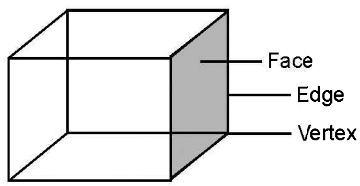
# MENSURATION

## SOLIDS

A solid has three dimensions, namely, length, breadth or width and height or thickness. The plane surfaces that binds it are called its faces and the solid so generated is known as a polyhedron.

The volume of any solid figure is the amount of space enclosed within its bounding faces. A solid has edges, vertices and faces which are shown in the figure given below:




A solid has two types of surface areas:

- *Lateral Surface Area (LSA)* LSA of a solid is the sum of the areas of all the surfaces it has except the top and the base.
- *Total Surface Area (TSA)* TSA of a solid is the sum of the lateral surface area and the areas of the base and the top.

**Note** In case of solids like the cube and cuboid, the lateral surface area consists of plane surface areas (i.e., area of all surfaces except the top and base) whereas in case of solids like cone and cylinder, it consists of curved surface areas (CSA). Thus, for such solids the LSA is also called CSA.

## Euler's Rule

Euler's rule states that for any regular solid:
Number of faces (F) + Number of vertices (V) = Number of Edges (E) + 2

### Cuboid

A cuboid is a rectangular solid having 6 rectangular faces. The opposite faces of a cuboid are equal rectangles. A cuboid has a length (l), breadth (b) and height (h).

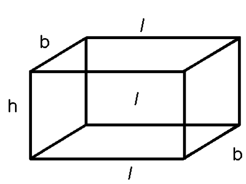



Fig. 1

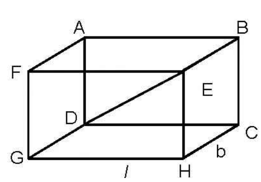


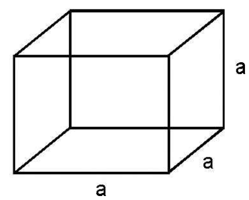

Fig. 2

In Fig. 2 ED is the diagonal of the cuboid. Moreover, the area of the surface GDCH is $x$, the area of the surface HEBC is $y$ and the area of the surface GFEH is z.

- Volume = Area of base × Height = $lbh$
- Volume = $\sqrt{xyz}$
- Volume = $xh = yl = zb$
- Lateral surface area (LSA) or area of the 4 walls = $2(l + b)h$
- Total surface area (TSA) = $2(x + y + z) = 2(lb + bh + lh)$
- Diagonal = $\sqrt{l^2 + b^2 + h^2}$

## CUBE

A cube is a solid figure having six faces. All the faces of a cube are equal squares (let us say of the side '$a$'). Thus the length, breadth and height of a cube are equal.



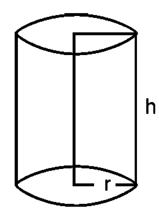

- Volume = $a^3$

(ii) Lateral surface area (LSA) or area of the 4 walls = $4a^2$

(iii) Total surface area (TSA) = $6a^2$

(iv) Diagonal = $a\sqrt{3}$

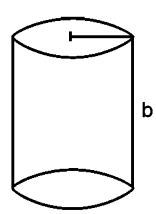
## RIGHT CIRCULAR CYLINDER

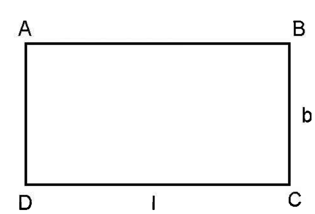



In the above figure, r is the radius of the base and h is the height of a right circular cylinder. A cylinder is generated by rotating a rectangle or a square by fixing one of its sides.

