GEOMETRY

Geometry, deals with measures and properties of points, lines, surfaces and solids. Here we will be discussing:

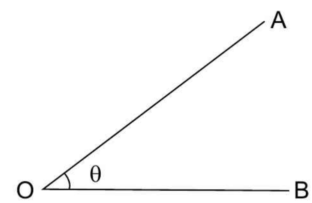
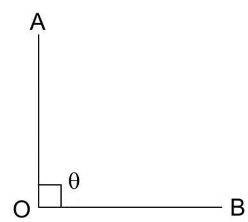
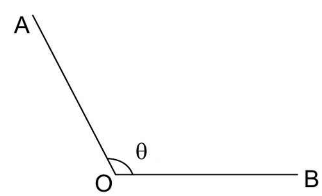
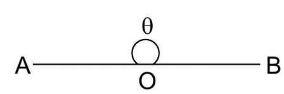
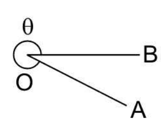
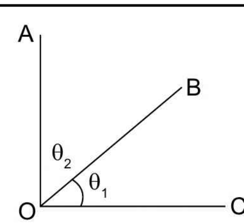
1. Lines and their properties
2. Polygons and their properties
3. Triangles and their properties
4. Quadrilaterals and their properties
5. Circles and their properties

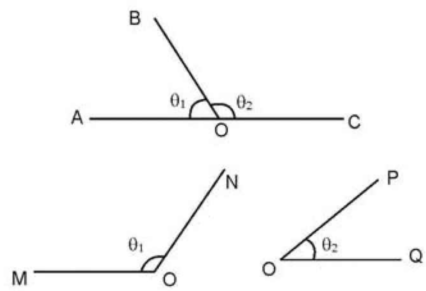
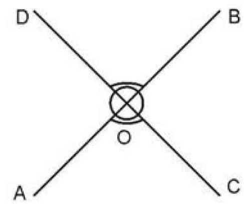
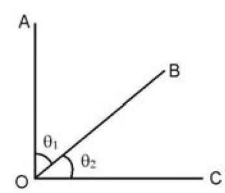
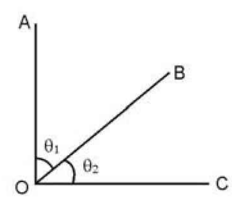
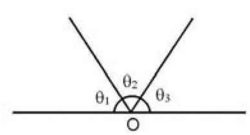
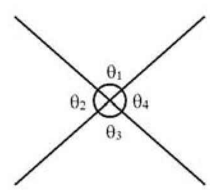
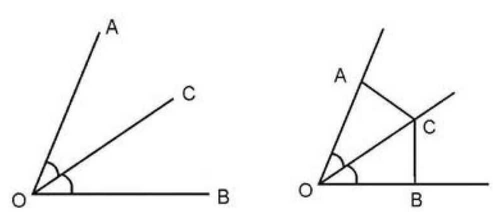
LINES AND THEIR PROPERTIES

A line is a set of points placed together that extends into infinity is both directions.

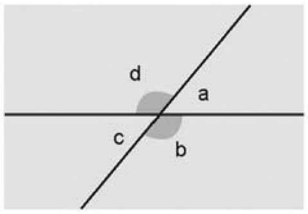
a) Different angles and pairs of angles

Measurement and nomenclature

Types of Angles	*Property*	*Diagram*
Acute Angle	$0° < \theta < 90°$ ($\angle AOB$ is an acute angle)	
Right Angle	$\theta = 90°$	
Obtuse Angle	$90° < \theta < 180°$	
Straight Line	$\theta = 180°$	
Reflex Angle	$180° < \theta < 360°$	
Complementary Angle	$\theta_1 + \theta_2 = 90°$ Two angles whose sum is 90° are complementary to each other	

Supplementary Angle	$\theta 1 + \theta 2 = 180°$ Two angles, whose sum is 180°, are supplementary to each other	
Vertically Opposite Angle	$\angle DOA = \angle BOC$ and $\angle DOB = \angle AOC$	
Adjacent Angles	$\angle AOB$ and $\angle BOC$ are adjacent angles Adjacent angles must have a common side (e.g., OB)	
Linear Pair	$\angle AOB$ and $\angle BOC$ are linear pair angles. One side must be common (e.g., OB) and these two angles must be supplementary.	
Angles on One Side of a Line	$\theta_1 + \theta_2 + \theta_3 = 180°$	
Angles Round the Point	$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 360°$	
Angle Bisector	OC is the angle bisector of $\angle AOB$. i.e., $\angle AOC = \angle BOC = \frac{1}{2} (\angle AOB)$ When a line segment divides an angle equally into two parts, then it is said to be the angle bisector (OC)	 (Angle bisector is equdistant from the two sides of the angle) i.e.,

b) Angles associated with two or more straight lines



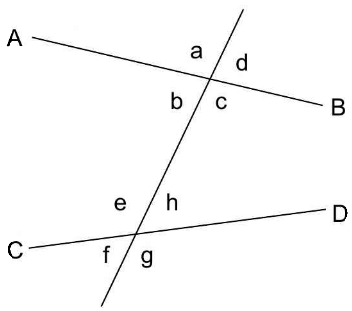
When two straight lines cross each other, ∠d and ∠b are the pair of vertical angles.

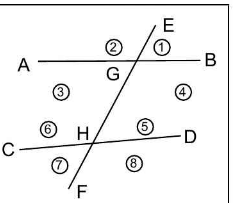
∠a and ∠c are the pair of vertical angles.

Vertical angles are equal in value.

Alternate angles and corresponding angles

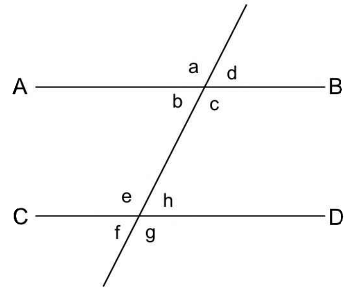
In the figure given below, corresponding angles are ∠a and ∠e, ∠b and ∠f, ∠d and ∠h, ∠c and ∠g. The alternate angles are ∠b and ∠h, ∠c and ∠e.



Corresponding angles	When two lines are intersected by a transversal, then they form four pairs of corre-sponding angles (a) ∠AGE, ∠CHG = (∠2, ∠6) (b) ∠AGH, ∠CHF = (∠3, ∠7) (c) ∠EGB, ∠GHD = (∠1, ∠5) (d) ∠BGH, ∠DHF = (∠4, ∠8)	

c) Angles associated with parallel lines

A line passing through two or more lines in a plane is called a transversal. When a transversal cuts two parallel lines, then the set of all the corresponding angels will be equal and similarly, the set of all the alternate angles will be equal.



In the figure given above, corresponding ∠a = ∠e and corresponding ∠b = ∠f

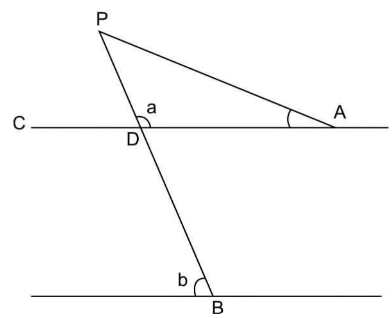
Similarly, alternate ∠b = ∠h and alternate ∠c = ∠e.

Now, ∠b + ∠c = 180°, so ∠b + ∠e = ∠h + ∠c = 180°

So we can conclude that the sum of the angles on one side of the transversal and between the parallel lines will be equal to 180°.

Converse of the above theorem is also true. When a transversal cuts two lines, and if the corresponding angles are equal in size, or if alternate angles are equal in size, then the two lines are parallel.

Example 1 In the figure given below, find the value of ∠b in terms of ∠a.



Solution In the given figure, ∠b = Alternate ∠PDC = 180° − ∠PDA = 180° − a

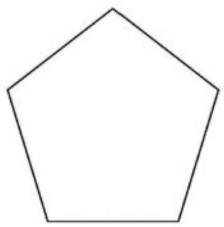
POLYGONS AND THEIR PROPERTIES

Any closed plane figure with *n* sides is known as a polygon. If all the sides and the angles of this polygon are equivalent, the polygon is called a regular polygon. Polygons can be convex or concave. The word "polygon" derives from the Greek *poly*, meaning "many," and *gonia*, meaning "angle." The most familiar type of polygon is the regular polygon, which is a convex polygon with equal side lengths and angles.

The generalization of a polygon into three dimensions is called a *polyhedron*, and into four dimensions is called a *polychoron*.

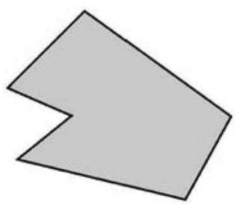
A **convex polygon** is a simple polygon that has the following features:

- Every internal angle is at most 180 degrees.
- Every line segment between the two vertices of the polygon does not go outside the polygon (i.e., it remains inside or on the boundary of the polygon). In other words, all the diagonals of the polygon remain inside its boundary.
- Every triangle is strictly a convex polygon.



Convex Polygon

If a simple polygon is not convex, it is called **concave**. At least one internal angle of a concave polygon is larger than 180°.



Concave Polygon

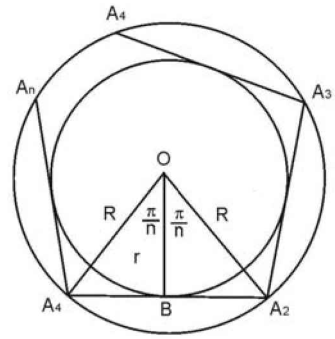
As we can see in the above figure, one of the internal angles is more than 180°.

Here onwards, all the discussion about polygon refer to regular polygons only.

Polygons are named on the basis of the number of sides they have. A list of some of the polygons are given below:

Number of sides	*Name of the polygon*
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon

Area and Perimeter of a Regular Polygon



Given, $A_1\ A_2\ A_3\ A_4 \ldots A_n$ is a with regular polygon, '*n*' sides.

$A_1A_2 = A_2A_3 = A_3A_4 = \ldots. = A_{(n-1)}A_n$ = a units

OB (in radius) = r and $OA_1 = OA_2$ (Circumradius) = R

(i) Perimeter (P) = *na*

(ii) Area $= \frac{na}{2} \times r = \frac{p}{2} \times r$

(iii) Area $= \frac{na}{2}\sqrt{R^2 - \left(\frac{a}{2}\right)^2} = \frac{p}{2}\sqrt{R^2 - \left(\frac{a}{2}\right)^2}$

(iv) Area $= \frac{na^2}{4} \times \cot\left(\frac{\pi}{n}\right)^{\circ}$

(v) Area $= nr^2 \times \tan\left(\frac{\pi}{n}\right)^{\circ}$

(vi) r (in radius) $= \frac{a}{2}\cot\left(\frac{\pi}{n}\right)^{\circ}$

Properties of a Polygon

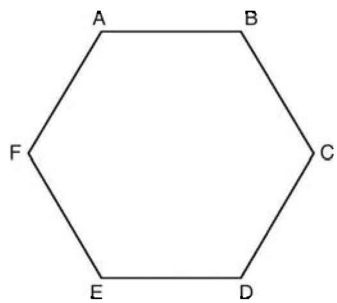
1. Interior Angle + Exterior Angle = 180°
2. The number of diagonals in an *n*-sided polygon = $n(n-3)/2$
3. The sum total of all the exteriors angles of any polygon = 360°
4. The measure of each exterior angle of a regular polygon $= \frac{360°}{n}$
5. The ratio of the sides of a polygon to the diagonals of a polygon is 2 : $(n-3)$
6. The ratio of the interior angle of a regular polygon to its exterior angle is $(n-2) : 2$
7. The sum total of all the interior angles of any polygon $= (2n-4) \times 90°$

So, each interior angle $= \frac{(2n-4)90°}{n}$

Some frequently used polygons

Apart from triangles and quadrilaterals, regular hexagon and regular octagon are also worth mentioning.

Regular hexagon In the figure given below, ABCDEF is a regular hexagon with each side measuring a unit. Point O inside the hexagon is the centre of the hexagon.



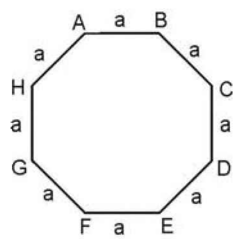
Sum of the interior angles = 720°

Each interior angle = 120°

Each exterior angle = 60°

Area $= \frac{3\sqrt{3}}{2} a^2$

Regular octagon In the figure given below, ABCD-EFGH is a regular octagon with each side measuring 'a' unit.



Sum of the interior angles = 1080°

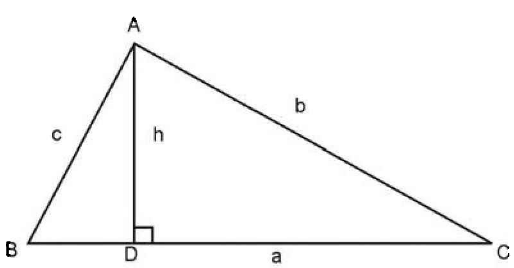
Each interior angle = 135°

Each exterior angle = 45°

Area $= 2a^2(1+\sqrt{2})$

TRIANGLES AND THEIR PROPERTIES

A triangle is a figure enclosed by three sides. In the figure given below ABC is a triangle with sides AB, BC and CA measuring c, a and b units respectively. Line AD represents the height of the triangle corresponding to the side BC and is denoted by h.



In any triangle ABC

Area $= \frac{1}{2} \times \text{BC} \times \text{AD} = \frac{1}{2}\, a \times h$

Properties of a Triangle

- The sum of all the angles of a triangle = 180°
- The sum of lengths of the two sides > Length of the third side
- The difference of any two sides of any triangle < length of the third side
- The area of any triangle can be found by several methods:

  (a) Area of any triangle $= \frac{1}{2} \times$ base $\times$ perpendicular to base from the opposite vertex.

  (b) Area of any triangle $= \sqrt{S(S-a)(S-b)(S-c)}$, where S is the semi-perimeter of the triangle and a, b and c are the sides of a triangle.

  (c) Area of any triangle $= \left(\frac{1}{2}\right) \times$ bc Sin A

  Besides, there are some formulae which we use exclusively in some particular cases.

Example 2 What is the number of distinct triangles with integral valued sides and perimeter as 14?

(a) 6 (b) 5 (c) 4 (d) 3

Solution The sum of the lengths of the two sides > the length of the third side

So, the maximum length of any particular side can be 6 units.

Now if a = 6, then b + c = 8, so the possible sets are (6, 6, 2), (6, 5, 3) and (6, 4, 4)

If a = 5, then b + c = 9, so the possible set is (5, 5, 4)

So, the number of distinct triangles = 4

CLASSIFICATION OF TRIANGLES

Based Upon Sides

1. Scalene triangle

A triangle whose all sides are of a different length is a scalene triangle.

Area $= \sqrt{s(s-a)(s-b)(s-c)}$, where

S (semi perimeter) $= \frac{a+b+c}{2}$

Example 3 What is the area of the triangle with side lengths 4 units, 5 units and 10 units?

