f(x)$ is divided by $(x - a)$, then the remainder is $f(a)$.

Proof : Suppose that when $f(x)$ is divided by $(x - a)$, the quotient is $g(x)$ and the remainder is $r(x)$.

Then,degree $r(x)$ < degree $(x - a)$

$\Rightarrow$ degree $r(x) < 1$

$\Rightarrow$ degree $r(x) = 0$ $\quad$ [$\because$ degree of $(x - a) = 1$]

$\Rightarrow r(x)$ is constant, equal to r (say).

Thus, when $f(x)$ is divided by $(x - a)$, then the quotient is $g(x)$ and the remainder is r.

$\therefore f(x) = (x - a) \cdot g(x) + r$ $\quad$ ... (i)

Putting $x = a$ in (i), we get $r = f(a)$.

Thus, when $f(x)$ is divided by $(x - a)$, then the remainder is $f(a)$.

Remarks

(i) If a polynomial $p(x)$ is divided by $(x + a)$, the remainder is the value of $p(x)$ at $x = -a$ i.e. $p(-a)$

$[\because x + a = 0 \Rightarrow x = -a]$

(ii) If a polynomial $p(x)$ is divided by $(ax - b)$, the remainder is the value of $p(x)$ at $x = \frac{b}{a}$ i.e. $p\left(\frac{b}{a}\right)$.

$[\because ax - b = 0 \Rightarrow x = \frac{b}{a}]$

(iii) If a polynomial $p(x)$ is divided by $(ax + b)$, then remainder is the value of $p(x)$ at $x = -\frac{b}{a}$ i.e. $p\left(-\frac{b}{a}\right)$

$[\because ax + b = 0 \Rightarrow x = -\frac{b}{a}]$

(iv) If a polynomial $p(x)$ is divided by $b - ax$, the remainder is the value of $p(x)$ at $x = \frac{b}{a}$ i.e. $p\left(\frac{b}{a}\right)$

$[\because b - ax = 0 \Rightarrow x = \frac{b}{a}]$

Factor Theorem

Let $p(x)$ be a polynomial of degree greater than or equal to 1 and a be a real number such that $p(a) = 0$, then $(x - a)$ is a factor of $p(x)$.

Conversely, if $(x - a)$ is a factor of $p(x)$,

then $p(a) = 0$

$\Rightarrow p(x)$, when divided by $(x - a)$ gives remainder zero.

But by Remainder theorem,

$p(x)$ when divided by $(x - a)$ gives the remainder equal to $p(a)$.

$\therefore p(a) = 0$

Remarks

(i) $(x + a)$ is a factor of a polynomial iff (if and only if) $p(-a) = 0$

(ii) $(ax - b)$ is a factor of a polynomial if $p\left(\frac{b}{a}\right) = 0$

(iii) $(ax + b)$ is a factor of a polynomial $p(x)$ if $p\left(-\frac{b}{a}\right) = 0$

(iv) $(x - a)$ $(x - b)$ are factors of a polynomial $p(x)$ if $p(a) = 0$ and $p(b) = 0$

ALGEBRAIC IDENTITIES

An algebraic identity is an algebraic equation which is true for all values of the variable (s).

IMPORTANT FORMULAE

1. $(a + b)^2 = a^2 + 2ab + b^2$
2. $(a - b)^2 = a^2 - 2ab + b^2$
3. $(a + b)^2 = (a - b)^2 + 4ab$
4. $(a - b)^2 = (a + b)^2 - 4ab$
5. $a^2 - b^2 = (a + b)(a - b)$
6. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
7. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
8. $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
9. $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
10. $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$
11. $a^3 - b^3 = (a - b)^3 + 3ab(a - b)$

12. $a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$

$= (a + b + c)\frac{1}{2}(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac)$

$= \frac{1}{2}(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$

13. If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$
14. $(a+b+c)^3 = a^3 + b^3 + c^3 + 3(b+c)(c+a)(a+b)$
15. $a^2 + b^2 = (a + b)^2 - 2ab$
16. $a^2 + b^2 = (a - b)^2 + 2ab$
17. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$
18. $a^4 + b^4 + a^2b^2 = (a^2 - ab + b^2)(a^2 + ab + b^2)$

GRAPHIC REPRESENTATION OF STRAIGHT LINES

Ordered Pair : A pair of numbers a and b listed in a specific order with a at the first place and b at the second place is called an ordered pair (a, b).