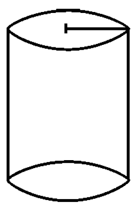
- Volume = area of base × height
  Volume = $\pi r^2 h$
- Curved surface area (CSA) = Perimeter of base × height
  LSA = $2\pi rh$
- Total surface area (TSA) = LSA + Area of the top + Area of the base
  TSA = $2\pi rh + \pi r^2 + \pi r^2$
  TSA = $2\pi r(r + h)$

### Some important deductions



- If the above rectangular sheet of paper (ABCD) is rolled along its length to form a cylinder, then the radius (r) of the cylinder will be (L/2π) and its height will be b and Volume of this cylinder = $\frac{L^2 b}{4\pi}$, where L is the length of the rectangle.
- 

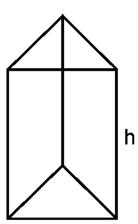



If the above rectangular sheet of paper (ABCD) is rolled along its breadth to form a cylinder, then the radius (r) of the cylinder will be $\frac{b}{2\pi}$ and its height will be L.

Volume of this cylinder = $\frac{b^2 L}{4\pi}$

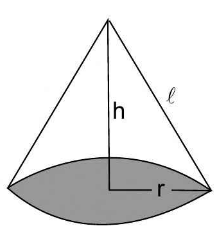
## PRISM

A prism is a solid having identical and parallel top and bottom faces, i.e., they will be identical polygons of any number of sides. The side faces of a prism are rectangular and are known as lateral faces. The distance between two bases is known as the height or the length of the prism.




- Volume = Area of base × Height
- Lateral Surface Area (LSA) = Perimeter of the base × height
- Total surface Area (TSA) = LSA + (2 × Area of the base)

## RIGHT CIRCULAR CONE




In the above figure 'r' is the radius of the base, h is the height and ℓ is the slant height of the right circular cone.

- Volume = $\frac{1}{3}$ × area of the base × height

  Volume = $\frac{1}{3}\pi r^2 h$

(ii) Slant height = $\ell = \sqrt{r^2 + h^2}$

(iii) Curved surface area (CSA) = $\pi r \ell$

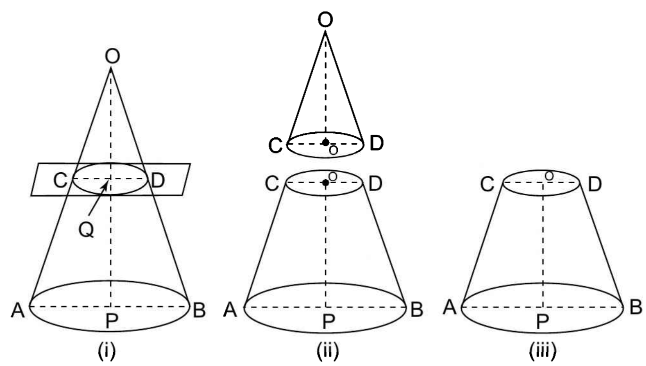
(iv) Total surface area (TSA) = (CSA + Area of the base)

TSA = $\pi r \ell + \pi r^2$

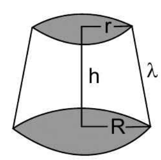
## Frustum of Cone

A cone whose top portion is sliced off by a plane which is parallel to the base is called frustum of cone.

Formation of frustum:




However, for the sake of representing the formula, we will use another form of frustum right now as given below:




In the above figure, r is the radius of the base, h is the vertical height of the frustum and λ is the slant height of the frustum.

- Volume = $\frac{\pi h}{3}\left(R^2 + r^2 + Rr\right)$

(ii) Slant height = $\lambda = \sqrt{(R - r)^2 + h^2}$

(iii) Curved Surface Area (CSA) = π(R + r) λ

(iv) Total Surface Area (TSA) = CSA + Area of the Top + Area of the base

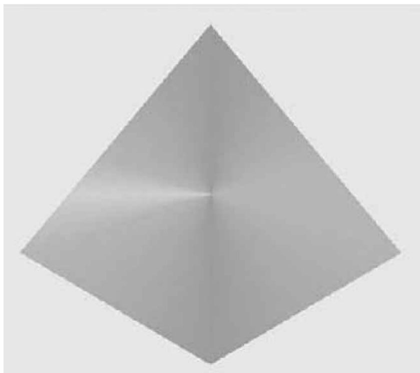
TSA = $\pi(R + r)\lambda + \pi r^2 + \pi R^2$

