PERMUTATION AND COMBINATION

MEANING OF PERMUTATION AND COMBINATION

If we go by the dictionary meaning of the words permutation and combination, then permutation is the number of ways in which a set or a number of things can be put in an order or arranged and; combination refers to the number of ways in which a group of things can be chosen from a larger group without regard to their arrangement.

Let us go through an example. Suppose there are four different batsmen A, B, C and D and we have to select a group of three batsmen out of these four. Now, we can select any combination of three batsmen so that no set of batsmen has all the same three batsmen. These set of batsmen will be — ABC, BCD, ABD, ACD. This is a case of combination as for every set of selection of three batsmen, order of selection does not play any role (i.e., we can select anybody—first or second or third—and it does not create any difference in the final selection as well as in the total number of selections).

Now, if we try to define their batting order also, i.e., who bats first and second and so on, then corresponding to every selection of a set of three batsmen, we will have six different arrangements of their batting order. It can be seen below that corresponding to the selection of ABC as a team, following is the list of different batting orders:

ABC, ACB, BAC, BCA, CAB, CBA.

This is a case of permutation since the order of occurrence has become important. Since there are four different ways of selecting a group of three batsmen and every selection can be arranged in 6 different ways, so the total number of ways of arranging 3 batsmen (or, distinct things) out of 4 batsmen (or, distinct things) = $4 \times 6 = 24$ ways.

Permutation and combination can be better understood through the examples of hand-shake and gifts exchange also. Assume that there are 20 persons in a party and everybody shakes hand with each other and also presents a gift. Now if we take a case of two persons A and B, then the event of shaking hand between them is a case of combination because when A shakes hand with B or B shakes hand with A, the number of hand shake is just one. So, there is no order as such and hence it is a case of combination.

Similarly, the event of presenting the gift is a case of permutation because the gift given to B by A and the gift given to A by B are two different gifts. So, the order of case plays a role here and hence it is a case of permutation.

> $n!$ = Product of all the natural numbers from n to 1 = $n(n-1)(n-2)(n-3) \ldots \times 3 \times 2 \times 1$.
>
> $0! = 1$
>
> Factorials are defined only for whole numbers, and not for negative numbers or fractions (≠ whole numbers).

FUNDAMENTAL PRINCIPLES OF COUNTING : TWO BASIC THEOREMS

1. Multiplication theorem

If there are two jobs in such a way that one of them can be done in m ways and when it is completed in any of the m ways, the second job can be completed in n ways, then the whole job can be done in $m \times n$.

2. Addition theorem

If there are two jobs in such a way that one of them can be done in m ways and the second one can be done in n ways independently, then either of the jobs can be done in $(m + n)$ ways.

Basically, there is one point where these two theorems differ—in multiplication the job does not get completed while in addition it gets completed. In a layman's language, we multiply the number of ways when the job has not been completed and we add the number of ways when the job has been completed.

Example 1 There are 10 girls and 15 boys in a class. In how many ways can

i. a class representative be selected?
ii. a team of two students be chosen with one girl and one boy?

Solution

i. A class representative can be a girl or a boy. Now, one girl can be selected from 10 girls in 10 ways (any of the girls can be selected) and one boy can be selected from 15 boys in 15 ways (any of the boys can be selected). So the ways of selecting a class representative includes either selecting a boy or a girl. So, the total number of ways of selecting a class representative $= 10 + 15 = 25$
In this case, the moment a girl gets selected, the job is completed. There are some more ways of doing this by selecting a boy. So, it is a case of addition.
ii. One girl can be chosen from 10 girls in 10 ways. Now corresponding to every selected girl, any one of the 15 boys can be selected in 15 ways.
It can be seen in the following presentation:
Girl selected (Assume name of the girls are $G_1, G_2, G_3, \ldots G_9, G_{10}$) – G_1

Boy selected (Assume the names of the boys are $B_1, B_2, B_3, \ldots B_{14}, B_{15}$)– B_1, Or B_2, or B_3, or B_{15}.

So, corresponding to G_1, the total number of selection of a boy = 15

Corresponding to G_2, the total number of selection of a boy = 15

Corresponding to G_3, the total number of selection of a boy = 15

… … … … …

Corresponding to G_{15}, the total number of selection of a boy = 15

So, the total number of ways of selecting a team of one boy and a girl = the total number of ways of selecting a girl × the total number of ways of selecting a boy = 10 × 15 = 150

In this case, just by selecting a girl or a boy, work has not been completed. So, it is a case of multiplication.

Another example of multiplication theorem If there are three cities A, B and C located in such a way that there are 3 roads joining A and B, and 4 roads joining B and C, then the number of ways one can travel from A to C is 3 × 4, i.e., 12.

PERMUTATIONS

As we have seen, the arrangements made by taking some or all elements out of a number of things is called a permutation. Permutation implies "arrangement" where "order of the things" is important.