Note that $(a, b) \neq (b, a)$.

Thus, (2, 3) is one ordered pair and (3, 2) is another ordered pair.

CO-ORDINATE SYSTEM

Co-ordinate Axes : The position of a point in a plane is determined with reference to two fixed mutually perpendicular lines, called the coordinate axes.

Let us draw two lines X'OX and YOY', which are perpendicular to each other and intersect at the point O.

These lines are called the coordinate axes or the axes of reference.

The horizontal line X'OX is called the x-axis.

The vertical line YOY' is called the y-axis.

The point O is called the origin.

The distance of a point from y-axis is called its x-co-ordinate or abscissa and the distance of the point from x-axis is called its y-co ordinate or ordinate.

If x and y, denote respectively the abscissa and ordinate of a point P, then (x, y) are called the coordinates of the point P.

The y–co-ordinate of every point on x-axis is zero. i.e. when a straight line intersects at x-axis, its y-co-ordinate is zero. So, the co-ordinates of any point on the x-axis are of the form (x, 0).

The x-co-ordinate of every point on y-axis is zero. So, the co-ordinates of any point on y-axis are of the form (0, y).

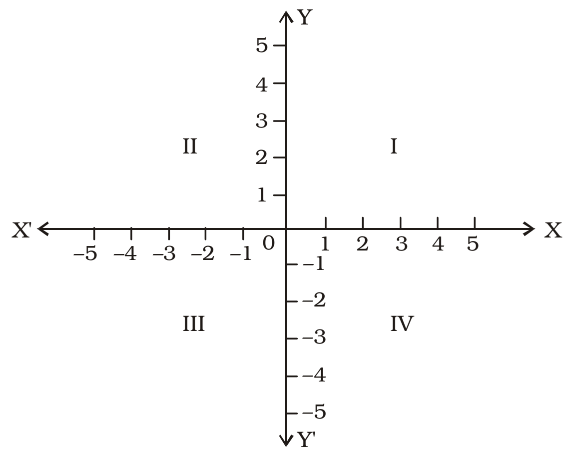
The co-ordinates of the origin are (0, 0).

$y = a$ where a is constant denotes a straight line parallel to x-axis.

$x = a$ where a is constant, denotes a straight line parallel to y-axis.

$x = 0$ denotes y-axis.

$y = 0$ denotes x-axis.



We can fix a convenient unit of length and taking the origin as zero, mark equal distances on the x-axis as well as on the y-axis.

Convention of Signs : The distances measured along OX and OY are taken as positive and those along OX' and OY' are taken as negative, as shown in the figure given above.

CO-ORDINATES OF A POINT IN A PLANE

Let P be a point in a plane.

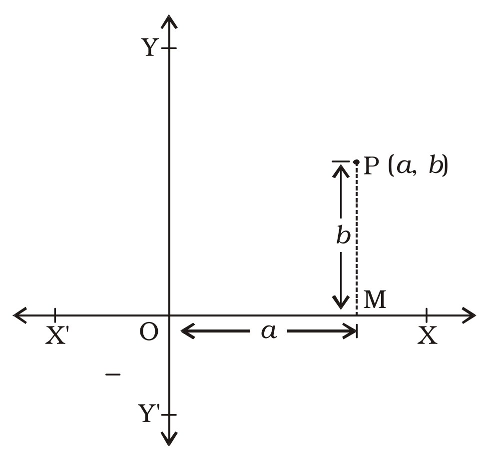
Let the distance of P from the y-axis = a units.

And, the distance of P from the x-axis = b units.

Then, we say that the co-ordinates of P are (a, b).

a is called the x-co-ordinate, or abscissa of P.

b is called the y co-ordinate, or ordinate of P.