TSA = $\pi(R\lambda + r\lambda + r^2 + R^2)$

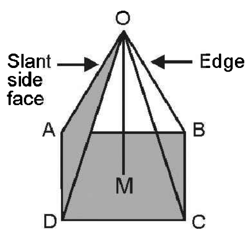
(v) To find the height (H) of original cone.

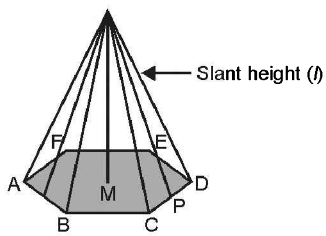
$$H = \frac{Rh}{R - r}$$

## PYRAMID



A pyramid is a solid having an n-sided polygon at its base. The side faces of a pyramid are triangular with the top as a point.




In the above figures OM is the height of the pyramid.

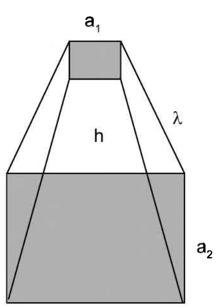
- Volume = $\frac{1}{3}$ × Area of the base × height

(ii) Lateral Surface Area (LSA) = $\frac{1}{2}$× (Perimeter of the base) × slant height

(iii) Total Surface Area (TSA) = LSA + Area of the base

### Frustum of Pyramid

A pyramid whose top portion is sliced off by a plane which is parallel to the base is called the frustum of a pyramid.




In the above figure, $A_1$ is the area of the top face of the frustum, $A_2$ is the area of the bottom face of the frustum, h is the height of the frustum and l is the slant height of the frustum.

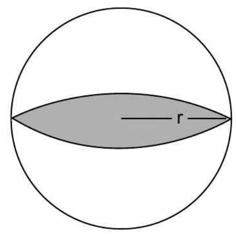
- Volume = $\frac{1}{3}h\left(A_1 + A_2 + \sqrt{A_1A_2}\right)$

(ii) Lateral Surface Area (LSA) = $\frac{1}{2}(P_1 + P_2)\ell$

where $P_1$ and $P_2$ are perimeters of the top and the bottom faces.

(iii) Total surface Area (TSA) = $\ell$S.A. + $A_1$ + $A_2$

## SPHERE

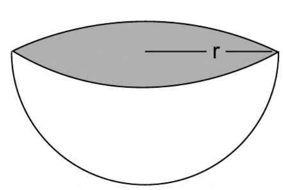



In the above figure r is the radius of the sphere.

- Volume = $\frac{4}{3}\pi r^3$

(ii) Surface Area = $4\pi r^2$

## HEMISPHERE




- Volume = $\frac{2}{3}\pi r^3$

(ii) Curved surface area (CSA) = $2\pi r^2$

(iii) Total surface area (TSA) = LSA + Area of the top face (read circle)

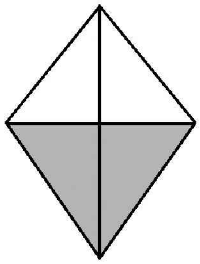
TSA = $2\pi r^2 + \pi r^2$

TSA = $3\pi r^2$

## SOME MORE SOLIDS

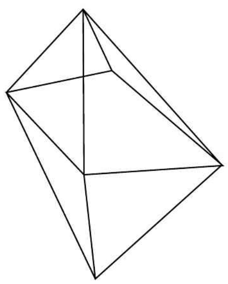
### Tetrahedron

A tetrahedron is a solid which has 4 faces. All the faces of a tetrahedron are equilateral triangles. A tetrahedron has 4 vertices and 6 edges.