The permutations of three things, *a b* and *c*, taken two at a time are *ab, ba, ac, ca, cb* and *bc*. Since the order in which the things are taken is important, *ab* and *ba* are counted as two different arrangements.

The number of permutations of *n* things taking *r* at a time is denoted by ${}^nP_r$.

COMBINATIONS

As we have seen, the groups or selections made by taking some or all elements out of a number of things is called a combination. In combination, the order in which the things are taken is not important.

The combination of three things, *a*, *b* and *c*, taken two at a time are *ab*, *bc* and *ca*. Here, *ab* and *ba* are same because the order in which *a* and *b* are taken is not important. What is required is only a combination including *a* and *b*. The words "combination" and "selection" can be used without any differentiation.

The number of combinations of *n* things taking *r* at a time is denoted by ${}^nC_r$.

Approaching a Problem

Mostly the questions asked in the CAT are self-explanatory, i.e., they clearly mention what process is to be used—permutation or combination.

In case, the question does not specify this, you should try to find out whether it is a case of permutation or a case of combination. Sometimes the problem very clearly states whether it is the number of permutations (or arrangements) or the number of combinations (or selections) that has to be found out. The questions can be as follows:

For permutations:

"What is the number of permutations that can be done … " or "What is the number of arrangements that can be made… " or "Find the different numbers of ways in which something can be arranged… etc.

For combinations:

"What is the number of combinations that can be done … " or "What is the number of selections that can be made … " or "Find the different numbers of ways in which things can be selected etc.

Some other standard examples of permutation and combination are:

Permutation Word formation, number formation and circular permutation etc.

Combination Selection of a team, forming geometrical figures, distribution of things (except some particular cases).

Still, sometimes the questions may not explicitly state what you have to find—permutation or combination. In that case, the nature of what is to be found out will decide whether it is the number of permutations or the number of combinations.

See the example given below

I have to invite two of my eight friends to my anniversary party. In how many different ways can I do this?

Assume my eight friends are A, B, C, D, E, F, G and H. Whether the two friends that I call for the party are A and B or B and A, does not make any difference. As we have discussed before, that what matters most in case of permutation is the order of occurrence of things. As order does not play any role here, it is clearly the case of combination.

Meaning and Derivation of ${}^nP_r$ and ${}^nC_r$

Number of permutations of n different things taking r at a time = ${}^nP_r$

In this statement, we take the following two assumptions:

- All the *n* things are distinct (or no two things are of the same type)
- Each thing is used at most once (i.e., nothing is repeated in any arrangement)

Let us assume that there are *r* boxes and each of them can hold one thing. When all the *r* boxes are filled, what we have is an arrangement of *r* things taken from the given *n* things. So, each time we fill up the *r* boxes with things taken from the given *n* things, we have an arrangement of *r* things taken from the given *n* things without repetition. Hence, the number of ways in which we can fill up the *r* boxes by taking things from the given *n* things is equal to the number of permutations of *n* things taking *r* at a time.

The first box can be filled in *n* ways (because this box can be filled by any one of the *n* things given). After filling the first box, we now have only $(n-1)$ things to fill the second box; any one of these things can be used to fill the second box and hence the second box can be filled in $(n-1)$ ways; Similarly, the third box can be filled in $(n-2)$ ways and so on the rth box can be filled in $(n-(r-1)$ ways, i.e., $[n-r+1]$ ways. Hence, from the fundamental rules of counting, all the r boxes together can be filled up in $\rightarrow n. (n-1). (n-2) \ldots (n-r+1)$ ways

Hence, $^nP_r = n. (n-1). (n-2) \ldots (n-r+1)$

This can be simplified by multiplying and dividing the right hand side be $(n-r)\ (n-\overline{r-1}) \ldots 3.2.1.$

$$^nP_r = n\ (n-1)(n-2) \ldots [n-\overline{r-1}]$$

$$= \frac{n!}{(n-r)!}$$

The number of arrangements of n distinct things taken r things at a time is

$$^nP_r = \frac{n!}{(n-r)!}$$

If we take n things at a time, then we get ^nP_n. From the discussion similar to that we had for filling the r boxes above we can find that ^nP_n is equal to $n!$ The first box can be filled in n ways, the second one in $(n-1)$ ways, the third one in $(n-2)$ ways and so on, then the *nth* box in 1 way; hence, all the n boxes can be filled in

$$^nP_n = n!$$

But if we substitute $r = n$ in the formula for ^nP_n then we get $^nP_n = \frac{n!}{0!}$; since we already found that $^nP_n = n!$