Solution This triangle is not possible. (Since the sum of lengths of the two sides > Length of the third side)

2. Isosceles triangle

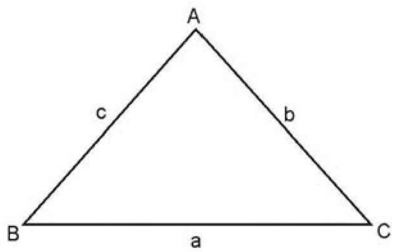
A triangle whose two sides are of an equal length is an isosceles triangle.

$$\text{Height} = \sqrt{\frac{4a^2 - b^2}{2}}$$

$$\text{Area} = \frac{b}{4}\sqrt{4a^2 - b^2}$$

3. Equilateral triangle

A triangle whose all sides are of an equal length is called an equilateral triangle.



In any equilateral triangle, all the three sides are of an equal length, so $a = b = c$

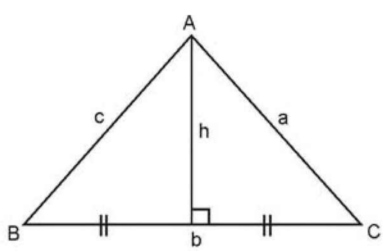
$$\text{Height} = (\text{side})\ \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}a$$

$$\text{Area} = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}a^2$$

Based Upon Angles

1. Right-angled triangle

A triangle whose one angle is of 90° is called a right-angled triangle. The side opposite to the right angle is called the hypotenuse.



$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{perpendicular}$$

Pythagoras Theorem

Pythagoras theorem is applicable in case of right-angled triangle. It says that, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

$(\text{Hypotenuse})^2 = (\text{Base})^2 + (\text{Perpendicular})^2$

$a^2 + b^2 = c^2$

For more proofs of Pythagoras theorem, go to www.cut-the-knot.org/pythagoras/index.shtml

The smallest example is $a = 3$, $b = 4$ and $c = 5$. You can check that

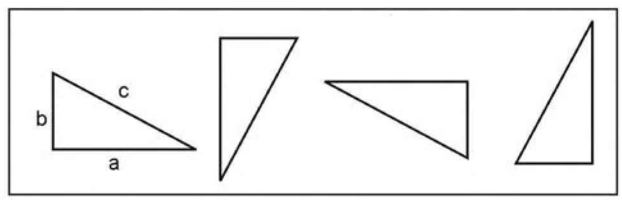
$3^2 + 4^2 = 9 + 16 = 25 = 5^2$.

Sometimes we use the notation (a, b, c) to denote such a triple.

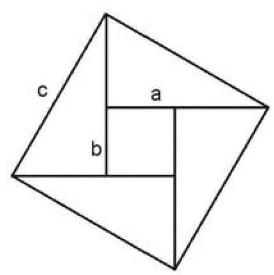
Notice that the greatest common divisor of the three numbers 3, 4 and 5 is 1. Pythagorean triples with this property are called *primitive*.

Proofs of Pythagoras theorem

Proof 1



Now we start with four copies of the same triangle. Three of these have been rotated at 90°, 180°, and 270°, respectively. Each has the area ab/2. Let's put them together without additional rotations so that they form a square with side c.

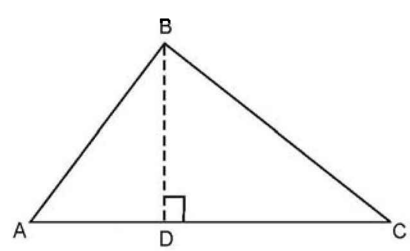


The square has a square hole with the side $(a - b)$. By summing up its area $(a - b)^2$ and 2ab, the area of the four triangles (4·ab/2), we get

$C^2 = (a-b)^2 + 2ab = a^2 + b^2$. QED

Proof 2 ABC is a right-angled triangle at B

To Prove: $AC^2 = AB^2 + BC^2$



Construction: Draw BD ⊥ AC

Proof: ΔADB ~ΔABC (Property 8.5)

$$\therefore \frac{AD}{AB} = \frac{AB}{AC} \text{ (Sides are proportional)}$$

Or $AB^2 = AD \times AC$ (1)

Also, $\Delta CDB \sim \Delta CBA$

$$\therefore \frac{CD}{BC} = \frac{BC}{CA}$$

$$\therefore \frac{BC^2}{AC^2} = \frac{BD \times BA}{AB \times AD} \text{ (Sides are proportional)}$$

or $BC^2 = CD \times CA$

Adding (1) and (2)

$AB^2 + BC^2 = AD \times AC + CD \times AC$

$= AC\,[AD + CD]$

$= AC \times AC = AC^2$ QED.

Pythagorean triplets

A *Pythagorean triplet* is a set of three positive whole numbers *a*, *b* and *c* that are the lengths of the sides of a right triangle.

$a^2 + b^2 = c^2$

It is noteworthy to see here that all of *a*, *b* and *c* cannot be odd simultaneously. Either of *a* or *b* has to be even and *c* can be odd or even.

The various possibilities for *a*, *b* and *c* are tabled below:

a	*b*	*c*
Odd	Odd	Even
Even	Odd	Odd
Odd	Even	Odd
Even	Even	Even

Some Pythagoras triplets are:

3	4	5	$(3^2 + 4^2 = 5^2)$
5	12	13	$(5^2 + 12^2 = 13^2)$
7	24	25	$(7^2 + 24^2 = 25^2)$
8	15	17	$(8^2 + 15^2 = 17^2)$
9	40	41	$(9^2 + 40^2 = 41^2)$
11	60	61	$(11^2 + 60^2 = 61^2)$
20	21	29	$(20^2 + 21^2 = 29^2)$

Note If each term of any pythagorean triplet is multiplied or divided by a constant (say P, P > 0) then the triplet so obtained will also be a pythagorean triplet. This is because if $a^2 + b^2 = c^2$, then $(Pa)^2 + (Pb)^2 = (Pc)^2$, where P > 0.

For example, 3×2 4×2 5×2 gives

6 8 10 $(6^2 + 8^2 = 10^2)$

Using Pythagoras theorem to determine the nature of triangle

If $c^2 = a^2 + b^2$, then the triangle is right-angled triangle.
If $c^2 > a^2 + b^2$, then the triangle is an obtuse-angled triangle.
If $c^2 < a^2 + b^2$, then the triangle is an acute-angled triangle.

Mechanism to derive a Pythagorean triplet If the length of the smallest side is odd, assume the length of the smallest side = 5

Step 1 Take the square of 5 (length of the smallest side) = 25

Step 2 Break 25 into two parts P and Q, where P – Q = 1. In this case, P = 13 and Q = 12. Now these two parts P and Q along with the smallest side constitute pythagorean triplet.

However, there is another general formula for finding out all the primitive pythagorean triplets:

$a = r^2 - s^2$,
$b = 2\,rs$,
$c = r^2 + s^2$,
$r > s > 0$ are whole numbers,
$r - s$ is odd, and
The greatest common divisor of *r* and *s* is 1.

Table of small primitive Pythagorean triplets Here is a table of the first few primitive Pythagorean triplets:

r	*s*	*a*	*b*	*c*
2	1	3	4	5
3	2	5	12	13
4	1	15	8	17
4	3	7	24	25
5	2	21	20	29
5	4	9	40	41
6	1	35	12	37
6	5	11	60	61
7	2	45	28	53

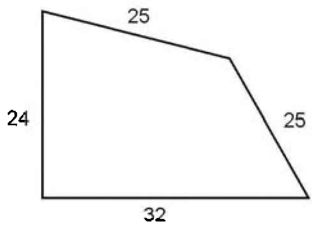
Perimeter, area, inradius and shortest side

The perimeter P and area K of a Pythagorean triple triangle are given by

$P = a + b + c = 2r(r + s)\,d.$

$K = ab/2 = rs\,(r^2 - s^2)\,d^2.$

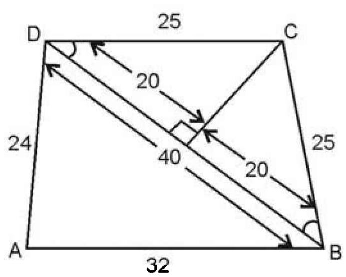
Example 4 Two sides of a plot measure 32 m and 24 m and the angle between them is a right angle. The other two sides measure 25 m each and the other three angles are not right angles.



What is the area of the plot (in m^2)?

(a) 768 (b) 534 (c) 696 (d) 684

Solution The figure given above can be seen as



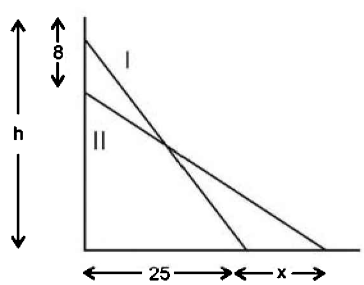
Since ABD is a right-angled triangle, so it will satisfy the Pythagoras theorem. And the triplet used here is – 3(×8), 4(×8) and 5(×8). Similarly the other part of the figure can also be bifurcated by drawing a perpendicular from C on BD.

So, the area of the plot is:

Area (ΔABD) + Area (ΔCBD) $= \frac{1}{2} \times 24 \times 32 + 2 \times (1/2 \times 20 \times 15) = 684\ m^2$

Example 5 A ladder of length 65 m is resting against a wall. If it slips 8 m down the wall, then its bottom will move away from the wall by N m. If it was initially 25 m away from it, what is the value of x?

(a) 60 m (b) 39 m (c) 14 m (d) 52 m



Solution Using Pythagorean triplets, (5, 12, 13),

$\Rightarrow$ h = 60

After it has slipped by 8 m, the new height = 52 m, and the length of the ladder = 65 m.

So $25 + x = 39$ (3, 4, 5 triplet)

$\Rightarrow x = 14$ m

2. Obtuse-angled triangle

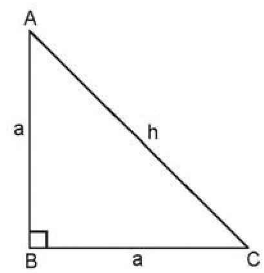
If one of the angles of the triangle is more than 90°, then the triangle is known as an Obtuse angled triangle. Obviously in this case, rest of the two angles will be less than 90°.

3. Acute-angled triangle

If all the angles of the triangle are less than 90°, then the triangle is known as acute angled triangle.

4. Isosceles right-angled triangle

A right angled triangle whose two sides containing the right angle are equal in length, is an isosceles right triangle.



In this case, Hypotenuse (h) $= a\sqrt{2}$

Perimeter $= 2a + h = 2a + a\sqrt{2}$

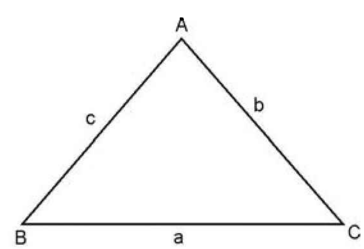
$= a\sqrt{2}(\sqrt{2} + a)$

$= h(1 + \sqrt{2})$

= Hypotenuse $(1 + \sqrt{2})$

Trigonometric formulae

In any Δ ABC,



Area of a triangle

Area of Δ $= \frac{1}{2}$ bc sin A, where ∠A = BAC

Area of Δ $= \frac{1}{2}$ ac Sin B, where ∠B = ABC

Area of Δ $= \frac{1}{2}$ ab Sin C, where ∠C = ACB

Cosine Rule and Sine Rule

In any Δ, we have six quantities namely the three angles and the three sides. Using the following rules, we can find any of the three quantities if we are provided with the remaining three quantities.

$$\text{Cos } A = \frac{b^2 + c^2 + a^2}{2bc}$$

$$\text{Cos } B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\text{Cos } C = \frac{a^2 + b^2 - c^2}{2ab}$$

Sine Rule $\frac{a}{\text{Sin } A} = \frac{b}{\text{Sin } B} = \frac{c}{\text{Sin } C}$

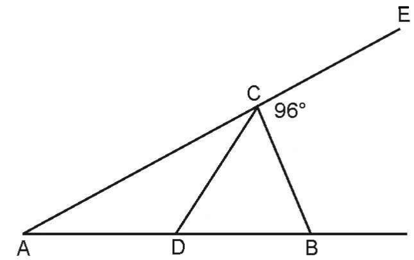
Sine and Cosine formulae are particularly more important in cases where we have one side and two angles out of the three angles of the triangle and we have to find out the value of all the sides and angles.

$$\frac{AB}{\sin\angle ACB} = \frac{BC}{\sin\angle BAC} = \frac{AC}{\sin\angle ABC} = 2R$$

Two important points

- The side opposite to the largest angle will be the largest.
- The side opposite to the smallest angle will be the smallest.

Example 6 In the figure given below, AD = CD = BC. What is the value of ∠CDB?

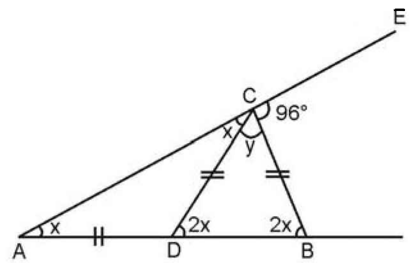


Solution $x + y = 180 - 96 = 84°$ (i)

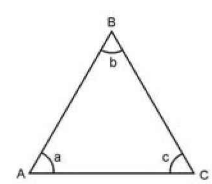
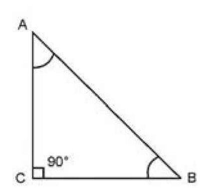
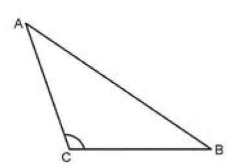
In ΔCDB, $4x + y = 180$ (ii)

Solving (i) and (ii), $x = 32°$

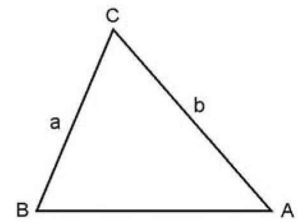
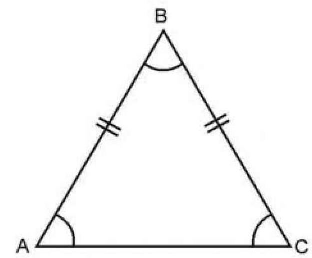
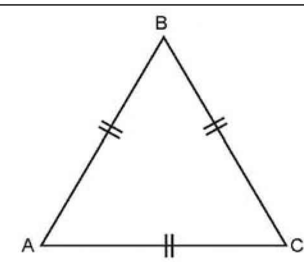
So, $2x = 64°$



Summarizing the Above Classification

Types of Triangles	*Property/Definition*	*Diagram*
Acute-angle Triangle	Each angle of a triangle is less than 90°, i.e., a < 90°, b < 90°, c < 90°	 {∠a, ∠b, ∠c} < 90°
Right-angled Triangle	If one of the angles is equal to 90°, then it is called a right-angled triangle. The rest two angles are complementary to each other.	 ∠C = 90°
Obtuse-angle Triangle	If one of the angles is obtuse (i.e., greater than 90°), then it is called an obtuse-angle triangle	 ∠C > 90°

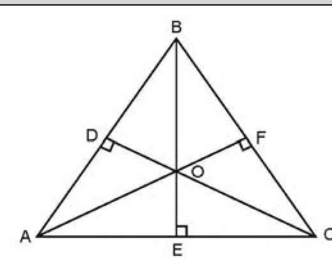
(b) According to the length of sides

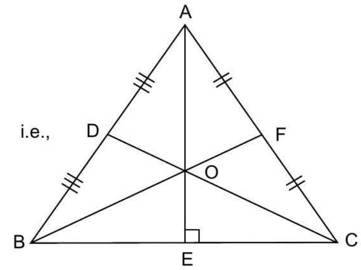
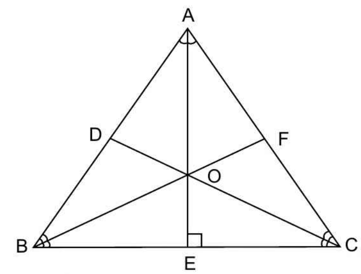
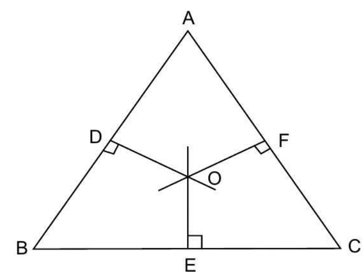
Types of Triangles	*Property/Definition*	*Diagram*
Scalene Triangle	A triangle in which none of the three sides are equal is called a scalene triangle (all the three angles are also different).	 $a \neq b \neq c$
Isosceles Triangle	A triangle in which two sides are equal is called an isosceles triangle. In this triangle, the angles opposite to the congruent sides are also equal.	 AB = AC $\angle B = \angle C$
Equilateral Triangle	A triangle in which all the three sides are equal is called an equilateral triangle. In this triangle each angle is congruent and equal to 60°.	 AB = BC = AC $\angle A = \angle B = \angle C = 60°$

Points of a Triangle

Before we move ahead to discuss different points inside a triangle, we need to be very clear about some of the basic definitions.