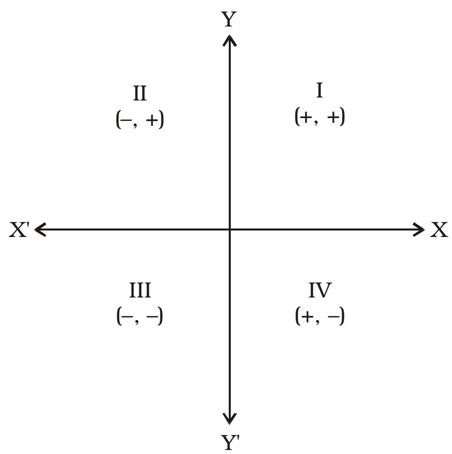


Quadrants : Let X' OX and YOY' be the co-ordinate axes.

These axes divide the plane of the paper into four regions, called quadrants. The regions XOY, YOX',X'OY' and Y'OX are respectively known as the first, second, third and fourth quadrants.

Using the convention of signs, we have the signs of the coordinates in various quadrants as given below.

Region	Quadrant	Nature of x and y	Signs of co-ordinates
XOY	I	$x > 0,\ y > 0$	(+, +)
YOX'	II	$x < 0,\ y > 0$	(–, +)
X'OY'	III	$x < 0,\ y < 0$	(–, –)
Y'OX	IV	$x > 0,\ y < 0$	(+, –)



Note : Any point lying on x-axis or y-axis does not lie in any quadrant.

Consistency and Inconsistency

A system of a pair of linear equations in two variables is said to be consistent if it has at least one solution.

A system of a pair of linear equations in two variables is said to be inconsistent if it has no solution.

The system of a pair of linear equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ has :

(i) a unique solution (i.e. consistent) if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$. The graph of the linear equations intersect at only one point.

(ii) no solution (i.e. inconsistent) if $\frac{a_1}{b_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

The graph of the two linear equations are parallel to each other i.e. the lines do not intersect.

(iii) an infinite number of solution if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

The graph of the linear equations are coincident.

Homogeneous equation of the form $ax + by = 0$ is a line passing through the origin. Therefore, this system is always consistent.

Rule 1. $(a + b)^2 = a^2 + 2ab + b^2$

$\Rightarrow\ a^2 + b^2 = (a + b)^2 - 2ab$

$(a - b)^2 = a^2 - 2ab + b^2$

$\Rightarrow\ a^2 + b^2 = (a - b)^2 + 2ab$

Rule 2. $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$

Rule 3. $(a + b)^2 - (a - b)^2 = 4ab$

or, $(a + b)^2 = (a - b)^2 + 4ab$

or, $(a - b)^2 = (a + b)^2 - 4ab$

Rule 4. $(a^2 - b^2) = (a + b)(a - b)$

Rule 5. $a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 - 2$ or, $\left(a - \frac{1}{a}\right)^2 + 2$

Rule 6. $a^4 - b^4 = (a^2 + b^2)(a + b)(a - b)$

Rule 7. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

or, $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$

Rule 8. $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

or, $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$

Rule 9. $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

or, $a^3 - b^3 = (a - b)^3 - 3ab(a - b)$.

Rule 10. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Rule 11. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Rule 12. $a^3 + \frac{1}{a^3} = \left(a + \frac{1}{a}\right)^3 - 3\left(a + \frac{1}{a}\right)$

Rule 13. $a^3 - \frac{1}{a^3} = \left(a - \frac{1}{a}\right)^3 + 3\left(a - \frac{1}{a}\right)$

Rule 14. If $a + \frac{1}{a} = 2$ then $a^n + \frac{1}{a^n} = 2$.

Rule 15. If $a + \frac{1}{a} = 2$ then, $a^n - \frac{1}{a^n} = 0$

(By putting a = 1)

Rule 16. If $a + \frac{1}{a} = 2$ then $a^m + \frac{1}{a^n} = 2$

(By putting a = 1), and $m \neq n$.

Rule 17. If $a + \frac{1}{a} = 2$ then $a^m - \frac{1}{a^n} = 0$

(By putting a = 1)

Rule 18. If $a + \frac{1}{a} = -2$, then $a^n + \frac{1}{a^n} = 2$ If n is even and $a^n + \frac{1}{a^n} = -2$, if n is odd.