We can conclude that $0! = 1$

The number of combinations of n distinct things taking r at a time = ^nC_r

Let the number of combinations ^nC_r be S. Consider one of these S combinations. Since this is a combination, the order of the r things is not important. If we now impose the condition that order is required for these r things, we can get $r!$ arrangement from this one combination. So each combination can give rise to $r!$ permutations. S combinations will thus give rise to $S \times r!$ permutations. But since these are all permutations of n things taking r at a time, this must be equal to ^nP_r. So,

$$S \times r! = {}^nP_r = \frac{n!}{(n-r)!}$$

$$\text{So, } S = {}^nC_r = \frac{n!}{(n-r)!} \times \frac{1}{r!}$$

It can also be deduced from here that the number of selections of n distinct things taken all at a time will be equal to 1 (since there is only one way in which all the articles can be selected).

$$\text{Alternatively } {}^nC_n = \frac{n!}{0! \times n!} = 1$$

Out of n things kept in a bag, if we select r things and remove them from the bag, we are left with $(n-1)$ things inside the bag i.e., whenever r things are selected out of n things, we automatically have another selection of $(n-1)$ things. Hence, the number of ways of making combinations taking r out of n things is the same as selecting $(n-r)$ things out of n given things, i.e.,

$$^nC_r = {}^nC_{n-r}$$

Before we move ahead, let us once again make it clear that whenever we are using ^nC_r and ^nP_r, our assumption is that all the things are distinct, i.e., no two of them are same.

Example 2 Munchun has 10 children. She takes 3 of them to the zoo at a time, as often as she can, but she does not take the same three children to the 0 more than once. How many times Munchun will be required to go to the zoo?

(a) 120 (b) 45 (c) 90 (d) 180

Solution Number of times (read ways) 3 children (read distinct things) can be selected from 10 children (read distinct things) = $^{10}C_3$.

So, she will be required to go to the zoo $^{10}C_3$ times.

So, option (a) is the answer.

Example 3 In the above question, how many times a particular child will go?

(a) 72 (b) 45 (c) 90 (d) 36

Solution Consider the case for any particular child C_1 Since C_1 has already been selected, so out of the rest 9 children Munchun will be required to select 2 more children. This can be done on $^9C_2$ ways.

So, option (d) is the answer.

Example 4 In a chess competition involving some boys and girls of a school, every student had to play exactly one game with every other student. It was found that in 45 games both the players were girls, and in 190 games both were boys. The number of games in which one player was a boy and the other was a girl is

(a) 200 (b) 216 (c) 235 (d) 256

Solution Let there be m boys and n girls

$$^nC_2 = 45 = \frac{n(n-1)}{2} \Rightarrow n(n-1) = 90 \Rightarrow n = 10$$

$$^mC_2 = 190 = \frac{m(m-1)}{2} = 190 \Rightarrow m\ (m-1) = 380 \Rightarrow m = 20$$

Number of games between one boy and one girl

$= {}^{10}C_1 \times {}^{20}C_1 = 10 \times 20 = 200$

Hence, option (a) is the answer.

Example 5 In how many ways can three persons be seated on five chairs?

Solution This question is a very fundamental problem of arrangements without repetition. The first person can sit in 5 ways (into any of the five chairs), the second person can take place in 4 ways (into any of the remaining 4 chairs) and the third person can sit in 3 ways.

So, the total number of ways in which these 3 persons can arrange themselves on 5 chairs is $5 \times 4 \times 3 = 60$.

Some Important Derivations

While deriving an expression for ^nP_r, we imposed two constraints, viz. distinct things and repetition being not allowed over it and learned how to find the number of permutations. Let us now see what will happen if we do not impose these two restrictions on ^nP_r.

Number of arrangements of *n* things of which *p* are of one type, *q* are of a second type and the rest are distinct When all the things are not distinct, then we cannot use the general formula for ^nP_r for any value of *r*. If we want to find out ^nP_r for a specific value of *r* in that given problem, we will be required to use it on the basis of the given situation.

The number of ways in which *n* things may be arranged taking all of them at a time, when *p* of the things are exactly alike of one kind, *q* of them exactly alike of another kind, *r* of them exactly alike of a third kind, and the rest all is distinct is

$$\frac{n!}{p!.q!.r!}$$

Number of permutations of *n* distinct things where each one of them can be used for any number of times (i.e., repetition allowed) Derivation for this is based upon common sense. If I have 5 friends and 3 servants and I have to send the invitation letters to all my friends through any of my servants, I obviously have 3 options for the invitation card to be sent to friend1, the same 3 options for the invitation card to be sent to friend2, and similarly 3 options for the invitation card to be sent to each of the friends. So the total number of ways of sending the invitation letters = 3^5 And it will not be 5^3, as friends are not going to the servants to get the letter.

In general, the number of perambulations of *n* things, taking *r* at a time when each of the thing may be repeated once, twice,... up to *r* times in any arrangement is n^r.

Total Number of Combinations

Out of *n* things, the number of ways of selecting one or more things:

where we can select 1 or 2 or 3 ... and so on *n* things at time; hence, the number of ways is $^nC_1 + {}^nC_2 + {}^nC_3 + \ldots {}^nC_n = 2^n - 1$, where *n* is the number of things.