Basic Definitions

Nomenclature	*Property/Definition*	*Diagram*
Altitude (or height)	The perpendicular drawn from the opposite vertex of a side in a triangle is called an altitude of the triangle. There are three altitudes in a triangle.	 AF, CD and BE are the altitudes

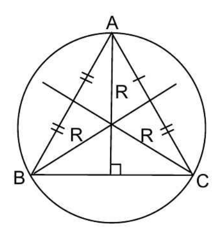
Nomenclature	*Property/Definition*	*Diagram*
Median	The line segment joining the mid-point of a side to the vertex opposite to the side is called a median. There are three medians in a triangle. A median bisects the area of the triangle. Area (ABE) = Area (AEC) $= \frac{1}{2}$ Area (ΔABC) etc.	 i.e., AE, CD and BF are the medians (BE = CE = AD = BD, AF = CF)
Angle bisector	A line segment which originates from a vertex and bisects the same angle is called an angle bisector. ($\angle$BAE = $\angle$CAE $= \frac{1}{2}$ $\angle$BAC) etc.	 AE, CD and BF are the angle bisectors
Perpendicular bisector	A line segment which bisects a side perpendicularly (i.e., at right angle) is called a perpendicular bisector of a side of triangle. All points on the perpendicular bisector of a line are equidistant from the ends of the line.	 DO, EO and FO are the perpendicular bisectors

Circumcentre

Circumcentre is the point of intersection of the three perpendicular bisectors of a triangle. The circumcentre of a triangle is equidistant from its vertices and the distance of the circumcentre from each of the three vertices is called circumradius (R) of the triangle. These perpendicular bisectors are different from altitudes, which are perpendiculars but not necessarily bisectors of the side.

The circle drawn with the circumcentre as the centre and circumradius as the radius is called the circumcircle of the triangle and it passes through all the three vertices of the triangle.

The circumcentre of a right-angled triangle is the mid-point of the hypotenuse of a right- angled triangle.



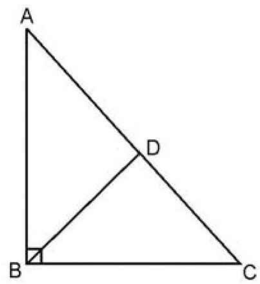
AB = c, BC = a, AC = b

The process to find the circumradius (R) For any triangle $R = \frac{abc}{4A}$, where a, b and c are the three sides and A = area of a triangle.

For equilateral triangle $R = \frac{\text{Side}}{\sqrt{3}}$

Positioning of the circumcentre

- If the triangle is acute-angled triangle, then the circumcentre will lie inside the triangle.
- If the triangle is obtuse-angled triangle, then the circumcentre will lie outside the triangle.
- If the triangle is a right-angled triangle, then the circumcentre will lie on the mid-point of the hypotenuse. This can be seen through the following diagram:

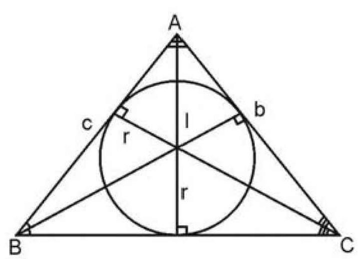


Here D is the circumcentre. So, AD = CD = BD

Incentre

Incentre is the point of intersection of the internal bisectors of the three angles of a triangle. The incentre is equidistant from the three sides of the triangle i.e., the perpendiculars drawn from the incentre to the three sides are equal in length and are called the inradius of the triangle.

The circle drawn with incentre as the centre and inradius as the radius is called the incircle of the triangle and it touches all the three sides from the inside.



AB = c, BC = a, CA = b

To find inradius (r)

For any triangle $r = \frac{A}{S}$, where

A = Area of triangle and

S = Semi-perimeter of the triangle $\frac{(a+b+c)}{2}$

For equilateral triangle $r = \frac{\text{side}}{2\sqrt{3}}$

$\angle BIC = 90° + \angle A/2$

Important derivation – In a right-angled triangle, Inradius = Semi-perimeter – Length of Hypotenuse.

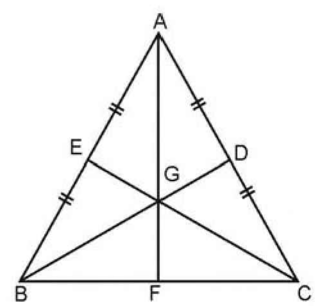
Euler's formula for inradius and circumradius of a triangle Let O and I be the Circumcentre and Incentre of a triangle with circumradius *R* and inradius *r*. Let *d* be the distance between O and I. Then

$d^2 = R(R - 2r)$

From this theorem, we obtain the inequality $r \geq 2r$. This is known as Euler's Inequality.

Centroid

Centroid is the point of intersection of the three medians of a triangle. The centroid divides each of the medians in the ratio 2 : 1, the part of the median towards the vertex being twice in length to the part towards the side.



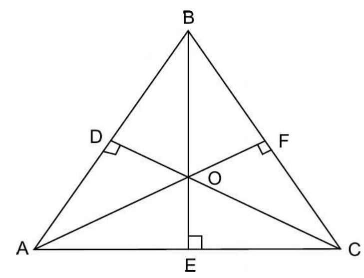
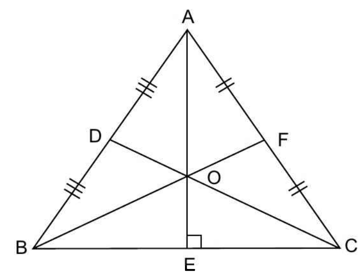
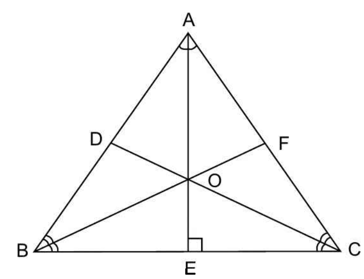
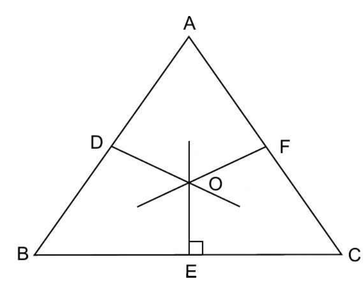
$$\frac{AG}{GF} = \frac{BG}{GD} = \frac{CD}{GE} = \frac{2}{1}$$

Median divides the triangle into two equal parts of the same area.

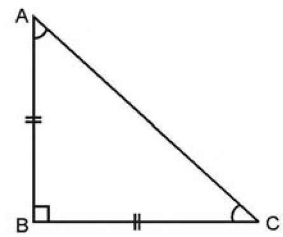
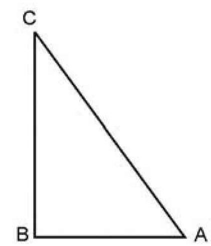
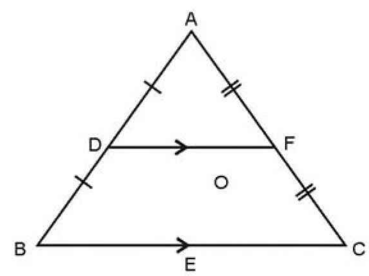
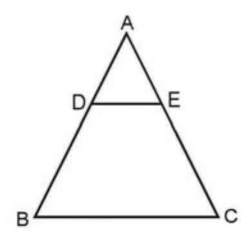
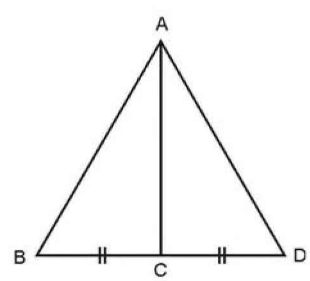
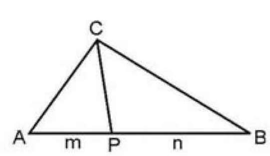
Orthocentre

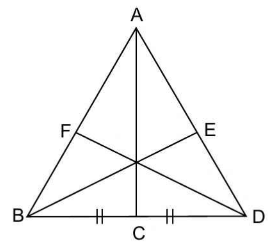
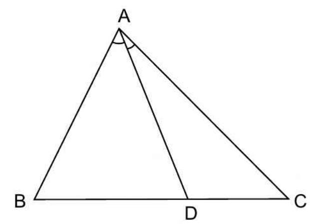
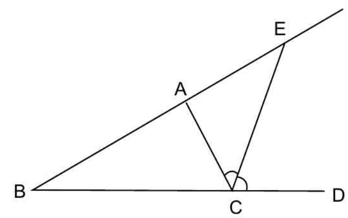
The point of concurrency of the altitudes is known as the orthocentre.

Summarizing the above discussion regarding the points of the triangle:

Orthocentre	The point of intersection of the three altitudes of the triangle is known as the orthocenter. $\angle BOC = 190 - \angle A$ $\angle COA = 190 - \angle B$ $\angle AOB = 180 - \angle B$	 'O' is the orthocenter
Centroid	The point of intersection of the three medians of a triangle is called the centroid. A centroid divides each median in the ratio 2 : 1 (vertex: base) $\frac{AO}{OE} = \frac{CO}{OD} = \frac{BO}{OF} = \frac{2}{1}$	 'O' is the centroid
Incentre	The point of intersection of the angle bisectors of a triangle is known as the incentre. Incentre O is the always equidistant from all three sides i.e., the perpendicular distance between the sides.	 'O' is the incentre
Circumcentre	The point of intersection of the perpendicular bisectors of the sides of a triangle is called the circumcentre. OA = OB = OC = (circum radius) Circumcentre O is always equidistant from all the three vertices A, B and C Perpendicular bisectors need not be originating from the vertices.	 'O' is the incentre

Important Theorems Related to Triangle

Theorem	*Statement/Explanation*	*Diagram*
$45° – 45° – 90°$	If the angles of a triangle are 45°, 45° and 90°, then the hypotenuse (i.e., longest side) is $\sqrt{2}$ times of any smaller side. Excluding hypotenuse rest two sides are equal. i.e., AB = BC and AC $= \sqrt{2}$ AB $= \sqrt{2}$ BC AB : BC : AC = 1 : 1 : $\sqrt{2}$	 $\angle A = 45°$ $\angle B = 90°$ $\angle C = 45°$
$30° – 60° – 90°$	If the angles of a triangle are 30°, 60° and 90°, then the sides opposite to 30° angle is half of the hypotenuse and the side opposite to 60° is $\frac{\sqrt{3}}{2}$ times the hypotenuse, e.g., AB $= \frac{AC}{2}$ and $= \frac{\sqrt{3}}{2}$ AC AB : BC: AC = 1: $\sqrt{3} : 2$	 $\angle C = 30°$, $\angle B = 90°$, $\angle A = 60°$
Basic Proportionality Theorem (BPT) or	Any line parallel to one side of a triangle divides the other two sides proportionally. So if DE is drawn parallel to BC, it would divide sides AB and AC proportionally i.e $\frac{AD}{DB} = \frac{AF}{FC}$ or $\frac{AD}{AB} = \frac{AF}{AC}$ $\frac{AD}{DF} = \frac{AB}{BC}$.	
Mid-point Theorem	Any line joining the mid-points of two adjacent sides of a triangle are joined by a line segment, then this segment is parallel to the third side, i.e., if AD = BD and AE = CE then DE‖BC.	
Apollonius' Theorem	In a triangle, the sum of the squares of any two sides of a triangle is equal to twice the sum of the square of the median to the third side and square of half the third side. i.e., $AB^2 + AD^2 = 2 (AC^2 + BC^2)$	
Stewarts Theorem/ Generalization of Apollonius Theorem	If length of AP = m and PB = n, then $m \times CB^2 + n \times AC^2 = (m + n) PC^2 + mn (m + n)$ Also understand that *m* and *n* here are length of segments, and not their ratio.	

Theorem	*Statement/Explanation*	*Diagram*
Extension of Apollonius' Theorem	In the given ΔABC, AC, BE and DF are medians. 3 (Sum of squares of sides) = 4 (Sum of squares of medians) 3 ($AB^2 + AD^2 + DB^2$) = 4 ($AC^2 + EB^2 + FD^2$)	
Interior Angle Bisector Theorem	In a triangle the angle bisector of an angle divides the opposite side to the angle in the ratio of the remaining two sides. i.e., $\frac{BD}{CD} = \frac{AB}{AC}$ and BD × AC − CD × AB = AD^2	
Exterior Angle Bisector Theorem	In a triangle the angle bisector of any exterior angle of a triangle divides the side opposite to the external angle in the ratio of the remaining two sides i.e., $\frac{BE}{AE} = \frac{BC}{AC}$	

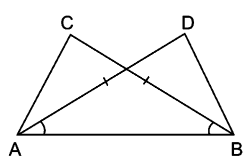
Congruency of Triangles

Two figures are said to be congruent if, when placed one over the other, they completely overlap each other. They would have the same shape, the same area and will be identical in all respects.

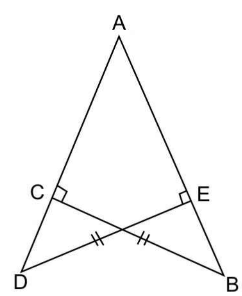
So, we can say that all congruent triangles are similar triangles, but vice-versa is not always true.

Rules for two triangles to be congruent

1. S – S – S
   If in any two triangles, each side of one triangle is equal to a side of the other triangle, the two triangles are congruent. This rule is S – S – S rule.
2. S – A – S
   In Δ ABC and Δ ABD,
   AB = AB (common side)
   ∠ABC = ∠ BAD (given)
   BC = AD (given)



   Thus by rule S – A – S the two triangles are congruent. This rule holds true, when the angles that are equal have to be included between the two equal sides (i.e., the angle should be formed between the two sides that are equal).
3. A – S – A
   In Δ ABC and Δ ADE,
   ∠ACB = ∠ AED (given)
   ∠ BAC = ∠ DAE (common angle)
   BC = DE (given)
   Thus by rule A – S – A the two triangles are congruent. For this rule, the side need not be the included side.