### Octahedron

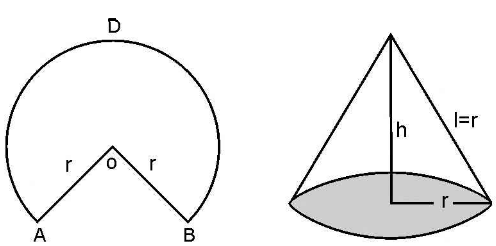
An octahedron is a solid which has 8 faces. All the faces of an octahedron are equilateral triangles. An octahedron has 6 vertices and 12 edges.



### Inscribed and circumscribed solids

- If a sphere of the maximum volume is inscribed in a cube of edge 'a', then the radius of the sphere = $\frac{a}{2}$
- If a cube of the maximum volume is inscribed in a sphere of radius 'r', then the edge of the cube = $\frac{2r}{\sqrt{3}}$
- If a cube of the maximum volume is inscribed in a hemisphere of radius 'r', then the edge of the cube = $\sqrt{\frac{2}{3}} \times r$

### Some important deductions



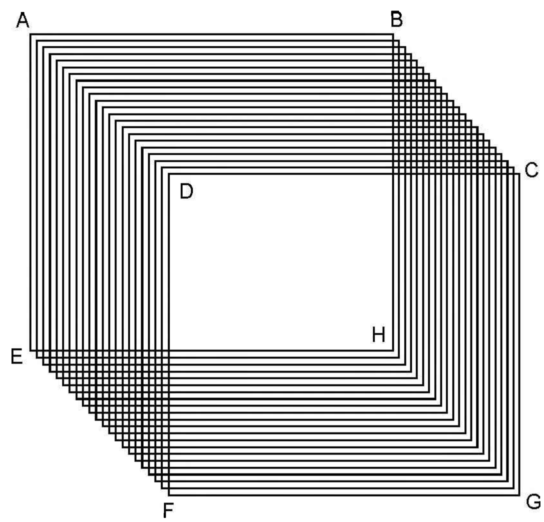

If a cone is made by a sector of a circle (AOBD), then the following two things must be remembered

- The area of the sector of a circle (AOBD) = The CSA of the cone
- Radius of the cirlce (r) = Slant height (l) of the cone.

## VISUAL MENSURATION

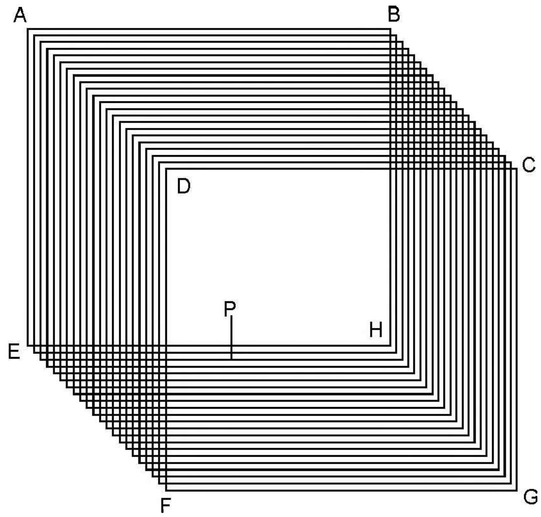
Under this section of mensuration, we will be required to visualize some of the unknown dimensions of any structure with the help of the given dimension. Some of the typical examples of Visual Mensuration are given below:

Given is a cube ABCDEFGH of side length 'a' units. Its top face is ABCD and its bottom face is EFGH. Since the side length of this cube is 'a' units, so, AB = BC = CD = AD = AE = EF = FD = FG = GH = EH = BH = CG = 'a'




**Minimum length between vertex A and vertex G** We can find the minimum distance between these two vertices in the following two ways:

- *Aerial distance* Aerial distance can be understood by assuming that there is a fly at vertex A and it has to reach vertex G through the minimum possible distance. This distance will be the diagonal distance between the vertices A and G $= \sqrt{a^2 + a^2 + a^2} = a\sqrt{3}$
- *Physical distance* Physical distance can be understood by assuming that there is an ant at vertex A and it has to reach vertex G through the minimum possible distance. Since it cannot fly, so ant will cover this distance along the two faces viz., face ABEH and face EFGH or face ADFE and face CDFG or face ABCD and face CDFG or face ABCD and face BCGH.