Above derivation can also be understood in the following manner:

Let there be *n* bags.

The first bag can be dealt in two ways—it is either included or not included. Similarly, the second bag can be dealt in two ways, the third one in two ways and so on, the *nth* bag in two ways. Using multiplication theorem of counting, the number of ways of dealing with all the bags together is $2 \times 2 \times 2 \times \ldots n$ times = 2^n ways. But out of these, there is one combination where we do not include any of the bags. This is not allowed because we have to select at least one thing.

Hence, the number of ways of selecting one or more things from *n* given things is $2^n - 1$.

Distributing the given things (*m* + *n*) into two groups where one group is having m things and other one *n* things If we select *m* things (which can be done in $^{m+n}C_m$ ways), then we will be left with *n* things, i.e., we have two groups of *m* and *n* things respectively. So, the number of ways of dividing (*m* + *n*) things into two groups of *m* and *n* things respectively is equal to $^{m+n}C_m$.

$$^{m+n}C_m = \frac{(m+n)!}{n!.m!}$$

If we take *m* = *n*, then the above expression will denote "Distributing 2m things equally between two distinct groups $= {}^{2m}C_m = \frac{(2m)!}{m!.m!}$

However, when the groups are identical, then we will be required to divide the above result by 2!.

Hence, in that case it becomes $\frac{(2m)!}{2!m!.m!}$

(Refer to word formation examples)

All the above derivations with their different applications can be seen below in a summarized form.

1. Fundamental Principle of counting:
   (a) *Multiplication rule* If a work is done only when all the number of works are done, then the number of ways of doing that work is equal to the product of the number of ways of doing separate works.
   (b) *Addition rule* If a work is done only when any one of the number of works is done, then the number of ways of doing that work is equal to the sum of the number of ways of doing separate works.
   Thus, if a work is done when exactly one of the number of works is done, then the number of ways of doing this work = sum of the number of ways of doing all the works.
2. If $^nC_x = {}^nC_y$ then either $x = y$ or $x + y = n$.
3. $\lfloor n = 1.2.3 \ldots n$; $\lfloor 0 = 1$
4. (a) The number of permutations of *n* different articles taking *r* at a time is denoted by ^nP_r and $^nP_r = \frac{\lfloor n}{\lfloor n-r}$
   (b) The number of permutations of n different articles taking all at a time = $\lfloor n$.
   (c) The number of permutations of *n* articles, out of which *p* are alike and are of one type, *q* are alike and are of second type and rest are all different $= \frac{\lfloor n}{\lfloor p = \lfloor q}$.

5. The number of permutations (arrangements) of n different articles taking r at a time when articles can be repeated any number of times = $n \times n \times \ldots r$ times = n^r.
6. Circular permutations:
   i. The number of circular permutations (arrangements) of n different articles = $\lfloor n-1$.
   ii. The number of circular arrangements of n different articles when clockwise and anticlockwise arrangements are not different i.e., when the observation can be made from both the sides $= \frac{\lfloor n-1}{2}$.
7. The number or combinations of n different articles taking r at a time is denoted by ${}^nC_r$ and ${}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}$.
8. The number of selections of r articles ($r \le n$) out of n identical articles is 1.
9. Total number of selections of zero or more articles from n distinct articles = ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots {}^nC_n = 2^n$.
10. Total number of selections of zero or more articles from n identical articles = 1 + 1 + 1 +… to (n + 1) terms = n + 1.
11. The number of ways of distributing n identical articles among r persons when each person may get any number of articles = ${}^{n+r-1}C_{r-1}$.
12. The number of ways of dividing $m + n$ different articles in two groups containing m and n articles respectively ($m \ne n$)

$$= {}^{m+n}C_n \times {}^mC_m = \frac{\lfloor m+n}{\lfloor m \lfloor n}.$$

13. The number of ways of dividing $2m$ different articles each containing m articles $= \frac{\lfloor 2m}{\lfloor m \lfloor m \lfloor 2}$.
14. The number of ways of dividing $3m$ different articles among three persons and each is getting m articles = $\frac{\lfloor 3m}{\lfloor m \lfloor m \lfloor m \lfloor 3} \lfloor 3$.
15. The number of ways of selecting n distinct articles taken r at a time when p particular articles are always included = ${}^{n-p}C_{r-p}$.
16. ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$
17. ${}^np_r = r.{}^{n-1}p_{r-1} + {}^{n-1}P_r$

SOME STANDARD FORMATS OF QUESTIONS

Word Formation

As we know that order of occurrence of letters decide the formation of words, so word formation is one standard example of permutation.

