CO - ORDINATE GEOMETRY

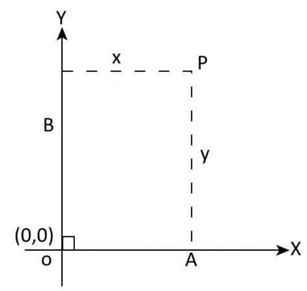
CONSIDER THE FOLLOWING CASE

Suppose there is an ant at point *O* in the figure and it wants to go to point *P*.

One way to reach P is that the ant travels along *OX*, reaches A and then travels along *AP* and reaches *P*.

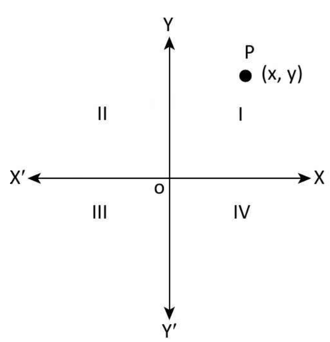
Hence, it first covers a distance '*X*' horizontally and then covers a distance '*Y*' vertically.

1st of all, we will assume a reference point, which is at a distance of 0 unit from both the X-axis and Y-axis and will call this Origin.



If we assume '*O*' as the origin, the distances *X* and *Y* are the distances of this point *P* from Y–axis and X-axis respectively. These are known as coordinates of point P and it is written as *P*(*X*, *Y*).

CO-ORIDNATE AXES AND REPRESENTATION OF A POINT



The figure given alongwith is called the *X*-*Y* Cartesian plane. The line XOX' is called the X-axis and YOY' the Y-axis.

If $P(x, y)$ is a point in this plane, then x is the X- coordinate of *P* or abscissa of *P* and y is called the Y-coordinate of P or the ordinate of *P*.

Remember that X-coordinate of the point is the distance of the point from Y-axis and Y-coordinate of the point is the distance of the point from X-axis.

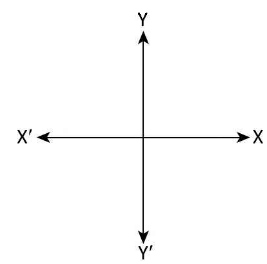
The X-Y Cartesian plane is divided into four equal parts called Quadrants (I, II, III, IV).

SIGN CONVENTION

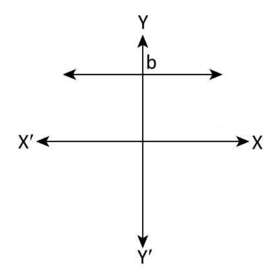
	1st Quadrant	*2nd Quadrant*	*3rd Quadrant*	*4th Quadrant*
X-axis	+ve	–ve	–ve	+ve
Y-axis	+ve	+ve	–ve	–ve

EQUATION AND GRAPH OF CO-ORDINATE AXES

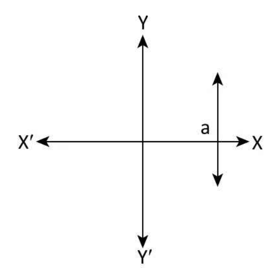
1. Equation of *X* and *Y* axes are $Y = 0$ and $x = 0$ respectively.



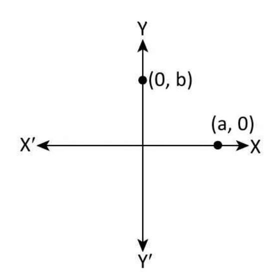
2. Equation of a line parallel to *X* axis is $Y = b$ (*b* is constant).