The shortest possible distance can be the diagonal only. Let us assume that the ant is going via face ABEH and face EFGH. So, the ant will cover first the diagonal distance between A and P, where P is the mid point of side EH, and then from P to vertex G.

Now $AP^2 = EP^2 + GP^2 = \frac{a\sqrt{5}}{2}$

And $GP^2 = HP^2 + GH^2 = \frac{a\sqrt{5}}{2}$

Hence, the minimum possible physical distance

$= 2\frac{a\sqrt{5}}{2} = a\sqrt{5}$

Let us find out why the distance AE + EG cannot be the shortest?

First calculate AE and EG.

AE = a

$EG = \sqrt{a^2 + a^2} = a\sqrt{2}$

So, $AE + EG = a + \sqrt{a^2 + a^2} = a\sqrt{2} + a = a\,(1 + \sqrt{2})$

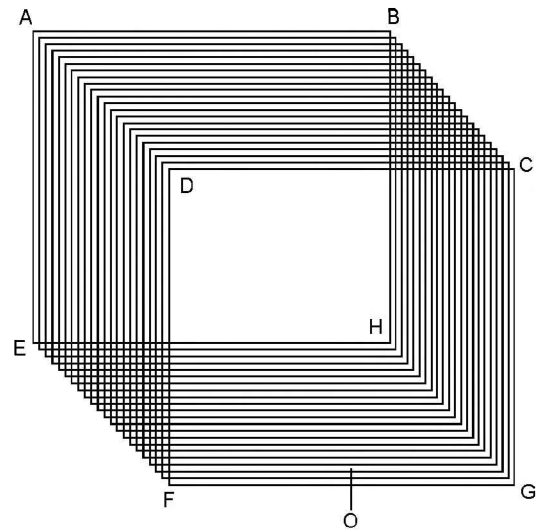
And obviously, $a\sqrt{5} < a(1 + \sqrt{2})$[For $a>0$]

We can understand this phenomenon by having a bit of *mental mapping*. As we have seen earlier that the minimum possible distance can be the diagonal distance only. Now let us cut open the face ABHE by making a cut mark at EH so that the faces ABHE and EFGH are in a plane, lying horizontal on the ground. Now the minimum possible distance between A and G will be the diagonal of the newly formed rectangle AFGB. This diagonal will pass through the mid-point of EH.

So, $AG^2 = AF^2 + FG^2 = (2a)^2 + (a)^2 = 5a^2$

$AG = a\sqrt{5}$

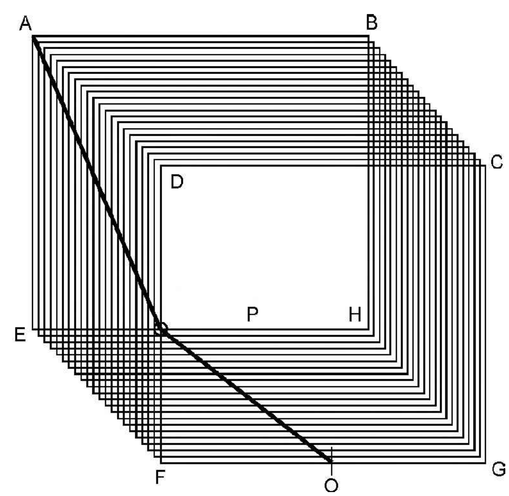
**Minimum length between vertex A and vertex O, where O is the mid-point of FG**




- *Aerial distance* We can see this situation vis-à-vis a cuboid of side lengths 'a' units, 'a' units and $\frac{a}{2}$ units.