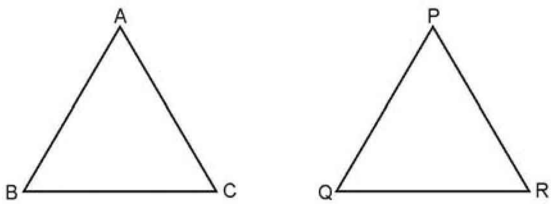
   A – S – A can be written as A – A – S or S – A – A also.
4. R – H – S
   This rule is applicable only for right-angled triangles.
   If two right-angled triangles have their hypotenuse and one of the sides as same, then the triangles will be congruent.

Similarity of the Triangles

If we take two maps of India of different sizes (breadths and lengths), then the map of all the 28 states of India will cover proportionally the same percentage area in both the maps.

Lets see this in geometry:

Criteria for similarity of two triangles



Two triangles are similar if (i) their corresponding angles are equal and/or (ii) their corresponding sides are in the same ratio. That is, if in two triangles, ABC and PQR, (i) $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$, and/or (ii) $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$, the two triangles are similar.

All regular polygons of the same number of sides such as equilateral triangles or squares, are similar. In particular, all circles are also similar.

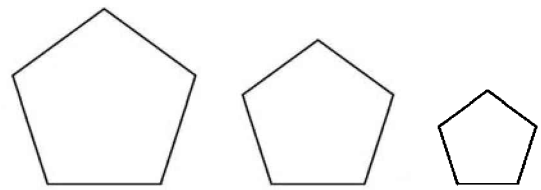
Theorems for Similarity

1. If in two triangles, the corresponding angles are equal, then their corresponding sides will also be proportional (i.e., in the same ratio). Thus the two triangles are similar.
   This property is referred to as the AAA similarity criterion for two triangles.
   Corollary If two angles of a triangle are respectively equal to two angles of another triangle, then the two triangles are similar. This is referred to as the AA similarity criterion for the two triangles. It is true due to the fact that if two angles of one triangle are equal to the two angles of another triangle, then the third angle of both the triangles will automatically be the same.
2. If the corresponding sides of two triangles are proportional (i.e., in the same ratio), their corresponding angles will also be equal and so the triangles are similar. This property is referred to as the SSS similarity criterion for the two triangles.
3. If one angle of a triangle is equal to one angle of the other and the sides including these angles are proportional, then the triangles are similar. This property is referred to as the SAS similarity criterion of the two triangles.
4. The ratio of the areas of the two similar triangles is equal to the ratio of the squares of their corresponding sides.
5. If a perpendicular is drawn from the vertex of the right angle of a right angled triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to each other.

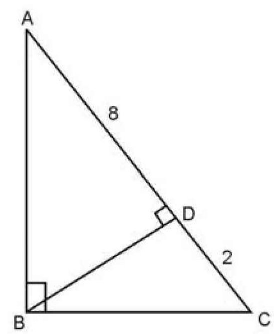
Similar Polygons

Two polygons of the same number of sides are similar, if

(i) Their corresponding angles are equal (i.e., they are equiangular) and
(ii) Their corresponding sides are in the same ratio (or proportional).
This can be seen in the figures given below:



Example 7 ΔABC is a right angled triangle $BD \perp AC$. If AD = 8 cm and DC = 2 cm, then BD = ?



(a) 4 cm	(b) 4.5 cm
(c) 5 cm	(d) Cannot be determined

Solution $\Delta ADB \sim \Delta BDC$

$\therefore \frac{AD}{BD} = \frac{BD}{DC}$

$\therefore BD^2 = AD \times DC = 8 \times 2$

$\therefore BD^2 = 16$

$\therefore BD = 4$ cm

Important Result of this question $BD^2 = AD \times DC$ can be used as a standard result also.

Example 8 Circles with radii 3, 4 and 5 units touch each other externally. If P is the point of intersection of the tangents to these circles at their point of contact, find the distance of P from the point of contacts of the circles.

Solution Let A, B and C be the centres of the three circles. So, the point P will be the incentre of DABC and distance of P from the point of contacts of the circles will be the inradius (r).

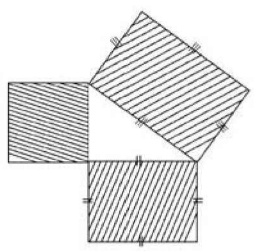
So, $r = \frac{A}{S}$

Sides of DABC will be 7 units, 8 units and 9 units.

So, $r = \sqrt{5}$

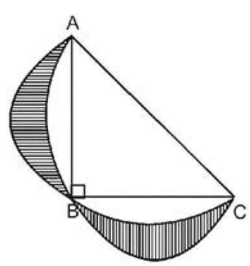
Some interesting facts

1. If we draw regular polygons on all the sides of a right angled triangle, taking the sides of the triangle as one of the sides of the figures, then the area of the shaded portion is equal to twice the area of the figure on the hypotenuse. This will also hold true for semi-circles whose diameters will be the side of the right-angled triangle.

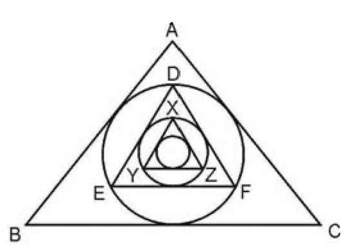


Area of the shaded region = 2 (area of figure on hypotenuse)

2. In the figure below, two semi-circles are drawn with diameters equal to the sides of the right angled triangle. The area of the shaded region (the crescents) is equal to the area of the right-angled triangle. Area of the shaded region (is equal to) the area of the right angled triangle.

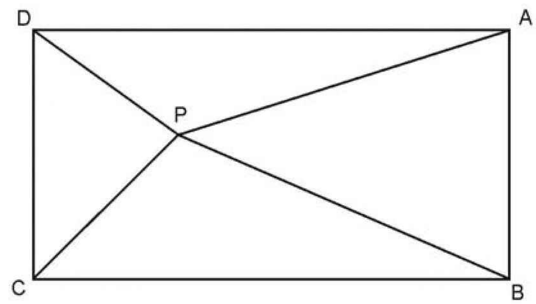


3. In the figure given below all triangles are equilateral triangles and circles are inscribed in these triangles. If the side of triangle ABC = a, then the side of triangle DEF $= \frac{a}{2}$ and the side of triangle $xyz = \frac{a}{4}$.



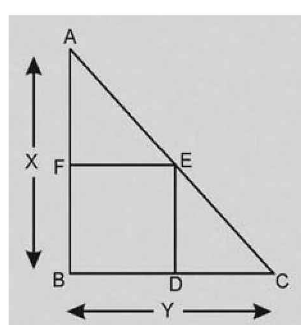
In other words, we can say in order to obtain the side of the next inner triangle divide the side of the immediate outer triangle by 2. The same algorithm holds true for the inscribed circles.

4. In the figure given below, if P is any point inside rectangle ABCD, then $PA^2 + PC^2 = PB^2 + PD^2$

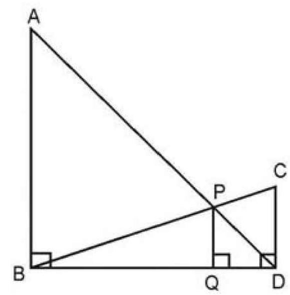


Some Important Points

1. (a) In an acute angled triangle, the circumcentre lies inside the triangle.
   (b) In a right-angled triangle, the circumcentre lies on the middle point of the hypotenuse.
   (c) In an obtuse angled triangle, the circum centre lies outside the triangle.
2. (a) In an acute angled triangle, the orthocenter lies inside the triangle.
   (b) In a right-angled triangle, the orthocenter lies on the vertex is where the right angle is formed i.e., the vertex opposite to the hy potenuse.
   (c) In an obtuse angled triangle, the orthocenter lies outside the triangle.
3. In a right angled triangle the length of the median drawn to the hypotenuse is equal to half the hypotenuse. This median is equal to the circumradius (R) of the right angled triangle.
4. In the figure given below triangle ABC is a right angle triangle, right angled at B. Side AB measures x units and BC measures y units. If a square (BDEF) the maximum area is inscribed in the triangle as shown, below then the side of the square is equal to $\frac{xy}{x+y}$.



Example 9 In the figure given below, $\angle ABD = \angle CDB = \angle PQD = 90°$. If AB:CD = 3:1, then what is the ratio of CD: PQ?

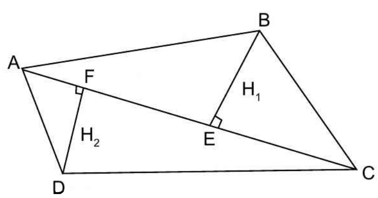


Solution Using the proportionality theorem, AB/PQ = BD/QD and PQ/CD = BQ/BD. Multiplying both these equations, we get AB/CD = BQ/QD = 3:1

Hence, CD/PQ = BD/BQ = 4 : 3.

QUADRILATERALS AND THEIR PROPERTIES

A quadrilateral is a figure bounded by four sides. In the figure given below ABCD is a quadrilateral. Line AC is the diagonal of the quadrilateral (denoted by d) and BE and DF are the heights of the triangles ABC and ADC respectively (denoted by h_1 and h_2).



AC = d, BE = h, and DE = h_2

(i) Area $= \frac{1}{2} \times$ one diagonal × (sum of perpendiculars to the diagonal from the opposite vertexes) $= \frac{1}{2}$ d $(h_1 + h_2)$

(ii) Area $= \frac{1}{2} \times$ product of diagonals × sine of the angle between them

(iii) Area of the cyclic quadrilateral $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$ where *a*, *b*, *c* and *d* are the sides of quadrilateral and *s* = semi-perimeter $= \frac{a+b+c+d}{2}$

(iv) *Brahmagupta's formula* For any quadrilateral with sides of length *a*, *b*, *c* and *d*, the area *A* is given by

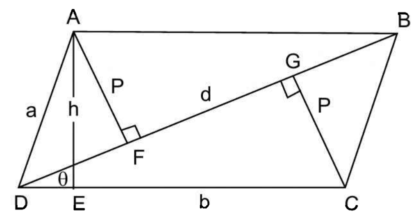
$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2 \frac{1}{2}(A+B)}$$

Where $s = \frac{a+b+c+d}{2}$ is known as the semi-perimeter, A is the angle between sides *a* and *d*, and B is the angle between the sides *b* and *c*.

Different Types of Quadrilaterals

Parallelogram

A parallelogram is a quadrilateral when its opposite sides are equal and parallel. The diagonals of a parallelogram bisect each other.



Given: AD = BC = a and AB = DC = b

BD = d

AF (height of ΔABD) = CG (height of ΔCBD) and AE = height of the parallelogram = h

$\angle ADC = \theta$

(i) Area = base × height

(ii) Area = (any diagonal) × (perpendicular distance to the diagonal from the opposite vertex)

(iii) Area = (product of adjacent sides) × (sine of the angle between them) Area = AB sin θ

(iv) Area $= 2\sqrt{s(s-a)(s-b)(s-d)}$, where a and b are the adjacent sides and d is the diagonal.

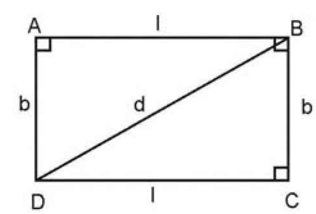
(v) $AC^2 + BD^2 = 2(AB^2 + BC^2)$

(vi) The parallelogram that is inscribed in a circle is a rectangle.

(vii) The parallelogram that is circumscribed about a circle is a rhombus.

(viii) A parallelogram is a rectangle if is diagonals are equal.

Rectangle



A rectangle is a quadrilateral when its opposite sides are equal and each internal angle equals 90°. The diagonals of a rectangle are equal and bisect each other.

Given: AD = BC = b and AB = DC = l, BD = d

(i) Area = length × breadth Area = lb

(ii) Perimeter = 2 (length + breadth) Perimeter = 2 (l + b)

(iii) Diagonal^2 = length^2 + breadth^2 (Pythagoras Theorem)

$d^2 = l^2 + b^2$ $d = \sqrt{l^2 + b^2}$

(iv) Finding area using Brahmagupta' Formula In this case, we know that $a = c$ and $b = d$, and A + B = π So, area of Rectangle

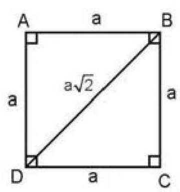
$$= \sqrt{(a+b-a)(a+b-b)(a+b-a)(a+b-b) - a.b.a.b\cos^2 90^\circ} = ab$$

$= ab$

(v) The quadrilateral formed by joining the mid points of intersection of the angle bisectors of a parallelogram is a rectangle.

Square

A square is a quadrilateral when all its sides are equal and each internal angle is of 90°. The diagonals of a square bisect each other at right angles (90°)



Given: AB = BC = CD = DA = a

BD (diagonal) = $a\sqrt{2}$

(i) Area = (side)^2 $= \dfrac{(\text{diagonal})^2}{2} = \dfrac{(\text{perimeter})^2}{16}$

Area $= a^2 = \dfrac{d^2}{2} = \dfrac{P^2}{16}$

(ii) Using Brahmagupta's formula to find out the area of a square:

We know that $a = b = c = d$ and A + B = π

So, area of square

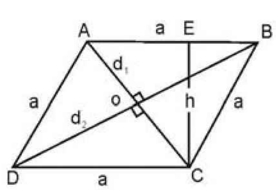
$$= \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\frac{1}{2}(A+B)} =$$

$$\sqrt{(2a-a)(2a-a)(2a-a)(2a-a) - a.a.a.a.\cos^2 90^\circ} = a^2$$

(ii) Perimeter = 4 (side) ⇒ Perimeter = $4a$

Rhombus

A rhombus is a quadrilateral when all sides are equal. The diagonals of a rhombus bisects each other at right angles (90°)



Given = AB = BC = CD = DA = a

∠AOB = ∠BOC = ∠COD − ∠DOA = 90°

AC = d, (AO = OC) and BD = d_2 (BO = OD) CE (height) = h

(i) Area $= \dfrac{1}{2} \times$ (product of the diagonals)

Area $= \dfrac{1}{2}\ d_1 d_2$

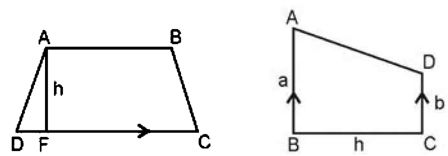
(ii) Area = base × height

Area = a × h

(iii) A parallelogram is a rhombus if its diagonals are perpendicular to each other. Remember, the sum of the square of the diagonals is equal to four time the square of the side i.e., $d_1^2 + d_2^2 = 4a^2$

Trapezium

A trapezium is a quadrilateral in which only one pair of the opposite sides is parallel



Given: AB = a and CD = b

In Fig. 1, AF (height) = h and in Fig. 2, BC (height) = h

(i) Area $= \dfrac{1}{2} \times$ (sum of the parallel sides) × (distance between the parallel sides)

Area $= \dfrac{1}{2}\ (a + b)$ h

(ii) The line joining the mid-points of the non-parallel sides is half the sum of the parallel sides and is known as Median.