(By putting a = –1)

Rule 19. If $a + \frac{1}{a} = -2$ then the value of $a^m \pm \frac{1}{a^n} = (-1)^m \pm \frac{1}{(-1)^n}$

Rule 20. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ or, $\frac{1}{2}(a + b + c)\ [(a - b)^2 + (b - c)^2 + (c - a)^2]$

Rule 21. If $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

Rule 22. If $a^3 + b^3 + c^3 = 3abc$, then $a + b + c = 0$ or $a = b = c$.

Proof $\because$ $a^3 + b^3 + c^3 = 3abc$

$\Rightarrow$ $a^3 + b^3 + c^3 - 3abc = 0$

Now, $a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c)\,[(a - b)^2 + (b - c)^2 + (c - a)^2]$

$\Rightarrow$ $0 = \frac{1}{2}(a + b + c)\,[(a - b)^2 + (b - c)^2 + (c - a)^2]$

$\therefore$ Either $a + b + c = 0$ or, $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$, i.e., $a - b = 0$

$\Rightarrow$ $a = b$, $b - c = 0$

$\Rightarrow$ $b = c$, $c - a = 0$

$\Rightarrow$ $c = a$

$\therefore$ $a = b = c$

Rule 23. If $a^2 + b^2 + c^2 = ab + bc + ca$, then $a = b = c$.

Rule 24. Componendo and Dividendo Rule, If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Rule 25. If $\frac{a+b}{a-b} = \frac{c}{d}$, then $\frac{a}{b} = \frac{c+d}{c-d}$.

Rule 26. If $\sqrt{x+\sqrt{x+\sqrt{x+\ldots\infty}}}$ where $x = n(n + 1)$

then $\sqrt{x+\sqrt{x+\sqrt{x+\ldots\infty}}} = (n+1)$

Rule 27. If $\sqrt{x-\sqrt{x-\sqrt{x-\ldots\infty}}}$ where $x = n(n + 1)$ then,

$\sqrt{x-\sqrt{x-\sqrt{x-\ldots\infty}}} = n.$

Rule 28. $(a + b + c)^3 = a^3 + b^3 + c^3 - 3(a + b)(b + c)(c + a)$

Rule 29. $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$

Rule 30. If $a + \frac{1}{a} = x$, then $a^3 + \frac{1}{a^3} = x^3 - 3x$.

Rule 31. If $a - \frac{1}{a} = x$, then $a^3 - \frac{1}{a^3} = x^3 + 3x$.

Rule 32. Binomial theorem :

$(a + b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_{n-1}a^1b^{n-1} + {}^nC_na^0b^n$, where, n is a positive number and ${}^nC_r = \frac{n!}{r!(n-r)!}$

Permutation and Combination

Permutation : It is used where we have to arrange things. Out of total n things, *r* things (taken at a time) can be arranged as ${}^np_r$ or $P(n,r)$

$P(n,r) = {}^nP_r = \frac{n!}{(n-r)!}$ where $n \ge r$

Combination : It is used where we have to select things. It is written as ${}^nC_r$ or $C(n,r)$

$C(n,r) = \frac{n!}{(n-r)!\,r!}$ $n \ge r$

Some important results.

${}^nP_o = 1$; ${}^nP_n = n!$

${}^nC_o = {}^nC_n = 1$; ${}^nC_r = {}^nC_{n-r} = {}^nC_1 = {}^nC_{n-1} = n.$

Ex. ${}^7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7.6.5.4!}{4!} = 210$

${}^5C_2 = \frac{5!}{(5-2)!2!} = \frac{5.4.3!}{3!\times 2\times 1} = 10$

n! (is called as n factorial)

5! = 5.4!

= 5.4.3!

= 5.4.3.2!

= 5.4.3.2.1!

$\boxed{5! = 120}$

Also $\boxed{0! = 1}$

COORDINATE GEOMETRY

Importance : Coordinate geometry is separate and important filled in mathematics but very rarely asked in competitive exams. However in two-dimensional (2–D) geometry introductory/easy questions should be practised for improving marks.

Scope of questions : Mostly questions are related to distance between two points, linear/non-linear these coplaner points, cutting a line a specific ratio by a given point.

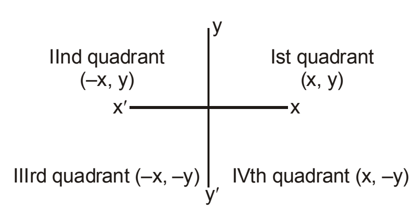
Way to success : The concept of coordinate geometry and practice of above mentioned questions is very important to solve questions.