  Minimum aerial distance between A and O

  $= \sqrt{\frac{a^2}{4} + a^2 + a^2} = \frac{3a}{2}$
- *Physical distance* Let us assume that the ant is moving through the faces ABHE and EFGH. Ant will first go the point Q, where Q is the mid-point of E and P. And then the ant will cover QO.

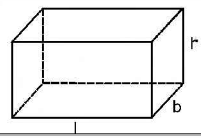
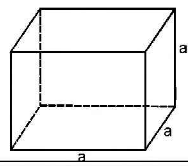
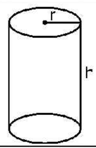
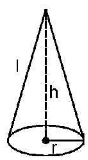
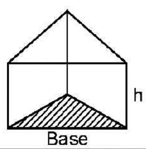
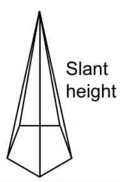
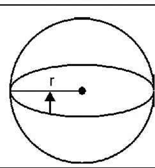
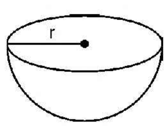
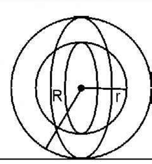
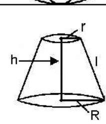



$AQ^2 = AE^2 + EQ^2 = a^2 + \frac{a^2}{16} = \frac{17a^2}{16}$

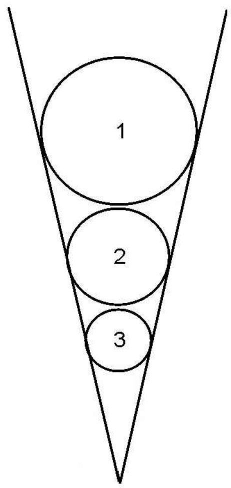
So, $AQ = QO = \frac{a\sqrt{17}}{4}$

So, the minimum distance between the vertex A and point $O = AQ + QO = \frac{a\sqrt{17}}{2}$.

## FORMULAE

| *S.No.* | *Name* | *Figure* | *Nomenclature* | *Volume* | ***Curved/* Lateral** | **Total *Surface Area Surface Area*** |
|---|---|---|---|---|---|---|
| 1. | Cuboid |  | l = length<br>b = breadth<br>h = height | lbh | 2(l+b) h | 2(lb+bh+hl) |
| 2. | Cube |  | a = edge/side | $a^3$ | $4a^2$ | $6a^2$ |
| 3. | Right Circular Cylinder |  | r = radius of base<br>h = height of the cylinder | $\pi r^2 h$ | $2\pi rh$ | $2\pi r\ (r + h)$ |
| 4. | Right Circular Cone |  | r = radius<br>h = height<br>l = slant height<br>$l = \sqrt{r^2 + h^2}$ | $\frac{\pi r^2 h}{3}$ | $\pi rl$ | $\pi r\ (l + r)$ |
| 5. | Right Triangular Prism |  | – | Area of the base × height | Perimeter of the base × height | Lateral surface area + 2 (area of base) |
| 6. | Right Pyramid |  | –<br>height | 1/3 × area of the base × of the base<br>× slant height | ½ × perimeter the base × slant height | Lateral surface area + area of base |
| 7. | Sphere |  | r = radius | $\frac{4}{3}\pi r^3$ | – | $4\pi r^2$ |
| 8. | Hemisphere |  | r = radius | $\frac{2}{3}\pi r^3$ | $2\pi r^2$ | $3\pi r^2$ |
| 9. | Spherical shell |  | r = inner radius<br>R = outer radius | $\frac{4}{3}\ \pi(R^3 - r^3)$ | – | $4\pi(R^2 + r^2)$ |
| 10. | Frustum of a Cone |  | – | – | – | Lateral surface area + Area of top + Area of base |

**Example 1** Three spheres are kept inside a cone, as given in the figure. Spheres are touching both the slant sides of the cone and the adjacent spheres. If the radius of the 1st sphere and the 3rd sphere are 5 units and 20 units respectively, find the radius of the 2nd sphere.