3. Equation of a line parallel to *Y* axis is $X = a$ (*a* is constant).



4. Any point on the X axis can be taken as (*a*, 0) and any point on the Y axis can be taken as (0, *b*).



5. To find out X and Y intercepts of a line, we will put Y = 0 and X = 0 respectively in the equation of the line.

Some standard formulae

1. Distance between two points:
   If there are two points $A\ (x_1,\ y_1)$ and $B\ (x_2, y_2)$ on the XY plane, then the distance between them is given by

$$AB = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

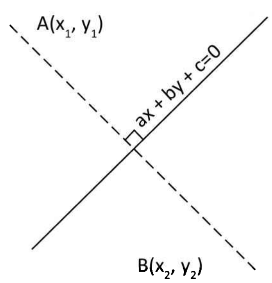
2. Division of a line segment:
   i. Internal - The Coordinates of a point P which divides the line joining $A\ (x_1\ y_1)$ and $B\ (x_2, y_2)$ internally in a ratio $l : m$ are given by

$$x = \frac{lx_2 + mx_1}{l + m}, y = \frac{ly_2 + my_1}{l + m}$$

   ii. External:
   The coordinates of a point P which divides the line joining the point $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $l : m$ are given by

$$x = \frac{lx_2 - mx_1}{l - m}, y = \frac{ly_2 - my_1}{l - m}$$

THE IMAGE OF A POINT ALONG THE MIRROR PLACE ON A STRAIGHT LINE



The image of $A\ (x_1, y_1)$ with respect to the line mirror $ax + by + c = 0$ be $B(x_2, y_2)$ is given by

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b}$$

$$= \frac{-2(ax_1 + by_1 + c)}{(a^2 + b^2)}$$

FOOT OF THE PERPENDICULAR

If the foot of the perpendicular from (x_1, y_1) to the line $lx + my + n = 0$ is (h, k) then

$$\frac{h - x_1}{l} = \frac{k - y_1}{m} = \frac{-(lx_1 + my_1 + n)}{l^2 + m^2}$$

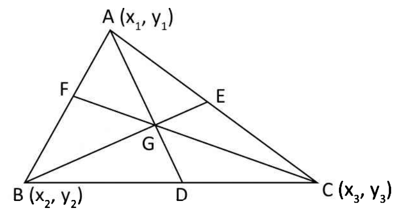
CENTROID OF A TRIANGLE

The point at which the medians of a triangle intersect is called the centroid of the triangle.

Let ABC be a given triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

Since D is the mid-point of BC, its coordinates are $[(x_2 + x_3)/2, (x_2 + x_3)/2]$

Let G (x, y) be a point dividing AD in the ratio 2:1.



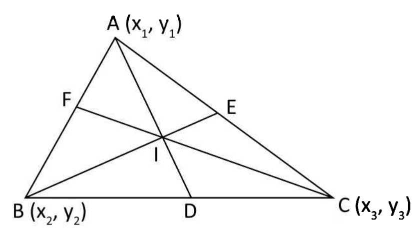
$$\text{Then, } x = \frac{2\left(\frac{x_2 + x_3}{2}\right) + 1.x_1}{(2+1)} = \left(\frac{x_1 + x_2 + x_3}{3}\right)$$

$$\text{and } \quad y = \frac{2\left(\frac{y_2 + y_3}{2}\right) + 1.y_1}{(2+1)} = \left(\frac{y_1 + y_2 + y_3}{3}\right)$$

Similarly, the coordinates of a point which divides BE in the ratio 2:1 as well as those of the point which divides CF in the ratio 2:1 are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

INCENTRE OF A TRIANGLE

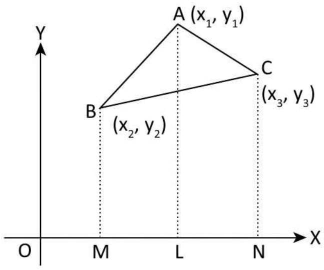


The point at which the bisectors of the angles of a triangle intersect, is called the incentre of the triangle. From geometry, we know that the bisector of an angle of a triangle divides the opposite side in the ratio of length of remaining sides. Hence, the bisectors of the angle of ΔABC are concurrent, and meet at a point, called incentre.