Let us understand with the help of some examples:

Example 6 How many words can be formed with the word 'LUCKNOW' which have

i. no restriction
ii. L as the 1st letter of the word
iii. L and W as the terminal letters
iv. all the vowels together
v. L always occuring before U
vi. L always occuring before U and U always occuring before W

Solution

i. The total number of distinct letters = 7 (L, U, C, K, N, O, W)
   So, the total number of words that can be formed = 7!
ii. Now we can arrange only 6 letters (as place of L is restricted),
   So, the total number of words that can be formed = 6!
iii. Now we can arrange only 5 letters (as place of L and W are restricted),
   So, the number of arrangements = 5!
   But the place of L and W can be interchanged between themselves.
   So, the total number of words that can be formed = 5! × 2!
iv. U and O should be together, so we will assume these two letters to be tied up with each other.
   Now we have 6 distinct things to be arranged– (L, U, O, C, K, N, W)
   So, the number of arrangements = 6!
   But place of U and O can be interchanged between themselves.
   So, the total number of words that can be formed = 6! × 2!
v. There is an equal likelihood occurrence of all the letters in the word, so in half of the cases L will occur before U and in the remaining half, U will occur before O.
   So, the total number of words that can be formed = 7!/2
vi. There are six possible arrangements (3!) corresponding to L, U and W. However, only one out of these six will be in the prescribed order: L always occurs before U and U always occurs before W.
   So, corresponding to 7! arrangements, the number of ways in which the condition will be satisfied = 7!/3! ways.

Example 7 How many new words can be formed with the word 'PATNA'?

Solution Total number of letters— P, T, N occur once while A occurs twice.

So, the total number of words that can be formed = 5!/2! = 60

Total number of new words = 60 – 1 = 59

Example 8 How many words can be formed with the word 'ALLAHABAD'?

Solution Letters are— A – Four times
L – Twice
H, B and D occur once.

So, the total number of words = $\frac{9!}{4!2!}$

Example 9 How many 4-lettered distinct words can be formed from the letters of the word 'EXAMINATION'?

Solution Letters are: A – Twice
I – Twice
N – Twice
E, X, M, T, O – Once
Words will be of three types:
i. All distinct, ii. Two same, two distinct, iii. Two same and of one kind: two same and of other kind.

i. All distinct = ${}^8P_4$ (Distinct letters are – A, I, N, E, X, M, T and O)
ii. Two same, two distinct
Selection of one pair out of the three pairs (A, I, N) can be chosen in ${}^3C_1$ ways. Now rest of the two distinct letters can be chosen in ${}^7C_2$ ways.
Total number of words = ${}^3C_1 \times {}^7C_2 \times \frac{4!}{2!}$
iii. Two same and are of one kind, two same and are of other kind = Out of the three pairs of letters (A, I, N), we can select two pairs in ${}^3C_2$ ways.
Total number of words = ${}^3C_2 \times \frac{4!}{2! \times 2!}$

Number Formation

Number formation is another standard example of permutation. Here we will discuss the box diagram method of solving the questions.

If a three-digit number is to be constructed, then we will use

Hundred's place	Ten's place	Unit's place

If a four-digit number is to be constructed, then we will use

Thousand's place	Hundred's place	Ten's place	Unit's place

and so on.

While solving the questions related to number formation, we should know two things very clearly:

- While using the box diagram, we should start with the digit which has restriction, i.e., some condition is imposed on that digit.
- When nothing about the repetition of digits is mentioned in the question, we will assume that the repetition is allowed.

Example 10 How many different 3-digit numbers can be formed using the digits 1, 2, 3, 4 and 5?
i. When repetition is not allowed
ii. When repetition is allowed

Solution The box given below represents the respective positioning of digits in a three-digit number.

Hundred's place	Ten's place	Unit's place

i. Since repetition of the digits is not allowed, we can fill the unit's place in 5 ways, ten's place in 4 ways and hundred's place in 3 ways.

3	4	5

Using Multiplication Theorem, the total number of numbers which can be formed = 5 × 4 × 3 = 60
Alternatively, 3 digits can be selected out of 5 digits in ${}^5P_3 = 60$

ii. Since repetition of the digits is allowed here, we can fill each of the hundred's, ten's and unit's place in 5 ways.

5	5	5

Using Multiplication Theorem, the total number of numbers which can be formed = 5 × 5 × 5 = 125

Example 11 How many 4-lettered numbers divisible by 4 can be formed from the digits 0, 1, 2, 3, 4, 5?

Solution Any number divisible by 4 will have the number formed by its last two digits divisible by 4.

In this case, last two digits of the number can be 00, 04, 12, 20, 24, 32, 40, 44, 52.

Corresponding to any one of 00, 04, 12, 20, 24, 32, 40, 44, 52, we can have the following digits at its hundred's and thousand's place:

5	6	1	1

Thousand's place cannot be filled by 0, so it can be filled in 5 ways.

Hundred's place can be filled by any of the 0, 1, 2, 3, 4, 5; hence 6 ways.