***Solution*** We can see in Fig. 1, that the AOA′P, BOB′Q and COC′R will be similar.

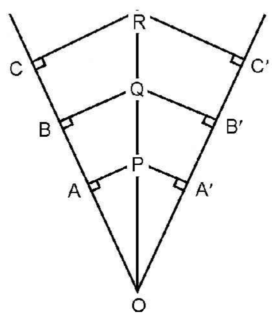
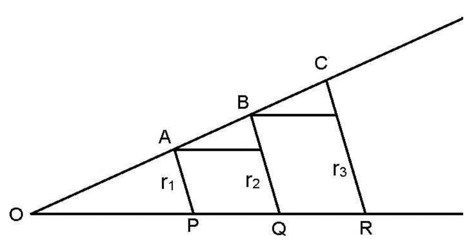



Fig. 1.




In Fig. 2, $(r_2 - r_1)/(r_2 + r_1) = (r_3 - r_2)/(r_3 + r_2) = K$
Using Componendo and Dividendo,
$r_2/r_1 = r_3/r_2$

Hence, the three radii are in a GP.

So, $r_2/20 = 5/r_2$, hence $r_2 = 10$ units.

Result of this question can be used as a formula also.

## SOLVED EXAMPLES

**Example 2** The diameter of a metallic sphere is 6 cm. The sphere is melted and drawn into a wire of uniform circular cross-section. If the length of the wire is 36 m, find its radius.

***Solution*** The diameter of the metallic sphere is 6 cm. Hence, radius of the sphere is 3 cm. Now, let the radius of the cross–section of the wire be r cm. And as we know, metallic sphere is converted into a cylindrical shaped wire, then their volumes will be equal.

So, $\frac{4}{3} \times \pi \times 3^3 = \pi \times r^2 \times 3600$
or, $4 \times 9 \pi \times 3^3 = 3600 \pi r^2$
This gives $r^2 = 0.01$
i.e., $r = 0.1$

**Example 3** Sardar Sarovar Dam which is rectangular in shape, can produce electricity only if the height of the water level in it is atleast 7cm. Now the water is pumped in at the rate of 5 km per hour through a pipe of diameter 14 cm into the dam area of dimensions 50 m × 44 m. In what time the dam will be able to produce electricity?

***Solution*** The volume of water flowing through the cylindrical pipe in one hour at the rate of 5 km (5000 m) per hour

$= \frac{22}{7} \times \frac{7}{100} \times \frac{7}{100} \times 5000 \, m^3$

[Since radius = 7 cm = $\frac{7}{100}$ m

$= 77 \, m^3$

Thus, 77 $m^3$ of water will fall into the tank in 1 h. Since the level of the water is required to be raised by 7 cm i.e., $\frac{7}{100}$ m,

Volume of the required quantity of water = $50 \times 44 \times \frac{7}{100} \, m^3 = 154 \, m^3$

Since 77 $m^3$ of water falls into the tank in 1 h, therefore, 154 $m^3$ of water will fall into the dam in $\frac{154}{77}$ h, i.e., 2 h.

So, the level of water will rise by 7 cm in 2 h.

**Example 4** A right angled triangle ABC, whose two sides other than the hypotonuse are 15 cm and 20 cm. The triangle is made to revolve about its hypotonuse. Find the volume of the double cone so formed.

***Solution*** Let ΔABC be the right angled triangle right angled at A, whose sides AB and AC measures 15 cm and 20 cm, respectively.

