PROBABILITY

PROBABILITY AND ITS APPLICATION

Probability is the measure of uncertainty. We turn our attention to one of the problems that is widely held responsible for the development of the theory of probability, namely, that of throwing a dice. As all of us know, a dice is a well-balanced cube with six faces marked with numbers from 1 to 6, one number on one face.

Let's throw a dice once. What are the possible outcomes? Clearly, a dice can fall with any of its faces uppermost. The number on each of the faces is, therefore a possible outcome and all the outcomes are equally probable. Hence, it is as likely to show up a number, say 2, as any other number 1, 3, 4, 5, or 6.

Since there are six equally likely outcomes in a single throw of a dice and there is only one way of getting a particular outcome (say 2). Therefore, the chance of the number 2 coming up on the dice is 1 by 6. In other words, the same phenomenon is known as probability of getting 2 in a single throw of dice is 1/6.

We write this as $P(2) = 1/6$

Similarly, when an ordinary coin is tossed, it may show up head (H) or tail (T). Hence, the probability of getting a head in a single toss of a coin is given by

$$P(H) = \frac{1}{2}$$

Before we define the process to finding out the probability, it is essential to understand the various terms associated with probability.

Trial and Elementary Events

When we repeat a random experiment under identical conditions, then the experiment is known as a trial and the possible outcomes of the experiment are known as elementary events.

For example,

- tossing of a coin is a trial and getting head or tail is an elementary event.
- throwing of a dice is a trial and getting 5 on its upper face is an elementary event.

Compound Events

Events obtained by combining two or more elementary events are known as the compound events. A compound event is said to occur if one of the elementary events associated with it occurs.

Exhaustive Number of Cases

The total number of possible outcomes of a random experiment in a trial is known as the exhaustive number of cases.

For example, in throwing of a dice the exhaustive number of cases is 6, since any one of the six face marked with 1, 2, 3, 4, 5, 6 may appear on its upper face.

Mutually Exclusive Events

Events are said to be mutually exclusive if the occurrence of any one of them prevents the occurrence of all the others, i.e., if no two or more than two events can occur simultaneously in the same trial.

Equally Likely Events

Events are equally likely if there is no reason for an event to occur in preference to any other event. For example, while throwing a dice, chances of occurring of Head or Tail are an equally likely event.

Favourable Number of Cases

The number of cases favourable to an event in a trial is the number of elementary events such that if any one of them occurs, we say that the event happens.

In other words, the number of cases favourable to an event in a trial is the total number of elementary events such that the occurrence of any one of them ensures the happening of the event.

For example, in throwing of a dice, the number of cases favourable to the appearance of a prime number is 3 viz., 2, 3 and 5.

Independent Events

Events are said to be independent if the happening (or non-happening) of one event is not affected by the happening (or non-happening) of others.

For example, if two cards are drawn from a well-shuffled pack of 52 cards one after the other with replacement, then getting an ace in the first draw is independent of getting a jack in the second draw. But, if the first card drawn in the first draw is not replaced, then the second draw is dependent on the first draw.

Sample Space

This is the most important factor in probability and is defined as the set of all possible outcomes of a random experiment associated with it.

These examples suggest the following definition of probability (assuming that outcomes are equally likely).

Now, Probability of an event E, written as P(E), is defined as

$$P(E) = \frac{\text{Number of outcomes favourable to E}}{\text{Total number of possible outcomes (sample space)}}$$

Points to Remember

1. $0 \le P(E) \le 1$
2. $P(E) + P'(E) = 1$

Example 1 In a single throw of two die, what is the probability that the sum on the top face of both the die will be more than 9?

Solution When two die are thrown, sum of the numbers appearing on the faces can be anything from 2 to 12. To find the number of favourable cases we will be required to find the cases in which the sum is more than 9.