(iii) If we make non-parallel sides equal, then the diagonals will also be equal to each other.

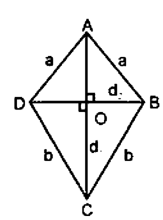
(iv) Diagonals intersect each other proportionally in the ratio of the lengths of the parallel sides.

(v) If a trapezium is inscribed inside a circle, then it is an isosceles trapezium with oblique sides being equal.

Kite

Kite is a quadrilateral when two pairs of adjacent sides are equal and the diagonals bisect each other at right angles (90°).

Given: AB = AD = a and BC = DC = b



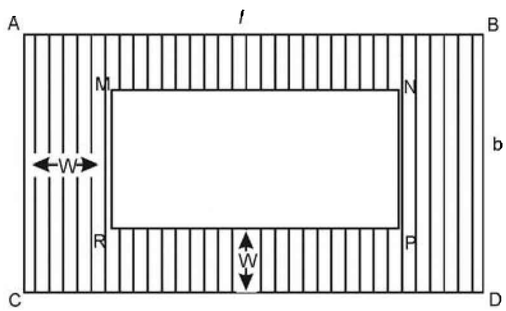
$AC = d_1$ (AO = OC) and $BD = d_2$ (BO = OD)
$\angle AOB = \angle BOC = \angle COD = \angle DOA = 90°$

(i) Area $= \frac{1}{2} \times$ (Product of the diagonals)

Area $= \frac{1}{2} d_1 d_2$

Area of Shaded Paths

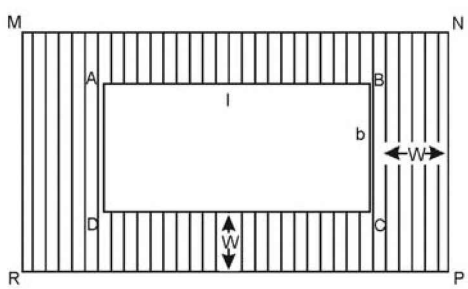
Case 1 When a pathway is made outside a rectangle having length = l and breadth = b



ABCD is a rectangle with length = l and breadth = b, the shaded region represents a pathway of uniform width = W

Area of the shaded region/pathway = 2w (l + b − 2w)

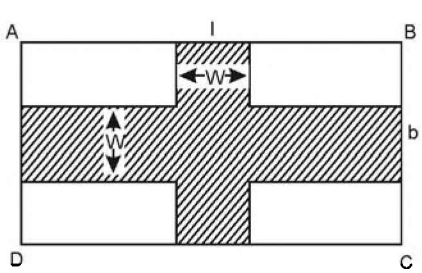
Case 2 When a pathway is made inside a rectangle having length = l and breadth = b



ABCD is a rectangle with length = l and breadth = b, the shaded region represents a pathway of uniform width = w

Area of the shaded region/pathway = 2w (l + b + 2w)

Case 3 When two pathways are drawn parallel to the length and breadth of a rectangle having length = l and breadth = b



ABCD is a rectangle with length = l and breadth = b, the shaded region represents two pathways of a uniform width = w

Area of the shaded region/pathway = W (l + b − w)

From the above figure we can observe that the area of the paths does not change on shifting their positions as long as they are perpendicular to each other.

We can conclude from here that:

1. Every rhombus is a parallelogram but the converse is not true.
2. Every rectangle is a parallelogram but the converse is not true.
3. Every square is a parallelogram but the converse is not true.
4. Every square is a rhombus but the converse is not true.
5. Every square is a rectangle but the converse is not true.

Construction of new figures by joining the mid-points

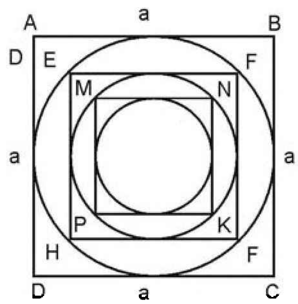
	Original Figure		Resulting Figure
	Quadrilateral		Parallelogram
Lines joining the mid-points of adjacent sides of	Parallelogram		Parallelogram
	Rectangle	form	Rhombus
	Rhombus		Rectangle
	Trapezium		Four similar Δ

Properties of diagonals

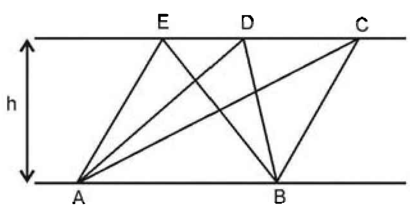
Properties		*Types of Quadrilaterals*				
Sl. No.		*Squ-are*	*Rect-angle*	*Paralle-logram*	*Rho-mbus*	*Trap-ezium*
1.	Diagonals are equal	Y	N	N	Y	N
2.	Diagonals bisect each other	Y	Y	Y	Y	N
3.	Diagonals bisect vertex angles	Y	N	N	Y	N
4.	Diagonals are at right angles	Y	N	N	Y	N
5.	Diagonals make congruent triangles	Y	N	N	Y	N

Some important points

1. In the figure given below, all the side quadrilaterals are squares and circles are inscribed in these squares. If the side of the square ABCD = a, then the side of square EFGH $= \frac{a}{\sqrt{2}}$ and the side of square MNKP $= \frac{a}{2}$. In other words we can say that in order to obtain the side of the next inner square, divide the side of the immediate outer square by $\sqrt{2}$. The same procedure will be applied for the inscribed circles i.e., we divide the radius of the immediate outer circle by $\sqrt{2}$ to obtain the radius of the next inner circle.



2. Triangles on the same base and between the same parallel lines are equal in area.

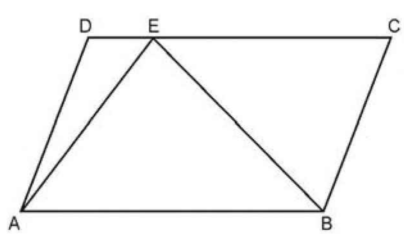


Area ($\angle$ABC) = Area (ΔABD) = Area (ΔABE)

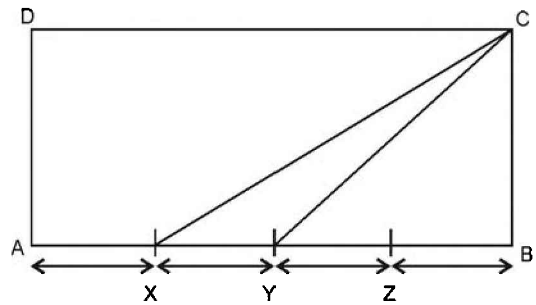
$= \frac{1}{2}$ (AB) × h

(Base AB is the same and the height ‘h’ is also the same.)

3. If a parallelogram and a triangle are drawn on the same base and between the same parallel lines, then the area of the parallelogram is twice the area of the triangle.
Area of ABCD = 2 × (Area of Δ ABE)



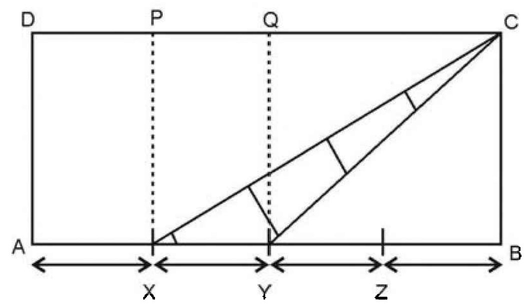
Example 10 Side AB of a rectangle ABCD is divided into four (4) equal parts as shown in the figure. Find the ratio of the area (ΔXYC) and area (€ABCD)?



Solution Let the area of the rectangle ABCD = A

Area of the rectangle XYQP = A/4

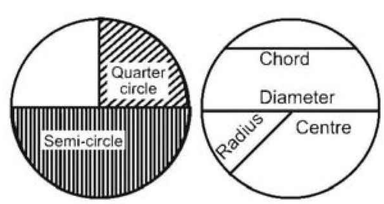
Rectangle XYQP and ΔXYC are on the same base and between the same parallel lines.



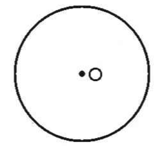
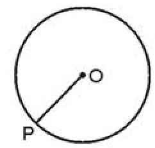
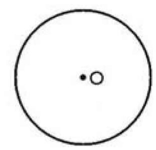
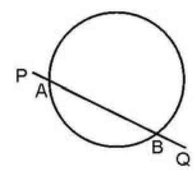
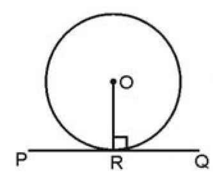
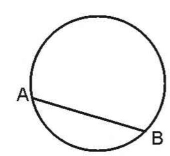
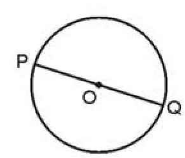
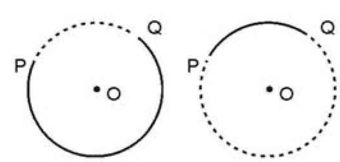
Thus area (XYC) = A/8

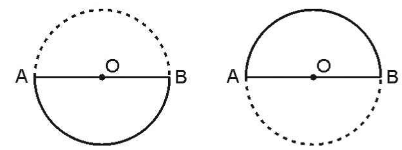
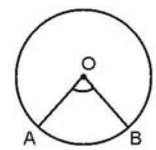
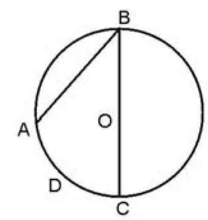
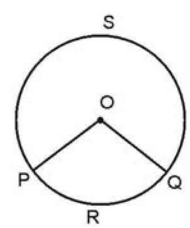
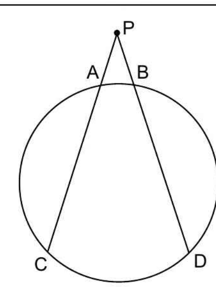
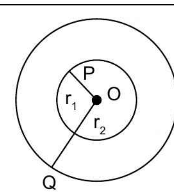
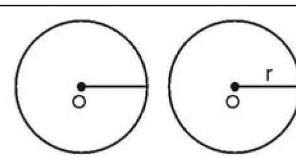
CIRCLES AND THEIR PROPERTIES

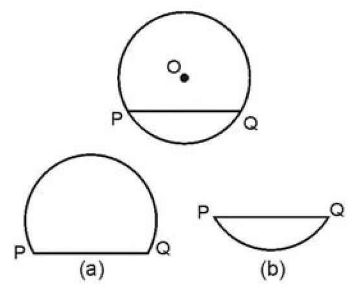
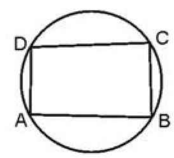
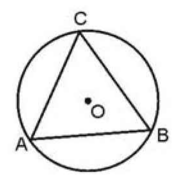
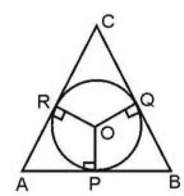
A circle is the path travelled by a point which moves in such a way that its distance from a fixed point remains constant. The fixed point is known as the centre and the fixed distance is called the radius.



Before we move ahead, let us understand the basics definitions of circle.

Nomenclature	*Definition*	*Diagram*
Centre	The fixed point is called the centre. In the given diagram ‘O’ is the centre of the circle.	
Radius	The fixed distance is called a radius. In the given diagram OP is the radius of the circle. (point P lies on the circumference)	
Circumference	The circumference of a circle is the distance around a circle, which is equal to $2\pi r$. (r →radius of the circle)	
Secant	A line segment which intersects the circle in two distinct points, is called as secant. In the given diagram secant PQ intersects circle at two points at A and B.	
Tangent	A line segment which has one common point with the circumference of a circle, i.e., it touches only at only one point is called as tangent of circle. The common point is called as point of contact. In the given diagram, PQ is a tangent which touches the circle at a point R.	 (R is the point of contact) Note: Radius is always perpendicular to tangent.
Chord	A line segment whose end points lie on the circle. In the given diagram AB is a chord.	
Diameter	A chord which passes through the centre of the circle is called the diameter of the circle. The length of the diameter is twice the length of the radius. In the given diagram PQ is the diameter of the circle. (O→is the centre of the circle)	
Arc	Any two points on the circle divides the circle into two parts the smaller part is called as minor arc and the larger part is called as major arc. It is denoted as ‘’. In the given diagram PQ is arc.	

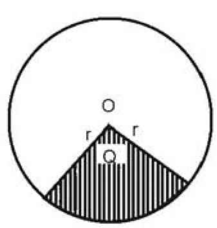
Nomenclature	*Definition*	*Diagram*
Semicircle	A diameter of the circle divides the circle into two equal parts. Each part is called a semicircle.	
Central Angle	An angle formed at the centre of the circle, is called the central angle. In the given diagram ∠AOB in the central angle.	
Inscribed Angle	When two chords have one common end point, then the angle included between these two chords at the common point is called the inscribed angle. ∠ABC is the inscribed angle by the arc ADC	
Measure of an Arc	Basically, it is the central angle formed by an arc. e.g., (a) measure of a circle = 360° (b) measure of a semicircle = 180° (c) measure of a minor arc = ∠POQ (d) measure of a major arc = 360 − ∠POQ	 m(arc PRQ = m ∠POQ m(arc PSQ) = 360° − m (arc PRQ)
Intercepted Arc	In the given diagram, AB and CD are the two intercepted arcs, intercepted by ∠CPD. The end points of the arc must touch the arms of ∠CPD, i.e., CP and DP.	
Concentric Circles	Circles having the same centre at a plane are called the concentric circles. In the given diagram, there are two circles with radii r_1 and r_2 having the common (or same) centre. These are called as concentric circl Ces.	
Congruent Circles	Circles with equal radii are called as congruent circles.	

Nomenclature	*Definition*	*Diagram*
Segment of a Circle	A chord divides a circle into two regions. These two regions are called the segments of a circle: (a) major segment (b) minor segment.	
Cyclic Quadrilateral	A quadrilateral whose all the four vertices lie on the circle.	
Circumcircle	A circle which passes through all the three vertices of a triangle. Thus the circumcentre is always equidistant from the vertices of the triangle. OA = OB = OC (circumradius)	
In Circle	A circle which touches all the three sides of a triangle i.e., all the three sides of a triangle are tangents to the circle is called an incircle. Incircle is always equidistant from the sides of a triangle.	

Now come to different formula and theorems attached to circle:

Circumference of a circle = $2\pi r$

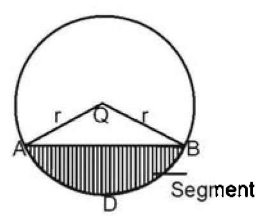
Area of a circle = πr^2, where r is the radius.