Important Points :

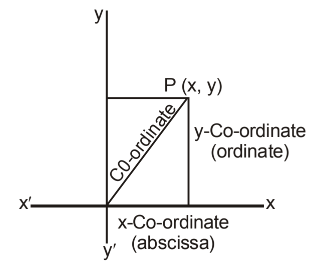
x–coordinate is called the abscissa of P, where (x, y) are co-ordinates of any point P.

y–co-ordinate is called the ordinate of P, where (x, y) are co–ordinates of any point P.

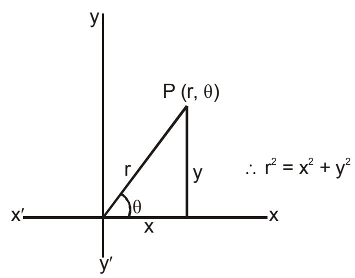
Quadrants :



Cartesian Co–ordinate System :



Polar Coordinate System :



RULE 1 : The distance between any two points in the plane is the length of the line segment joining them. The distance between two points P (x_1, y_1) and Q (x_2, y_2) is

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \text{ or,}$$

$$PQ = \sqrt{(\text{difference of abscissa})^2 + (\text{difference of ordinates})^2}$$

RULE 2 : The area of a triangle, the Co-ordinates of whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\text{Area } \Delta = \left(\frac{1}{2}\right) | x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) |$$

$$= \left(\frac{1}{2}\right) \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If all three points are collinear,
then area of $\Delta = 0$

RULE 3 : The Co-ordinates of the point which divides the line segment joining the points (x_1, y_1) and (x_2, y_2) internally in the ratio m : n are given by

$$x = \frac{mx_2 + nx_1}{m+n} \qquad y = \frac{my_2 + ny_1}{m+n}$$

RULE 4 : If P is the mid-point of AB, such that it divides AB in the ratio 1 : 1, then its Co-ordinates are (x,y) = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ also called mid point formula.

RULE 5 : The Co–ordinates of the point which divides the line segment joining the points (x_1, y_1) and (x_2, y_2) externally in the ratio m : n, are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

RULE 6 : The Co-ordinates of the centroid of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

RULE 7 : The Co–ordinates of the in–centre of a triangle whose vertices are A (x_1, y_1), B(x_2, y_2), C(x_3, y_3) are given by

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right) \text{ where } a = BC,$$

b = CA and c = AB.

Equation of straight line.

A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

RULE 8 : If (x_1, y_1) and (x_2, y_2) are the Co-ordinates of any two points on a line, then its slope is

$$(\tan\theta) = m = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{\text{difference of ordinates}}{\text{difference of abscissa}}$$

RULE 9 : The angle θ between the lines having slopes m_1 and m_2 is given by $\tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2}$

RULE 10 : If two lines having slopes m_1 and m_2 are (i) parallel if $m_1 = m_2$ (ii) Perpendicular if $m_1 \times m_2 = -1$

RULE 11 : (Slope–Intercept) The equation of a line with slope m and making an intercept c on y-axis is $y = mx + c$.

RULE 12 : (Point-Slope form) The equation of a line which passes through the point (x_1, y_1) and has the slope 'm' is $(y - y_1) = m(x - x_1)$

RULE 13 : (Two-point form) The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_2}$$

RULE 14 : (Intercept form) The equation of a line which cuts off intercepts a and b respectively on the x and y–axes is

$$\frac{x}{a} + \frac{y}{b} = 1$$

RULE 15 : (i) The slope of a line whose general quation is given by $Ax + By + C = 0$ is $\frac{-A}{B}$

(ii) The intercepts of a line on x and y axes respectively whose general equation is $Ax + By + C = 0$ is given by :-

x-intercept = $\frac{-C}{A}$ and y-intercept = $\frac{-C}{B}$

RULE 16 : General equation of straight line is $ax + by + c = 0$

∴ Now the area of the triangle made by the given straight line and its intercepts is

$$\Delta = \frac{1}{2} \times \left(\frac{-c}{a}\right) \times \left(\frac{-c}{b}\right) \text{ sq. units}$$