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right)$$

Area of triangle

Let ABC be a given triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.



Then the area of the triangle is given by

$$= \frac{1}{2}\left[x_1\left(y_2 - y_3\right) + x_2\left(y_3 - y_1\right) + x_3\left(y_1 - y_2\right)\right].$$

If we interchange the order of any two vertices of the ΔABC, we obtain a negative value of the area. However, the area shall always be taken to be positive.

EQUATION OF A CURVE

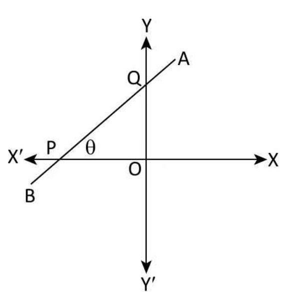
An equation in two variables X and Y with the degree of the equation being equal to or more than two is called the equation of a curve. If the graph of that equation plotted on the XY Cartesian plane, it will give a shape of a curve, and not a straight line

For example, $x^2 + y^2 = 16, y = x^2$.

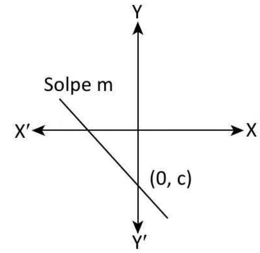
STRAIGHT LINE

Any equation with the degree of equation being one is known as an equation of straight line. General equation of straight line is given by $aX + bY + c = 0$, where X and Y are variables and a, b, c are constants.

Any point lying on this line will satisfy the equation of the line.



If AB is a straight line on the XY plane, then the angle θ which the line makes with the X axis in the anticlockwise direction is called the inclination of the line and tangent of this angle θ ($\tan \theta$) is called the slope of the line AB. It is denoted by 'm'. The lengths OP and OQ are respectively known as the intercepts on X-axis and Y-axis, made by the line.



DIFFERENT FORMS OF REPRESENTING A STRAIGHT LINE

1. Slope- intercept form

   $$y = mx + c$$

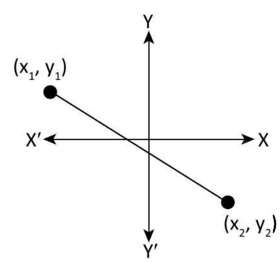
   If 'm' is the slope of the line and 'c' the intercept made by the line on Y axis, the equation is

   $$y = mx + c$$

2. Point-slope form

   If 'm' is the slope of the line and it passes through the point, the equation is (x_1, y_1), then the equation of the line is given by:

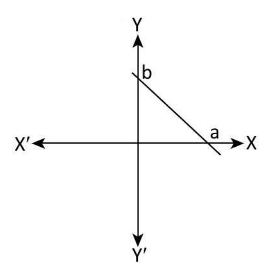
   $$y - y_1 = m\,(x - x_1)$$



3. Two-point form

   If the line passes through two points (X_1, Y_1) and (X_2, Y_2) the equation is

   $$(Y - Y_1) = \frac{Y_2 - Y_1}{X_2 - X_1}(X - X_1)$$



Using Point-slope form and Two-point form, we can find out the formula for slope also. Comparing the two equations, we get m = $\frac{Y_2 - Y_1}{X_2 - X_1}$

4. Slope-intercept form
   If the line makes an intercept of 'a' units on X-axis and b units on Y-axis, then the equation is:

$$\frac{X}{a}+\frac{Y}{b}=1$$

Angle between two intersecting lines

The angle between two lines whose slopes are m_1 and m_2 is given by a formula such that $\tan \theta = \frac{m_1 - m_2}{1+m_1 m_2}$ (where θ is the angle between the lines)

Condition for two straight lines to be parallel

It can be visualized that two straight lines can be parallel only if they make an equal inclination with the X-axis. This will, in turn, ensure that their slopes are equal.

The lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are parallel if and only if $m_1 = m_2$.