Area of a sector $= \pi r^2 \dfrac{\theta}{360°}$

Circumference of a sector $= 2\pi r \dfrac{\theta}{360} +$

Perimeter of a sector $= 2r\left(\dfrac{\pi\theta}{360} + 1\right)$



Area of a segment = Area of a sector OADB – Area of triangle OAB

Area of a segment $= \pi r^2 \dfrac{\theta}{360} - \dfrac{1}{2} r^2 \text{Sin}\theta$

Common Tangents and Secants of Circles

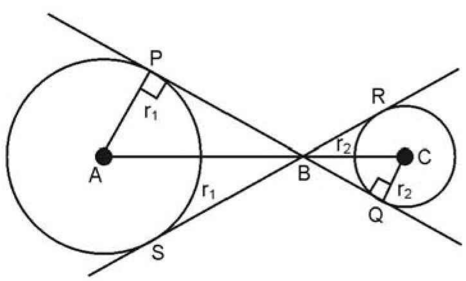
Depending upon the positioning of the circles, two or more than two circles can have a common tangent. Following is a list indicating the number of common tangents in case of two circles:

Sl. No.	*Position of two circles*	*Number of common tangents*
1.	One circle lies entirely inside the other circle	Zero
2.	Two circles touch internally	One
3.	Two circles intersect in two distinct points	Two
4.	Two circles touch externally	Three
5.	One circle lies entirely outside the other circle	Four

Direct common tangents and transverse common tangents

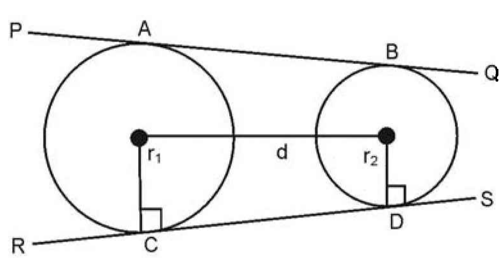
1. *Transverse common tangent* In the figure given below, PQ and RS are the transverse common tangents. Transverse common tangents intersect the line joining the centre of the two circles. They divide the line in the ratio $r_1 : r_2$.
   $AB : BC = r_1 : r_2$

   Assume AC = Distance between centres = d



   $PQ^2 = RS^2 = d^2 - (r_1 + r_2)^2$

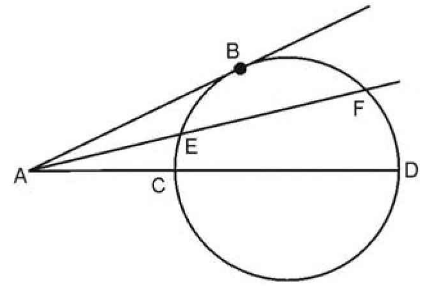
2. *Direct common tangent*



   In the figure given above, PQ and RS are direct common tangents.

   Points A and C are the point of tangency for the first circle and similarly points B and D are the point of tangency for the second circle. AB and CD are known as lengths of the direct common tangents and they will be same.
   $CD^2 = AB^2 = d^2 - (r_1 - r_2)^2$

Secants

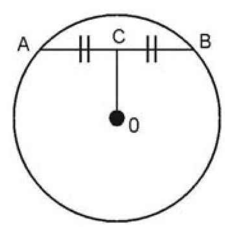


In the figure given above, AB is a tangent and ACD is a secants

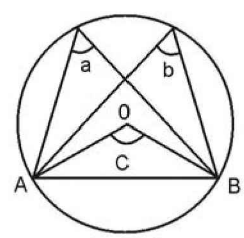
i. $AB^2 = AC \times AD$
ii. $AE \times AF = AC \times AD$

Important theorems related to circle

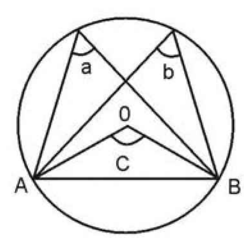
1. If C is the mid-point of AB, then OC is perpendicular to AB. And vice versa is also true.



2. Angles in the same segment will be equal.

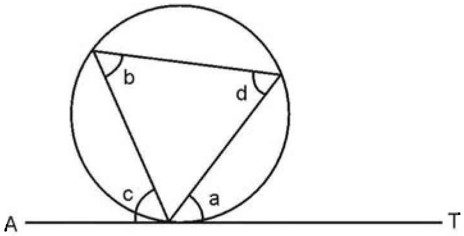


   In the figure given above, $a = b$.
3. Angle subtended by a chord at the centre is two times the angle subtended on the circle on the same side. In the figure given below, $2a = 2b = c$.



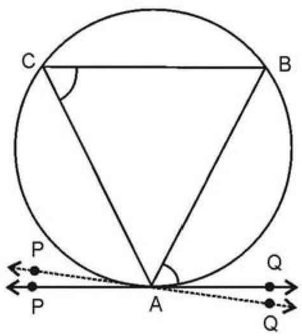
4. Angle subtended by a diameter of the circle is a right angle.

5. Alternate segment theorem



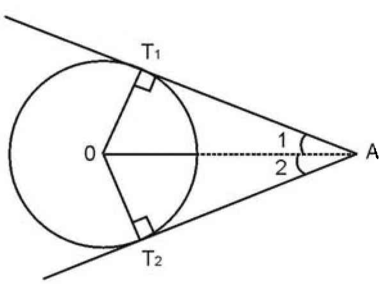
In the figure above, AT is the tangent. $\angle a$ = Alternate segment $\angle b$ $\angle c$ = Alternate segment $\angle d$

6. Converse of alternate segment theorem If a line is drawn through an end point of a chord of a circle so that the angle formed by it with the chord is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.



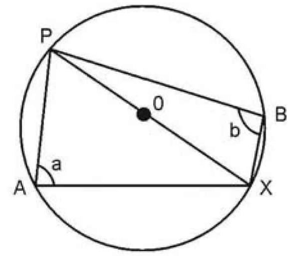
AB is a chord of a circle and a line PAQ such that $\angle BAQ = \angle ACB$, where C is any point in the alternate segment ACB, then PAQ is a tangent to the circle.

7. Tangent drawn to a circle from a point are same in length.
In the figure below, tangents are drawn to the circle from point A and AT_1 and AT_2 are the tangents.
i. $AT_1 = AT_2$ ii. $\angle 1 = \angle 2$
iii. $AT_1^2 + OT_1^2 = AT_2^2 + OT_2^2 = AO^2$



Cyclic Quadrilateral

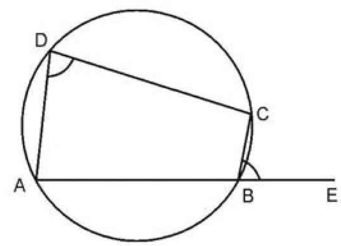
Consider the figure given below:



If we have $a + b = 180°$ and quadrilateral AXBP has all its vertices on a circle, then such a quadrilateral is called a Cyclic quadrilateral.

For a cyclic quadrilateral, the sum of the opposite angles of a quadrilateral in a circle is 180°.

It can also be seen that exterior $\angle CBE$ = internal $\angle ADC = 180° - \angle ABC$.



Using Brahmagupta's formula to find out the area of a cyclic quadrilateral We know that A + B = p So, area of cyclic quadrilateral

$$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Where terms used are having their meaning.
[Cos 90° = 0]

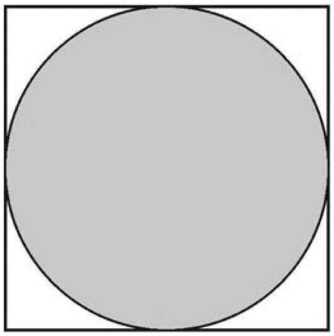
Circle Packing

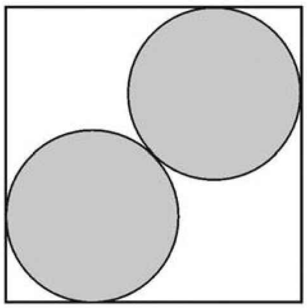
N circles have been packed inside a square of side length R unit and radius of the circle is to be calculated.

Case 1 When *n* = 1

Obviously in this case, diameter of the circle = side of the square.

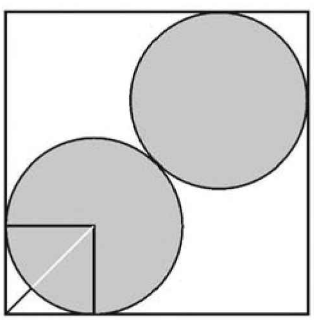
So, the radius of the circle $= \frac{1}{2}$ R unit





Case 2 When $n = 2$

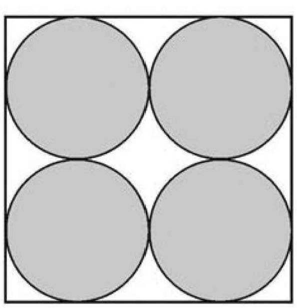
Consider this figure



This is a right-angled triangle with sides r, r and r √2. Hence the diagonal of the square = 4r + 2 r √2 = √2 R

So, $r = \dfrac{\sqrt{2}}{4+2\sqrt{2}} R$

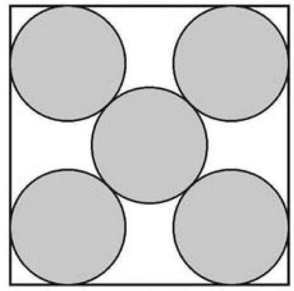
Case 3 When $n = 4$



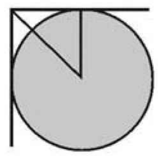
Obviously, in this case 2 × diameter of a circle = side of square.

So, the radius of a circle $= \dfrac{1}{4}$ R

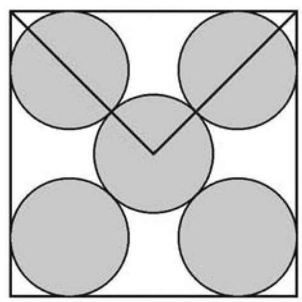
Case 4 When $n = 5$



Consider this figure:



If the radius of the circle is r, then the distance between the centre of the circle and the vertex of a square = r √2

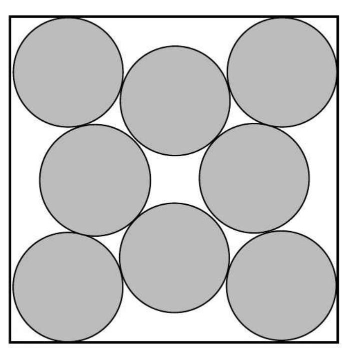


Now, consider this figure, the triangle formed here is a right-angled triangle:

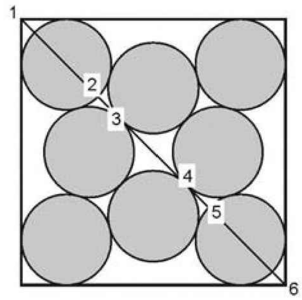
1/√2(1/2 of the diagonal of the square) = r + r + r √2

= r(2 + √2), so, $r = \dfrac{1}{\sqrt{2}\times(2+\sqrt{2})} R$

Case 5 When $n = 8$

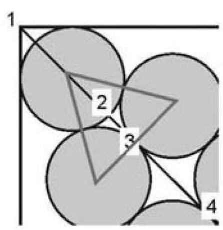


We will find out the value of r in 3 steps here:

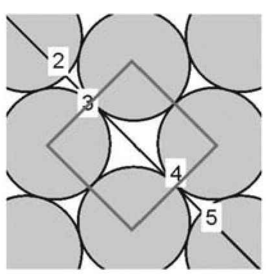


Step 1 Find 1 - 2 = 5 - 6
Step 2 Find 2 - 3 = 4 - 5
Step 3 Find 3 - 4
Step 1 1 - 2 = 5 - 6 = r + r √2
Step 2 For 2 - 3

This is an equilateral triangle with side length = 2r. Height of this triangle r √3

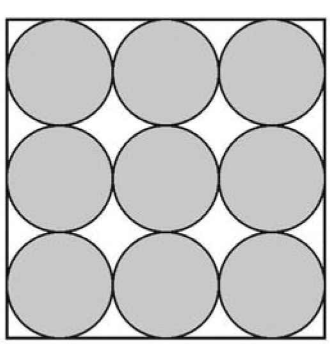


So, 2 - 3 = 4 - 5 = r√3 – r
Step 3



This is a square with side length 2 r and diagonal = 2r √2
So, 3 - 4 = 2r
Hence 1 – 6 = 2(r + r√2) + 2(r√3 – r) + 2r = R√2. Now r can be calculated.

Case 6 When *n* = 9

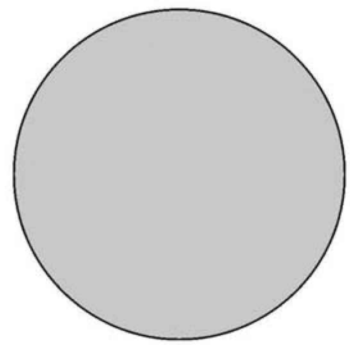


Radius of circle = r
So, the side of a square = 3 × diameter of a circle = 6r
So, 6r = R unit, r = R/6 unit

Circles Inside Circles

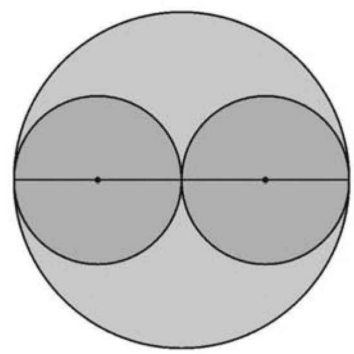
Radius of the outer circle R = 1 unit and n similar circles of 'r' radius have been inserted inside this outer circle.

N = 1



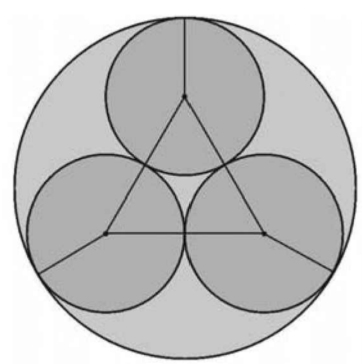
Obviously R = r

N = 2



Obviously, 4 r = 2 R Hence r = R/2

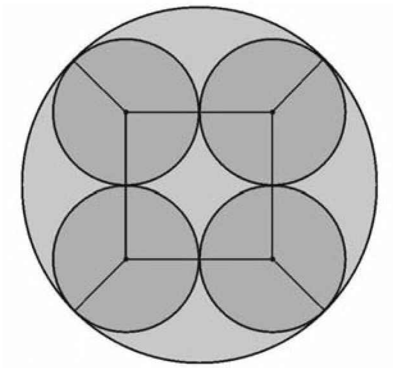
N = 3



The triangle formed inside will be an equilateral triangle of the side length 2 r. The centre of this circle will be incentre/centroid/circumcentre of this equilateral triangle. So, the distance between the centre of any smaller circle to the centre of the bigger circle = 2/3 median = x = 2/3 [√3/2 × a], where a = 2r.

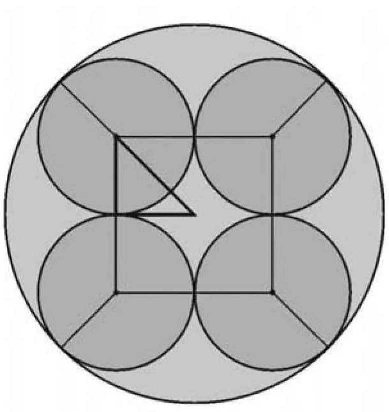
Now, r + x = R. Now put the values of x from the above condition to find the value of r.