So, corresponding to any one of 00, 04, 12, 20, 24, 32, 40, 44, 52, the total number of ways = 5 × 6 = 30

So, the total number of numbers which can be formed = 30 × 9 = 270

Example 12 In the above question, how many numbers can be formed if repetition of the digits is not allowed?

Solution Last two digits of this number can be—04, 12, 20, 24, 32, 40, 52.

At this point, we will have to divide the process of solving this question—one part will have those numbers which contain '0' as one of its last two digits viz., 04, 20, 40 and other part will have the remaining numbers viz., 12, 24, 32, 52.

1st part – Last two digits are 04, 20, 40.

4	3	1	1

$= 4 \times 3 = 12$ ways

Hence, the total number of numbers which can be formed $= 12 \times 3 = 36$

2nd part — Last two digits are 12, 24, 32, 52.

'0' cannot occur at thousand's place.

3	3	1	1

$= 3 \times 3 = 9$ ways

Hence, the total number of numbers which can be formed $= 9 \times 4 = 36$

Total numbers $= 36 + 36 = 72$

Example 13 How many odd integers from 1000 to 8000 have none of its digits repeated?

Solution There are two restrictions operating in this questions:

i. For a number to be odd, unit digit should be either 1 or 3 or 5 or 7 or 9.
ii. Thousand's place cannot be filled with 8 or 9.

For unit's digit — When it is filled with 9, thousand's place can be filled in 7 ways namely any digit from 1 to 7, and the remaining two places can be filled in $8 \times 7 = 56$ ways.

So, the total number of numbers formed in this way $= 56 \times 7 = 392$

Now, if the unit's place is filled with any of the four digits 1, 3, 5 or 7, the thousand's place can be filled in 6 ways (0 will be excluded), and the remaining two places can be filled in $8 \times 7 = 56$ ways.

So, the total number of numbers formed in this way $= 56 \times 6 \times 4 = 1344$

So, the total number of numbers $= 392 + 1344 = 1736$

Example 14 How many integers from 6000 to 6999 have atleast one of its digits repeated?

Solution Total number of numbers = None of its digits repeated numbers + atleast one of its digits repeated number (i.e., either the digits will be repeated or not repeated).

Total numbers with none of its digits repeated $= 1 \times 9 \times 8 \times 7 = 504$

So, the numbers having atleast one of its digits repeated $= 1000 - 504 = 496$

Example 15 How many natural numbers less than a million can be formed using the digits 0, 7 and 8?

Solution The numbers formed would be of a single digit, two digits, three digits, four digits, five digits and six digits.

Single-digit numbers = 7 and 8

For two-digit numbers,

2	3

$= 2 \times 3 = 6$ numbers

For three-digit numbers,

2	3	4

$= 2 \times 3 \times 3 = 18$ numbers

For four-digit numbers,an

2	3	3	3

$= 2 \times 3 \times 3 \times 3 = 54$ numbers

For five-digit numbers,

2	3	3	3	3

$= 2 \times 3 \times 3 \times 3 \times 3 = 162$ numbers

For six-digit numbers,

2	3	3	3	3	3

$= 2 \times 3 \times 3 \times 3 \times 3 \times 3 = 486$ numbers

So, the total number of numbers = 728

CIRCULAR PERMUTATION

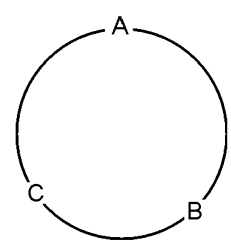
When n distinct things are to be arranged in a straight line, we can do this in $n!$ ways. However, if these n things are arranged in a circular manner, then the number of arrangements will not be $n!$.

Let us understand this:

The number of ways A, B and C can be arranged in a straight line $= 3! = 6$

The possible arrangements are — ABC, ACB, BAC, BCA, CAB, CBA

Now arrange these three people A, B and C in a circle



What we observe here is that the arrangements ABC, BCA and CAB are the same. And similarly the arrangements ACB, CBA and BAC are the same.

So, there are only two permutations in this case of circular permutation.

To derive the formula for circular permutation, we first fix the position of one thing then the remaining $(n - 1)$ things can be arranged in $(n - 1)!$ ways.

Hence, the number of ways in which n distinct things can be arranged in a circular arrangement is $(n - 1)!$

It can be seen in the following way also:

If n things are arranged along a circle, then corresponding to each circular arrangement the number of linear arrangement = n

So, the number of linear arrangements of n different things = $n \times$ (number of circular arrangements of n different things)

Hence, the number of circular arrangements of n different things = $(1/n) \times$ number of linear arrangements of n different things = $(1/n) \times n! = (n-1)!$

Clockwise and Anti-clockwise Circular Arrangements

If we take the case of four distinct things A, B, C and D sitting around a circular table, then the two arrangements ABCD (in clockwise direction) and ADCB (the same order but in anti-clockwise direction) will be different and distinct. Hence, we can conclude that the clockwise and anti-clockwise arrangements are different. However, if we consider the circular arrangement of a necklace made of four precious stones A, B, C and D, the two arrangements discussed as above will be the same because we take one arrangement and turn the necklace around (front to back), then we get the other arrangement. Hence, the two arrangements will be considered as one arrangement because the order of stones is not changing with the change in the side of observation. So, in this case there is no difference between the clockwise and the anti-clockwise arrangements.