Condition for two straight lines to be perpendicular

The lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular if and only if $m_1\, m_2 = -1$.

The equation of a line parallel to a given line $ax + by + c = 0$ will be $ax + by + k = 0$, where k is any constant which can be found by additional information given in the question.

The equation of a line perpendicular to a given line $ax + by + c = 0$ will be $bx - ay + k = 0$, where k is any constant which can be found by additional information given in the question.

Point of intersection of two lines

The coordinates of the point of intersection of the two intersecting lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)$$

However, to obtain the point of intersection, we are required to just solve the equations of the straight lines given as we do in the case of Simultaneous Equations.

Condition of concurrency of three lines

Three lines are said to be concurrent, if they pass through a common point i.e. if they meet at a point. The condition for three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ is $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$

Length of perpendicular

The length of perpendicular (p) from (X_1, Y_1) on the line $AX + BY + C = 0$ is:

$$P = \left|\frac{AX_1 + BY_1 + C}{\sqrt{A^2 + B^2}}\right|$$

Distance between two parallel lines

The distance between two parallel lines $AX + BY + C_1 = 0$ and $AX + BY + C_2 = 0$ is given by

$$\left|\frac{C_1 - C_2}{\sqrt{A^2 + B^2}}\right|$$

Conditions for points to be collinear

If three points A, B, C are co-linear, then any one of the following conditions should be true :

i. Area of triangle $ABC = 0$
ii. Slope of AB = slope of BC = slope of AC
iii. $AB + BC = AC$

Depending upon the points given, we can use any one of three to check if the points are collinear. It should also be mentioned that if one of these conditions is true, then other two will definitely be true.

WORKED OUT PROBLEMS

Example 1 Find the coordinates of the point which divides the line segment joining the point (5, –2) and (9,6) in the ratio 3:1.

Solution Let the required point be (x, y). Then,

$$x = \left(\frac{3\times 9 + 1\times 5}{3+1}\right) = 8$$

and

$$y = \left(\frac{3\times 6 + 1\times(-2)}{3+1}\right) = 4$$

The coordinates of the required point are (8, 4).

Example 2 Find the ratio in which the point (2, y) divides the join of (–4, 3) and (6, 3) and hence find the value of y.

Solution Let the required ratio be k : 1.

Then, $$2 = \frac{6k - 4\times 1}{k+1} \Rightarrow k = \frac{3}{2}$$

∴ The required ratio is $\frac{3}{2}$:1, i.e., 3:2.

Also, $$y = \frac{3\times 3 + 2\times 3}{3+2} = 3$$

When asked for ratio m : n, for convenience we take ratio as m/n : 1 or k : 1.

Example 3 Two vertices of a triangle are (–1, 4) and (5, 2) and its centroid is (0, –3). Find the third vertex.

Solution Let the third vertex be (x, y). Then

$$\frac{x+(-1)+5}{3}=0 \text{ and } \frac{y+4+2}{3}=-3$$

$\therefore$ $x = -4$ and $y = -15$

Hence the third vertex of the triangle is (–4, –15).

Example 4 Find the co-ordinates of incentre of the triangle whose vertices are (4, –2), (–2, 4) and (5, 5).

Solution $a = BC = \sqrt{(5+2)^2+(5-4)^2} = \sqrt{50} = 5\sqrt{2}$

$$b = AC = \sqrt{(5-4)^2+(5+2)^2} = \sqrt{50} = 5\sqrt{2}$$

and $c = AB = \sqrt{(-2-4)^2+(4+2)^2} = \sqrt{72} = \sqrt{2}6$

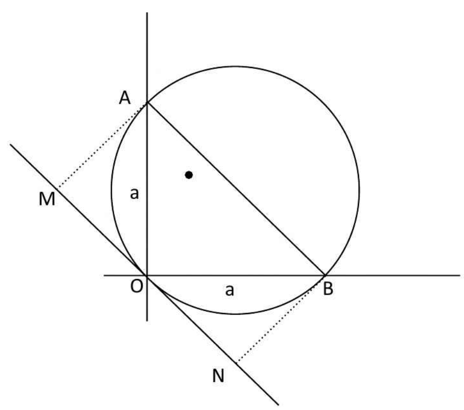
Let (x, y) be the coordinates of incentre of ΔABC, Then,