The length of the side BC (hypotenuse)

$= \sqrt{15^2 + 20^2}$ cm = 25

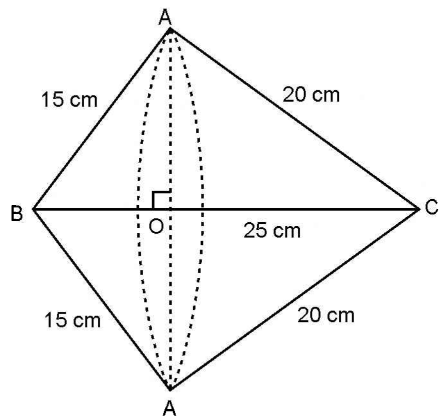
Here, AO (and A'O) is the radius of the common base of the double cone formed by revolving the ΔABC about BC.

Height of the cone BBA' is BO and the slant height is 15 cm.

Height of the cone CAA' is CO and the slant height is 20 cm

Using AA similarity,

Now, ΔAOB ~ ΔCAB




Therefore, $\frac{AO}{20} = \frac{15}{25}$

This gives $AO = \frac{20 \times 15}{25}$ cm $= 12$ cm

Also, $\frac{BO}{15} = \frac{15}{25}$ cm

This gives $BO = \frac{15 \times 15}{25}$ cm $= 9$ cm

Thus, CO = 25 cm – 9 cm = 16 cm

Now, volume of the double cone

$= \left(\frac{1}{3} \times 3.14 \times 12^2 \times 9 + \frac{1}{3} \times 3.14 \times 12^2 \times 16\right) \text{cm}^2$

$= \frac{3.14}{3} \times 12^2 \times (9 + 16) \text{cm}^2$

= 3768 $\text{cm}^3$

**Example 5** When shopping in Big Bazar, I saw a peculiar solid toy in the form of a hemisphere surmounted by a right circular cone. Height of the cone was 2 cm and the diameter of the base was 4 cm. If a right circular cylinder circumscribed the solid, find out how much more space will it have, provided the height of the cone was 2 cm and diameter of the base was 4 cm respectively?

***Solution*** See the figure below.

Assume BPC be the hemisphere and ABC is the cone standing on the base of the hemisphere.

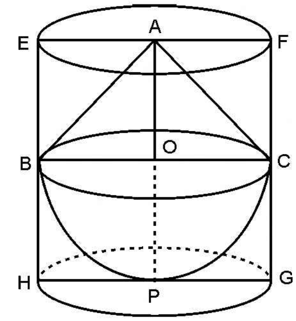
Radius BO of the hemisphere (as well as of the cone) = $\frac{1}{2} \times 4$ cm = 2 cm.

Now, let the right circular cylinder EFGH circumscribe the given solid.

Radius of the base of the right circular cylinder = HP = BO = 2 cm.

Height of the cylinder = AP = AO + OP = 2 cm + 2 cm = 4 cm

Now, volume of the right circular cylinder – volume of the solid




$= \left[\pi \times 2^2 \times 4 - \left(\frac{2}{3} \times \pi 2^3 + \frac{1}{3} \times \pi \times 2^3\right)\right] \text{cm}^3$

= (16π – 8π) $\text{cm}^3$

= 8π $\text{cm}^3$

Hence, the right circular cylinder is having 8p $\text{cm}^3$ more space than the solid.

**Example 6** A toy consists of a base that is the section of a sphere and a conical top. The volume of the conical top is 30π sq. units and its height is 10 units. The total height of the toy is 19 units. The volume of the sphere (in cubic units), from which the base has been extracted, is

(a) $\frac{256}{3}\pi$ (b) $\frac{64}{3}\pi$

(c) $\frac{108}{3}\pi$ (d) $\frac{500}{3}\pi$

***Solution*** Height of the cone = 10 units

Volume of the cone = 30π cubic units

= diameter of the cross section from where the sphere has been sectioned = 6 units

$= r + \sqrt{2^2 - 3^2} = 9$

= r = 5 units

= volume of the original sphere from which the base has been sectioned = $\frac{4}{3}\pi r^3 = \frac{500}{3}\pi$