Summarizing the above discussion, the number of circular arrangements of n distinct things is $(n-1)!$ if there is a difference between the clockwise and anti-clockwise arrangements and $(n-1)!/2$ if there is no difference between the clockwise and anti-clockwise arrangements.

Example 16 In how many ways 5 Indians and 4 Americans can be seated at a round table if

i. There is no restriction
ii. All the four Americans sit together
iii. No two Americans sit together
iv. All the four Americans do not sit together

Solution

i. Total number of persons = 9. These 9 persons can be arranged around a circular table in 8! ways.
ii. Assuming all the Americans to be one group, we have 6 things (5 Indians + 1 group) to be arranged around a circular table which can be arranged in 5! ways. However, these 4 Americans can be arranged in 4! ways among themselves.
So, the total number of arrangements = $5! \times 4!$
iii. Since there is no restriction on Indians. The 5 Indians can be seated around a table in 4! ways. The Americans will now be seated between two Indians, i.e., 5 places. 4 Indians can be seated on these 5 places in ${}^5P_4$ ways.
iv. The total number of arrangements when there is no restriction = 8! and the number of arrangements when all the four Americans sit together = $5! \times 4!$
So, the total number of arrangements when all the four Americans do not sit together = $8! - 5! \times 4!$.

PERMUTATION AND COMBINATION IN GEOMETRY

It is quite difficult to quantify the importance of P and C in geometry, a good number of P and C questions which use the concepts of geometry (and vice-versa) have been asked in the CAT and other premier B-school exams.

Example 17 How many diagonals will be there in an n-sided regular polygon?

Solution An n-sided regular polygon will have n vertices. And when we join any of these two vertices (${}^nC_2$) we get a straight line, which will be either a side or a diagonal.

So, ${}^nC_2$ = Number of sides + number of diagonals = n + number of diagonals

Hence, the number of diagonals = ${}^nC_2 - n = \dfrac{n(n-3)}{2}$

Above written result can be used as a formula also.

Example 18 Ten points are marked on a straight line and 11 points are marked on another parallel straight line. How many triangles can be constructed with vertices among the above points?

Solution Triangles will be constructed by taking one point from the 1st straight line and two more points from the 2nd straight line, and vice versa.

So, the total number of Δ formed = ${}^{10}C_2 \times {}^{11}C_1 + {}^{11}C_2 \times {}^{10}C_1 = 1045$

Example 19 There is an n-sided polygon (n>5). Triangles are formed by joining the vertices of the polygon. How many triangles can be constructed which will have no side common with the polygon?

Solution An n-sided polygon will have n vertices. Triangles constructed out of these n vertices will be of three types:

i. Having two sides common with the polygon,
ii. Having one side common with the polygon,
iii. Having no side common with the polygon. And total number of triangles formed will be ${}^nC_3$.

i. Having two sides common with the polygon — Out of total n vertices, any combination of three consecutive vertices will give us the triangle which is having two sides common with polygon = n
ii. Having one side common with the polygon — Number of selection of three vertices out of which two are consecutive (If we select A_5 and A_6 as the two vertices, then A_7 or A_4 should not be the third vertex because it will constitute the two sides of the common triangle).
$= n \times {}^{(n-4)}C_1$

iii. So, the total number of triangles having no side common with polygon = ${}^{n}C_3 - n \times {}^{(n-4)}C_1 - n$

Some More Important Results:

1. Maximum No.of points of Intersection among n Straight Lines = ${}^{n}C_2$
2. Maximum No. of points of Intersection among n Circles = ${}^{n}P_2$

FINDING THE RANK OF A WORD

To find the rank of a word out of all the possibilities using all the letters given in the word is nothing but the extension of the concept of alphabetically arranging the words in a dictionary. However, unlike the case of the dictionary, we can have 'meaningless' words also in the case of finding the rank.

Example 20 All the letters of the word 'LUCKNOW' are arranged in all possible ways. What will be the rank of the word LUCKNOW?

Solution Alphabetical order of occurrence of letters— C, K, L, N, O, U, W.

Number of words starting with C = 6!
Number of words starting with K = 6!
All the words starting with LC – 5!
All the words starting with LK – 5!
All the words starting with LN – 5!
All the words starting with LO – 5!
Next word will start with LU – C – K – N – O – W.
So, rank of LUCKNOW – $2 \times 6! + 4 \times 5! + 1 = 1921$.