N = 4



The square formed inside will be of the side length 2r. Now consider this figure,

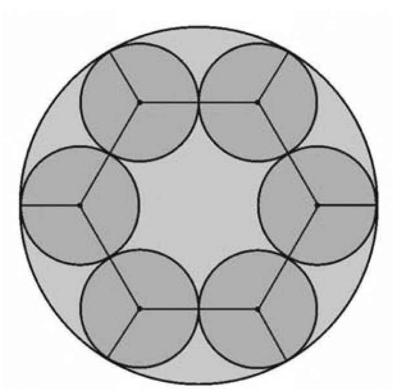
N = 5



The figure inside is a right angled triangle, with base and height being equal to r, and hypotenuse = $r\sqrt{2} = r + x$. Hence $x = r(\sqrt{2} - 1)$

Now 2R = 4 r + 2 x. Hence $r = 2(R - x)/4$[Now put the value of x from the above condition.]

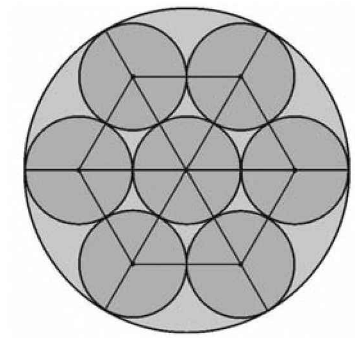
N = 6



Here we can put a similar circle inside all the six circles making it a 7 circle figure. Now, the situation is the same as that of a circle packing with N = 7

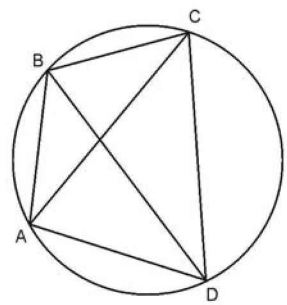
In this case, 6r = R, so, r = 1/6 R

N = 7



In this case, 6r = R, so, r = 1/6 R

Ptolemy's Theorem of Cyclic Quadrilateral



For a cyclic quadrilateral, the sum of the products of the two pair of the opposite sides equal the product of the diagonals.

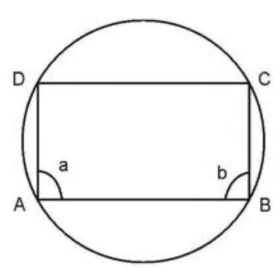
In the figure given above, AB × CD + BC × DA = AC × BD

In any cyclic quadrilateral of side lengths a, b, c and d, the length of the diagonals p and q will be equal to:

$$p = \sqrt{\frac{(ab+cd)(ac+bd)}{ad+bc}} \quad q = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+dc}}$$

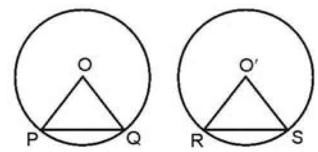
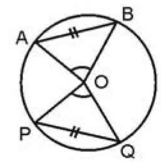
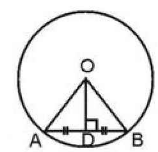
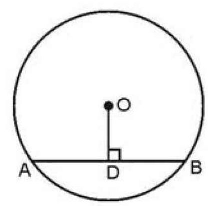
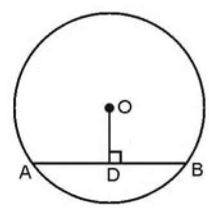
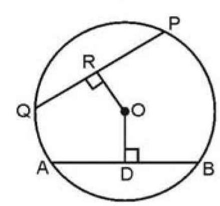
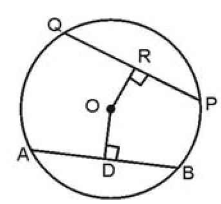
Example 11 Under some special conditions, it is given that a cyclic quadrilateral ABCD is a parallelogram. What kind of figure will ABCD be?

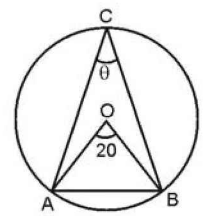
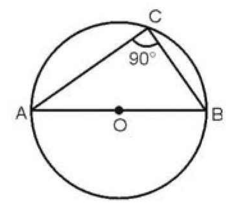
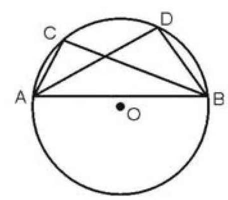
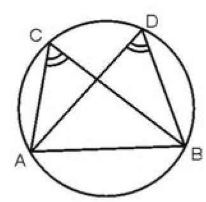
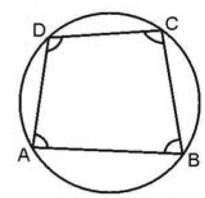
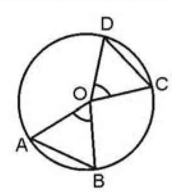
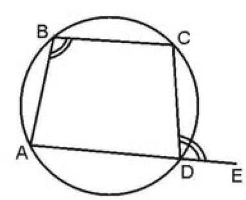
Solution ABCD is a parallelogram. Thus $a + b = 180°$. And $\angle A + \angle C = 180°$

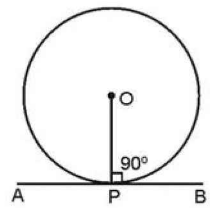
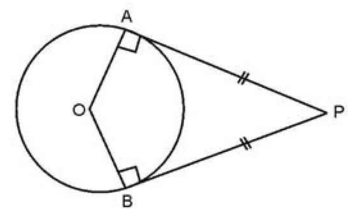
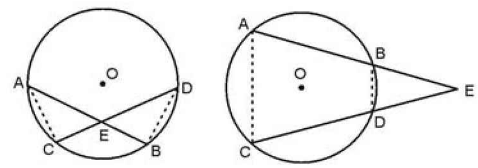
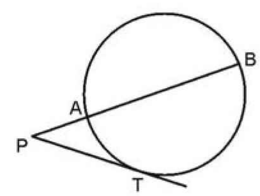
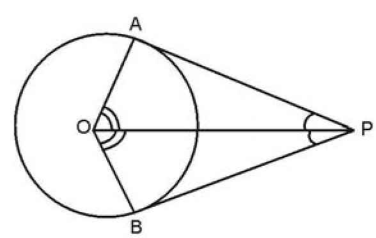
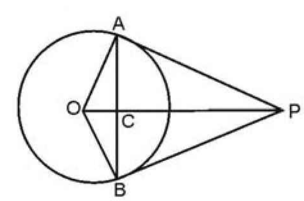
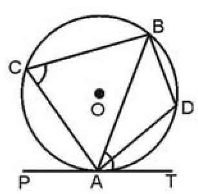


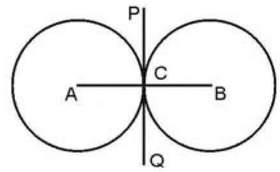
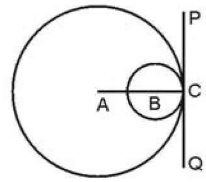
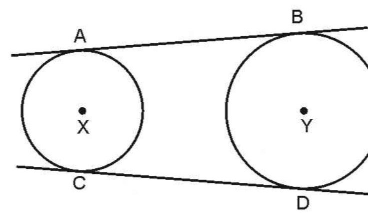
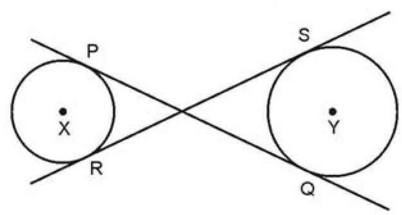
This means that a parallelogram drawn inside a circle is always a rectangle.

Summarizing the discussion regarding circle

Sl. No.	*Theorem/Property*	*Diagram*
1.	In a circle (or congruent circles) equal chords are made by equal arcs. {OP = OQ} = {O'R = O'S) PQ = RS and PQ = RS	
2.	Equal arcs (or chords) subtend equal angles at the centre PQ = AB (or PQ = AB) ∠POQ = ∠AOB	
3.	The perpendicular from the centre of a circle to a chord bisects the chord i.e., if OD ⊥ AB (OD is perpendicular to AB).	
4.	The line joining the centre of a circle to the mid-point of a chord is perpendicular to the chord. AD = DBOD ⊥ AB	
5.	Perpendicular bisector of a chord passes through the centre i.e., OD ⊥ AB and AD = DB ∴ O is the centre of the circle	
6.	Equal chords of a circle (or of congruent circles) are equidistant from the centre ∴ AB = PQ ∴ OD = OR	
7.	Chords of a circle (or of congruent circles) are equidistant from the centre ∴ OD = OR ∴ AB = PQ	

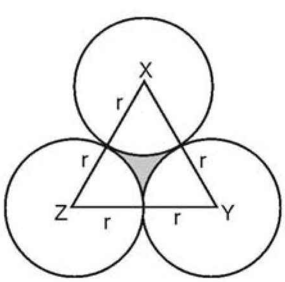
Sl. No.	*Theorem/Property*	*Diagram*
8.	The angle subtended by an arc (the degree measure of the arc) at the centre of a circle is twice the angle subtended by the arc at any point on the remaining part of the circle. m ∠AOB = 2m ∠ACB.	
9.	Angle in a semicircle is a right angle.	
10.	Angles in the same segment of a circle are equal i.e., ∠ACB = ∠ADB	
11.	If a line segment joining two points subtends equal angle at two other points lying on the same side of the line containing the segment, then the four points lie on the same circle. ∠ACB = ∠ADB ∴ Points A, C, D, B are concyclic i.e., lie on the circle	
12.	The sum of pair of opposite angles of a cyclic quadrilateral is 180°. ∠DAB + ∠BCD = 180° and ∠ABC + ∠CDA = 180° (Inverse of this theorem is also true)	
13.	Equal chords (or equal arcs) of a circle (or congruent circles) subtended equal angles at the centre. AB = CD (or AB = CD) ∠AOB = ∠COD (Inverse of this theorem is also true)	
14.	If a side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle. m ∠CDE = m ∠ABC	

Sl. No.	*Theorem/Property*	*Diagram*
15.	A tangent at any point of a circle is perpendicular to the radius through the point of contact. (Inverse of this theorem is also true)	
16.	The lengths of two tangents drawn from an external point to a circle are equal i.e., AP = BP	
17.	If two chords AB and CD of a circle, intersect inside a circle (outside the circle when produced at a point E), then AE × BE = CE × DE	
18.	If PB be a secant which intersects the circle at A and B and PT be a tangent at T then PA × PB = $(PT)^2$	
19.	From an external point from which the tangents are drawn to the circle with centre O, then (a) they subtend equal angles at the centre (b) they are equally inclined to the line segment joining the centre of that point $\angle AOP = \angle BOP$ and $\angle APO = \angle BPO$	
20.	If P is an external point from which the tangents to the circle with centre O touch it at A and B then OP is the perpendicular bisector of AB. OP $\perp$ AB and AC = BC	
21.	If from the point of contact of a tangent, a chord is drawn then the angles which the chord makes with the tangent line are equal respectively to the angles formed in the corresponding alternate segments. In the adjoining diagram. $\angle BAT = \angle BCA$ and $\angle BAP = \angle BDA$	

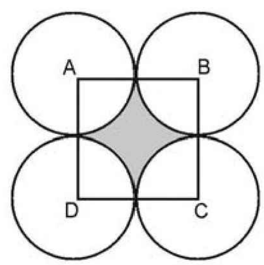
Sl. No.	*Theorem/Property*	*Diagram*
22.	The point of contact of two tangents lies on the straight line joining the two centres. (a) When two circles touch externally then the distance between their centres is equal to sum of their radii, i.e., AB = AC + BC (b) When two circles touch internally the distance between their centres is equal to the difference between their radii. i.e., AB = AC – BC	 
23.	For the two circles with centre X and Y and radii r_1 and r_2. AB and CD are two Direct Common Tangents (DCT), then the length of DCT $= \sqrt{(\text{Distance between centres})^2 - (r_1 - r_2)^2}$	
24.	For the two circles with centre X and Y and radii r_1 and r_2 PQ and RS are two transverse common tangent, then length of TCT $= \sqrt{(\text{Distance between centres})^2 - (r_1 + r_2)^2}$	

Some important points

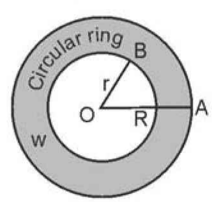
1. If three circles, each of radius r, are so kept that each circle touches the other two, then the area of the shaded region is $r^2 [\sqrt{3} - (\pi/2)]$ which is approximately equal to $(4/25) r^2$.



2. If four circles, each of the radius r, are so kept that each circle touches the other two, then the area of the shaded region is $r^2 (4 - \pi)$ which is equal to $(6/7)r^2$.



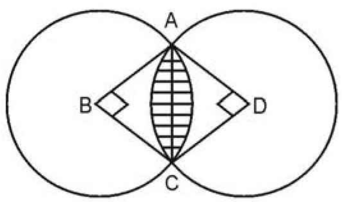
3. In the diagram given below, if the radius (OA) of the outer circle is R and the radius (OB) of the inner circle is r, then the width (w) of the ring is (R – r) and the area of the shaded region is $\pi(R^2 - r^2)$ or $\pi(R - r)(R + r)$ or $\pi(w)(R + r)$



Example 12 Two identical circles intersect so that their centers, and the points at which they intersect, form a square of side 1 cm. The area in sq. cm of the portion that is common to the two circles, is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}-1$ (c) $\frac{\pi}{5}$ (d) $\sqrt{2}-1$

Solution



Shaded area = 2 × (area of sector ADC – area of ΔADC)

$$=2\times\left(\frac{\pi}{4}\times1^2-\frac{1}{2}\times1\times1\right)$$

$$=\frac{\pi}{2}-1$$

Hence, option (b)

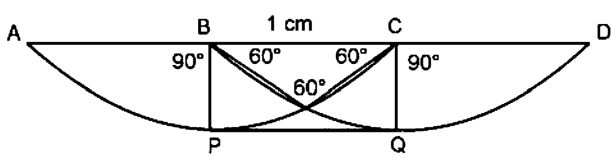
Example 13 Four points A, B, C and D lie on a straight line in the X – Y plane, such that AB = BC = CD, and the length of AB is 1 cm. An ant at A wants to reach a sugar particle at D. But there are insect repellents kept at points B and C. The ant would not go within one metre of any insect repellent. The minimum distance the ant must traverse to reach the sugar particle (in m) is

(a) $3/\sqrt{2}m$ (b) $1+\pi$

(c) $\frac{4\pi}{3}$ (d) 5

Solution

In the drawn figure, ant will go along APQD since it cannot be within a distance of 1 cm from the repellents kept at B and C.



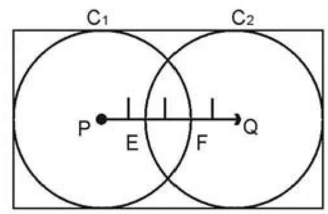
$$AP=\frac{90}{360}\times2\pi\times1=\frac{\pi}{2}$$

Also AP = QD $=\frac{\pi}{2}$

So the minimum distance = AP + PQ + QD

$$=\frac{\pi}{2}+1+\frac{\pi}{2}=1+\pi$$

Example 14 Two circles C_1 and C_2, having the same radius of 2 cm and centers at P and Q respectively, intersect each other such that the line of centers PQ intersects C_1 and C_2 at F and E respectively. EF = 1 cm. The whole assembly is enclosed in a rectangle of the minimum area. The perimeter of the rectangle is



Hence, breadth of the rectangle = 4 cm

And length = 7 cm

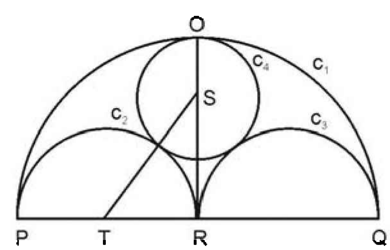
Perimeter = 2 × (7 + 4) = 22 cm

Example 15 Semi-circle C_1 is drawn with a line segment PQ as its diameter with centre at R. Semicircles C_2 and C_3 are drawn with PR and QR as diameters respectively, both C_2 and C_3 lying inside C_1. A full circle C_4 is drawn in such a way that it is tangent to all the three semicircles C_1, C_2 and C_3. C_4 lies inside C_1 and outside both C_2 and C_3. The radius of C_4 is

(a) $\frac{1}{3}PQ$ (b) $\frac{1}{6}PQ$

(c) $\frac{1}{\sqrt{2}}PQ$ (d) $\frac{1}{4}PQ$

Solution



Assume that the radius of C_4 = r and PQ = k.