$$x = \frac{ax_1+bx_2+cx_3}{a+b+c} = \left(\frac{20\sqrt{2}-10\sqrt{2}+30\sqrt{2}}{5\sqrt{2}+5\sqrt{2}+6\sqrt{2}}\right) = \frac{40\sqrt{2}}{16\sqrt{2}} = \frac{5}{2}$$

$\therefore$ The coordinates of the incentre are $\left(\frac{5}{2},\frac{5}{2}\right)$

Example 5 A line makes equal intercepts of length 'a' on the coordinate axes, intersecting the X axis and Y-axis at A and B respectively. A circle is circumscribed about the triangle OAB, where O is the origin of the coordinate system. A tangent is drawn to this circle at the point O, the sum of the perpendicular distances of the vertices A, B and O from this tangent is:

Solution



$$\text{AM + BN + OO} = \frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}+0=\sqrt{2}a$$

Example 6 What is the area of the triangle whose vertices are (4, 4), (3, –16) and (3, –2)?

Solution Let $x_1 = 4, x_2 = 3, x_3 = 3$ and $y_1 = 4, y_2 = -16, y_3 = -2$.

Then area of the EXAMPLE ven triangle

$$=\frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$$

$$=\frac{1}{2}\left[4.\{-16-(-2)\}+3.(-2-4)+3.\{4-(-16)\}\right]$$

$$=\frac{1}{2}\left[4.\{-14+3(-6)\}+3\times 20\right]=\frac{1}{2}[-56-18+60]=-7$$

Since area of a triangle cannot be negative, so area = 7 sq. units.

Example 7 What is the area of the triangle formed by the points (–5, 7), (–4, 5) and (1, –5)?

Solution Let $x_1 = -5, x_2 = -4, x_3 = 1$ and $y_1 = 7, y_2 = 5, y_3 = -5$.

Area of the triangle formed by given points

$$=\frac{1}{2}\left[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right]$$

$$=\frac{1}{2}[(-5)\{5-(-5)\}+(-4)(-5-7)+(7-5)]$$

$$= 0$$

Hence the EXAMPLE ven points are not forming any triangle, rather they are collinear.

Example 8 Find the equation of the straight line which passes through (3,4) and the sum of whose X and Y intercepts is 14.

Solution Let the intercepts made by the line x-axis and y-axis and y-axis be α and $(14-\alpha)$ respectively.

Then, its equation is $\frac{X}{\alpha}+\frac{Y}{14-\alpha}=1$ ...(i)

Since it passes through (3,4), we have:

$$\frac{3}{\alpha}+\frac{4}{14-\alpha}=1 \Rightarrow \alpha^2-13\alpha+42=0 \Rightarrow (\alpha-6)(\alpha-7)=0.$$

$\alpha = 6$ and $\alpha = 7$.

So, the required equation is: $\frac{x}{6}+\frac{y}{8}=1$ or $\frac{x}{7}+\frac{y}{7}=1$ i.e. $4x + 3y = 24$ or $x + y = 7$.

Example 9 What is the equation of a line which passes through the point (–1, 3) and is perpendicular to the straight-line $5x + 3y + 1 = 0$?

Solution The equation of any line perpendicular to the line $5x + 3y + 1 = 0$ is

$3x - 5y + K = 0$ ... (i)

Since the required line passes through the point (–1, 3), we have

$3 \times (-1) - 5 \times 3 + K = 0$, or, $K = 18$

Hence, the required equation is $3x - 4y + 18 = 0$.