Example 21 In a chess competition involving some boys and girls of a school, every student had to play exactly one game with every other student. It was found that in 45 games both the players were girls, and in 190 games both were boys. The number of games in which one player was a boy and the other was a girl is

(a) 200 (b) 216 (c) 235 (d) 256

Solution Let there be m boys and n girls

$${}^{n}C_2 = 45 = \frac{n(n-1)}{2} \Rightarrow n(n-1) = 90 \Rightarrow n = 10$$

$${}^{m}C_2 = 190 = \frac{m(m-1)}{2} = 190 \Rightarrow m(m-1) = 380 \Rightarrow m = 20$$

Number of games between one boy and one girl
$= {}^{10}C_1 \times {}^{20}C_1 = 10 \times 20 = 200$
Hence, option (a).

Example 22 If each permutation of the digits 1, 2, 3, 4, 5, 6 is listed in the increasing order of the magnitude, the 289th term will be

(a) 326541 (b) 341256
(c) 356241 (d) 314256

Solution $289 = (2 \times 5!) + (2 \times 4!) + 1$
So, the number will be 341256.

Example 23 There are 12 intermediate stations between two places A and B. In how many ways can a train be made to stop at 4 of these 12 intermediate stations provided no two of them are consecutive?

Solution

1st Method Let $S_1, S_2, \ldots, S_8$ denote the stations where the train does not stop. The four stations where the train stops should be at any four of the nine places indicated by cross.

$$\therefore \text{ Required number} = {}^{9}C_4 = \frac{9.8.7.6}{\underline{|4}} = 126$$

2nd Method Let S_1, S_2, S_3, S_4 be the four intermediate stations where the train stops.

$$\boxed{a}\ S_1\ \boxed{b}\ S_2\ \boxed{c}\ S_3\ \boxed{d}\ S_4\ \boxed{e}$$

Let a, b, c, d, e be the number of stations between A and S_1, S_1 and S_2, S_2 and S_3, S_3 and S_4 and B respectively.

Then, $a + b + c + d + e = 8$ ...(1)
Where $a \geq 0, b \geq 1, c \geq 1, d \geq 1, e \geq 0$
Let $x = a, y = b - 1, z = c - 1, t = d - 1, w = e$
Now $x + y + z + t + w = a + b + c + d + e - 3$
$= 8 - 3 = 5$
Or $x + y + z + t + w = 5$, where $x, y, z, t, w \geq 0$ ...(2)
Required number = number of non negative integral solutions
$= {}^{n+r-1}C_r = {}^{5+5-1}C_5 = {}^{9}C_5 = 126$

Example 24 Find the number of integral solutions of equation $x + y + z + t = 25, x > 0, y > 1, z > 2$ and $t \geq 0$.

Solution Given, $x + y + z + t = 25$, ...(1)
Where $x \geq 1, y \geq 2, z \geq 3, t \geq 0$
Let $p = x - 1, q = y - 2, r = z - 3, s = t$
Then $p + q + r + s = x + y + z + t - 6 = 25 - 6 = 19$, where, $p, q, r, s \geq 0$
$\therefore p + q + r + s = 19, p, q, r, s \geq 0$ ...(2)
$\therefore$ Required number = number of ways in which 19 identical things can be distributed among 4 persons when each person can get any number of things
$= {}^{n+r-1}C_{r-1} = {}^{22}C_3$

Example 25 There are 4 oranges, 5 apricots and 6 alphonso in a fruit basket. In how many ways can a person make a selection of fruits from among the fruits in the basket?

Solution Whenever we are talking about fruits, we assume them to be identical. However when we are talking about men, we treat them to be distinct.

Zero or more oranges can be selected out of 4 identical oranges in 4 + 1 = 5 ways.

Zero or more apricot can be selected out of 5 identical apricots in 5 +1 = 6 ways.

Zero or more can be selected out of 6 identical alphanso in 6 + 1 = 7 ways.

∴ The total number of selections when all the three types of fruits are selected (the number of any type of fruit may also be zero)

$= 5 \times 6 \times 7 = 210$.

But in one of these selections the number of each type of fruit is zero and hence this selection must be excluded.

∴ Required number = 210 – 1 = 209.

Example 26 Twelve different letters of alphabet are given. Words with six letters are formed from these given letters. Find the number of words which have at least one letter repeated.

Solution The total number of letters = 12. Words of six letters are to be framed.

The total number of words of 6 letters when any letter may be repeated any number of times (This also includes the number of words formed when no letter is repeated.)

$= 12 \times 12 \times 12 \times 12 \times 12 \times 12 = 12^6$

Number of words of 6 letters when no letter is repeated $= {}^{12}P_6$.

So, Number of words of 6 letters which have at least one letter repeated $= 12^6 - {}^{12}P_6$