Now, PR = k/2 = RQ = RO

⇒ RS = (k/2) – r

RT = k/4

ST = (k/4) + r

Applying Pythagoras theorem in triangle STR.

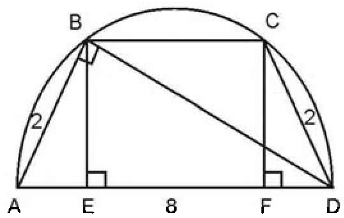
$$\left(\frac{k}{4}+r\right)^2=\left(\frac{k}{4}\right)^2+\left(\frac{k}{2}-r\right)^2$$

⇒ r = k/6 = PQ/6

Example 16 On a semicircle with diameter AD, chord BC is parallel to AD. Further each of the chords AB and CD has length 2 units, while AD = 8 units. What is the length of BC?

(a) 7.5 (b) 7
(c) 7.75 (d) 8

Solution



Finding area of DABD $\frac{1}{2}$ AB × BD $= \frac{1}{2}$ AD × BE

So, BE = (√15)/2

Hence AE $= \frac{1}{2}$

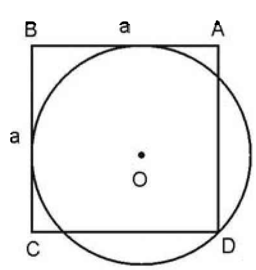
Now, BC = EF = 8 – $\left(\frac{1}{2}+\frac{1}{2}\right)$ = 7

Alternatively, this question can be solved by using Ptolemy's theorem also.

Example 17 The adjacent sides AB, BC of a square of side 'a' units are tangent to a circle. The vertex D of the square lies on the circumference of the circle. The radius of the circle could be:

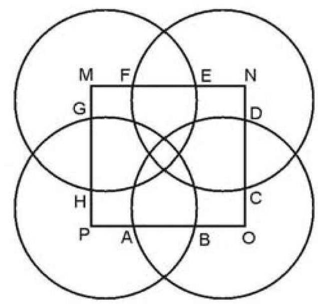
(a) a(2 – √2) (b) a(√2 – 1)
(c) a(√2 + 1.5) (d) a(√2 + 1)

Solution



OD = r, OB = r√2
Hence, r + r√2 = √2 a
Hence, r = a(2 –√2)

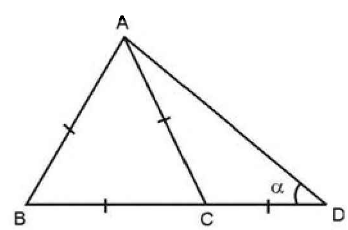
Example 18 M, N, O and P are centres of four intersecting circles each having a radius of 15 cm. If AB = 7 cm, CD = 5 cm, EF = 6 cm, GH = 8 cm, what is the perimeter of the quadrilateral MNOP?



(a) 84 cm (b) 45 cm
(c) 94 cm (d) 124 cm

Solution MN = 30 – 6 = 24 cm
NO = 30 – 5 = 25 cm
OP = 30 – 7 = 23 cm
PM = 30 – 8 = 22 cm
So, the perimeter = 24 + 25 + 23 + 22 = 94 cm

Example 19

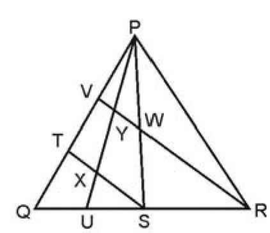


AB = BC = AC = CD. Find ∠a.

(a) 30° (b) 60°
(c) 15° (d) None of these

Solution ∠ACB = 60° ∠ACD = 180 – 60 = 120° ∠CAD = ∠CDA = 30° ∠α = 30°

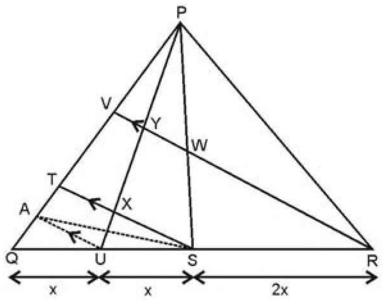
Example 20 In the figure shown here QS = SR, QU = SU, PW = WS and ST||RV. What is the value of the $\frac{\text{Area of } \Delta PSX}{\text{Area of } \Delta PQR}$?



(a) $\frac{1}{5}$ (b) $\frac{1}{7}$

(c) $\frac{1}{6}$ (d) $\frac{1}{9}$

Solution



Draw a line form U such that it is parallel to ST (and hence RV, also) joining AS, to get ΔQSA.

Now, let us first find the area of ΔSUX.

PW = WS (W is the mid point of PS)

Consider the ΔPXS, SX||YW and PX and PS are transversals to those parallel lines, we must have

$$\frac{PY}{YX} = \frac{PW}{WS} = \frac{1}{1} = 1$$

⇒ PY = YX

Similarly we also get

PV = TV (in ΔPTS; VW||ST) and QA = AT (in ΔQTS: AU||ST) and QT = VT (in ΔVQR; ST||VR)

Combining all these QA = AT $= \frac{1}{2}(VT) + \frac{1}{2}(VP)$

Now, in ΔRUY

$$\frac{TS}{SR} = \frac{x}{2x} = \frac{1}{2} = \frac{UX}{XY} \Rightarrow UX = \frac{1}{2}(XY)$$

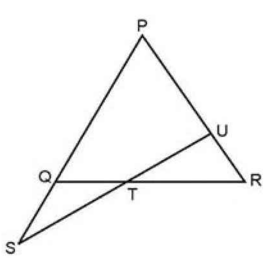
so, in ΔPUS, $UX = \frac{1}{2}XY = \frac{1}{2}PY \Rightarrow \frac{\text{Area }(\Delta SUX)}{\text{Area }(\Delta PUS)} = \frac{1}{5}$

$\Rightarrow$ Area $(\Delta SUX) = \frac{1}{5}$ Area (ΔPUS)

$= \frac{1}{5}\left(\frac{1}{4}\text{area }\Delta PQR\right) \Rightarrow$ Area $(\Delta SUX) = \frac{1}{2}(\Delta SUX)$, then

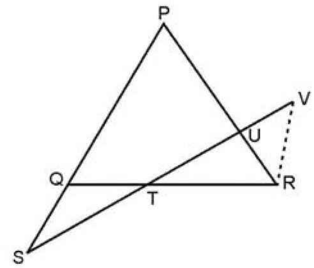
Area (ΔPSX) = Area (ΔPSU) − area (Δ SUX) $= \frac{1}{4} = \frac{1}{5}$ $(\Delta\, PQR)$

Example 21 Through T, the mid-point of the side QR of a ΔPR, a straight line is drawn to meet PQ produced to S and PR at U, so that PU = PS. If length of UR = 2 units then the length of QS is:



(a) $2\sqrt{2}$ Units (b) $\sqrt{2}$ Units
(c) 2 Units (d) Cannot be determined

Solution We have QT = TR and PU = PS and UR = 2 units



Draw RV||PS that meets SU extended at V.

Now, in ΔQST and ΔTVR

∠QST = ∠TVR (alternate angles as PS||VR) and ∠QTS = ∠VTR

QT = TR

∴ DQST and DTVR are congruent.

∴ QS = VR … (1)

Now ∠QST = ∠PUS = ∠VUR = ∠UVR

In ΔUVR, ∠VUR = ∠RVU

or, RV = UR = 2 …(2)

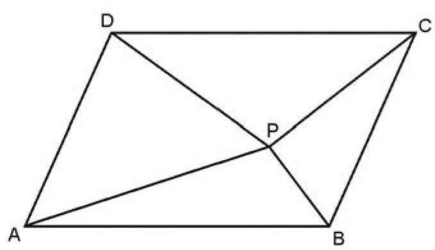
From (1) and (2)

QS = VR = UR = 2 units

Example 22 ABCD is a parallelogram and P is any point within it. If the area of the Parallelogram ABCD is 20 units, then what is the sum of the areas of the DPAB and DPCD?

(a) 5 units (b) 10 units
(c) 12 units (d) Cannot be determined

Solution



Let AB = CD = a and x, y be the lengths of the perpendiculars from P on AB and CD respectively, then

Area of (ΔPAB + ΔPCD) $= \frac{1}{2}ax + \frac{1}{2}ay$

$= \frac{1}{2}a(x + y) = \frac{1}{2}ah$

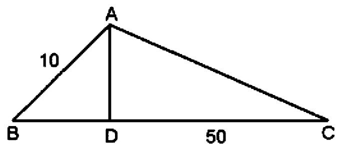
Given area of the parallelogram = 20 units

Area of (ΔPAB + ΔPCD) $= \frac{1}{2} \times 20 = 10$ units

Example 23 Nayantara bought a triangular piece of land of area 150 m^2. He took a piece of rope and measured the two sides of the plot and found the largest side to be 50 m and another side to be 10 m. What is the exact length of the third side?

(a) $40\sqrt{3}$ m (b) $30\sqrt{2}$ m
(c) $\sqrt{1560}$ m (d) 32 m

Solution



Let the triangle be ABC.

Now, area of ΔABC = (1/2) (BC) (AD) (Where D is a point on BC such that $AD \perp BC$)

Now, AD has to be equal to $\frac{\text{Area} \,\Delta\, 2}{BC}$

$$AD = \frac{(150)(2)}{50} \text{ m} = 6 \text{ m}$$

Now ΔBDA is a right-angled triangle

$\therefore BD^2 = AB^2 - AD^2 = 10^2 - 6^2 = 8^2 \text{ m}^2$

$\Rightarrow$ BD = 8 m and DC = BC – BD = 42 m

In a right-angled ΔADC, $AC^2 = AD^2 + DC^2 = (6)^2 + (42)^2$

$\Rightarrow AC = \sqrt{1800} = 30\sqrt{2}$

Alternatively, going through the options

$30\sqrt{2} = 42$

$40\sqrt{3} = 68$

$\sqrt{1560} = 39$

$\therefore$ Using the principle that the sum of any two sides of a triangle is greater than the third side. The two given sides are 50 and 10.

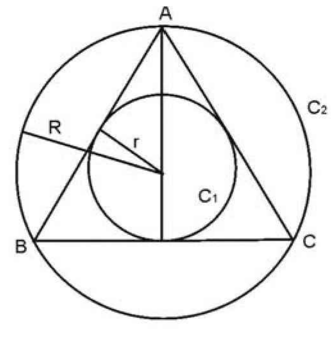
From the choices the above condition is satisfied only for choice (b).

In Choice (a) 10 + 50 < 68
In Choice (c) 39 + 10 < 50
In Choice (d) 32 + 10 < 50

Example 24 C_1 and C_2 are two concentric circles with radii 5 cm and 9 cm respectively. If A, B and C are points on C_2 such that AB and AC are tangent to C_1 at how many points does BC intersect C_1?

(a) 0 (b) 1
(c) 2 (d) Cannot be determined

Solution

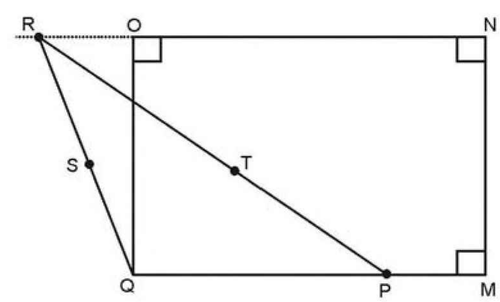


Consider two concentric circles with radii r and R (r < R). If $r = \frac{R}{2}$ and A, B and C are points on the outer circle such that AB and AC are tangent to the inner circle, then BC is also a tangent.

(In this case, C_1 and C_2 become the incircle and circumcircle of ABC respectively, which will be an equilateral triangle)

If $r < \frac{R}{2}$, BC does not intersect or touch the inner circle and if $r > \frac{R}{2}$ then BC intersects the inner circle at two points.

Example 25 A rectangle MNOQ is drawn and length 'NO' is extended to point R and a triangle QPR is drawn with QP $= \frac{2}{3}$ QM. Angle QRP = 45° and side QR $4\sqrt{7}$ cm, S and T are the midpoints of sides QR and PR respectively. If ST = 6 units, the area (in sq. cm) of the rectangle is



(a) 112 (b) 144
(c) 288 (d) 256

Solution Since the line joining the mid-points of two sides of a triangle is parallel and equal to half the third side, we have PQ = 2(ST), $\Rightarrow$ PQ = 12 cm

Since, PQ $= \frac{1}{5}$ QM

Now, $\angle ORP = 45°$

Draw $PV \perp ON$

In ΔRVP, $\tan 45° = \frac{PV}{VR} = \frac{PV}{OR+OV} = \frac{\text{breadth}}{12+OR}$

(Since OV = PQ = 12 cm) also, breadth^2 + OR^2 = (QR)^2

In ΔROQ, $\Rightarrow$ Breadth = x + 12 where OR = x

$\Rightarrow (x + 12)^2 = \left(4\sqrt{17}\right)^2 - x^2$ (given that QR = $\left(4\sqrt{17}\right)$

$\Rightarrow 2x^2 + 24x - 128 = 0$

$\Rightarrow x^2 + 12x - 24 = 0$

$\Rightarrow x = 4$ or -16

Since $x > 0$, $x = 4$

Breadth = x + 12 i.e., 16 cm, and the area of MNOQ = 16 × 18 = 288 sq. cm.

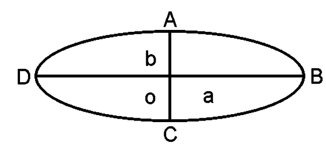
Example 26 ΔABC has sides AB, AC measuring 2001 and 1002 units respectively. How many such triangles are possible with all integral sides?

(a) 2001 (b) 1002 (c) 2003 (d) 1004

Solution Value of BC will lie in between 999 and 3003. Hence 999 < BC < 3003.

So, the total values possible for BC = 2003

Ellipse The path of a moving point which moves in such a way that its distance from a fixed point (focus) bears a constant ratio with its distance from a fixed line (directrix)



Given: OB = OD = a (Semi-major axis)

OA = Ol = b (Semi = minor axis)

- Area = πab
- Perimeter = π(a + b)

Maths behind the formula (Area = πab) One way to see why the formula is true is to realize that the ellipse is just a unit circle that has been stretched by a factor 'a' in the x-direction and by factor 'b' in the y-direction. Hence the area of the ellipse is just $a \times b$ times the area of unit circle.