Following are the cases—(5, 5), (6, 4), (4, 6), (6, 5), (5, 6), (6, 6)

So, the total number of favourable cases = 6

The total number of possible outcomes = $6 \times 6 = 36$

Hence probability = 6/36 = 1/6

Example 2 Six dice are thrown simultaneously. Find the probability that all of them show the same face.

Solution Sample space of throwing six dice = $6 \times 6 \times 6 \times 6 \times 6 \times 6 = 6^6$.

All dice are showing the same face implies that we are getting same number on the entire six dice. The number of ways for which is ${}^6C_1$.

Hence, the required probability = $\frac{{}^6C_1}{6^6} = \frac{1}{6^5}$

Example 3 In the above question, find the probability that all of them show a different face.

Solution The total number of ways in which all the dice show different numbers on their top-faces is the same as the number of arrangement of 6 numbers 1, 2, 3, 4, 5, 6 by taking all at a time.

So, the number of favourable cases = 6!

Hence, the required probability = $\frac{6!}{6^6}$

Example 4 Five persons enter a lift on the ground floor of an 8-floor apartment. Assuming that each one of them independently and with equal probability can leave the lift at any floor beginning with the first. What is the probability that all the five persons are leaving the lift at different floors?

Solution Apart from the ground floor, there are 7 floors.

A person can leave the lift at any of the seven floors. Hence the total number of ways in which each of the five persons can leave the lift at any of the 7 floors = 7^5.

So, the sample space = 7^5

Five persons can leave the lift at five different floor = ${}^7P_5$ ways.

So, the favourable number of ways = ${}^7P_5$.

Hence, the required probability = $\frac{{}^7P_5}{7^5}$

Example 5 If you have 3 tickets of a lottery for which 10 tickets were sold and 5 prizes are to be given, the probability that you will win at least one prize is: (JMET 2005)

(a) $\frac{7}{12}$ (b) $\frac{9}{12}$ (c) $\frac{1}{12}$ (d) $\frac{11}{12}$

Solution Probability that you will win at least one prize = 1 – probability that you will not win any prize.

$$= 1 - \frac{5C_3}{10C_3} = \frac{11}{12}$$

Odds in Favour and Odds Against

$$\text{Odds in Favour} = \frac{\text{Number of favourable cases}}{\text{Number of unfavourable cases}}$$

$$\text{Odds in Against} = \frac{\text{Number of unfavourable cases}}{\text{Number of favourable cases}}$$

Understanding And/Or To understand the role played by And/Or in our calculation, let us take the example of throwing an unbiased dice. Let A and B be the two events associated with it such that

A = getting an even number, B = getting a multiple of 3.

Then A = {2, 4, 6}, and B = {3, 6}.

We now define a new event "A or B" which occurs if A or B or both occur i.e., at least one of A or B occurs. Clearly the event "A or B" occurs if the outcome is any one of the outcomes {2,3,4,6}. Thus, the event "A or B" is represented by the subset A U B.

Similarly, "A and B" means occurrence of both A and B which is possible if the outcome is {6}.

Hence, it is represented by the subset $A \cap B$.

ADDITION THEOREM

If A and B are two events associated with a random experiment, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Corollary: If the events are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$

Addition theorem can be extended for any number of events.

Example 6 A basket contains 20 apples and 10 oranges out of which 5 apples and 3 oranges are defective. If a person takes out 2 at random what is the probability that either both are apples or both are good?

Solution Out of 30 items, two can be selected in ${}^{30}C_2$ ways.
So, exhausted number of cases = ${}^{30}C_2$.
Consider the following events:

A = getting two apples; B = getting two good items

Required probability = $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

There are 20 apples, out of which two can be drawn in ${}^{20}C_2$ ways.

$\therefore P(A) = \frac{20C_2}{{}^{30}C_2}$

There are 8 defective pieces and the remaining 22 are good. Out of 22 good pieces, two can be selected in ${}^{22}C_2$ ways.

$$\therefore P(B) = \frac{22C_2}{{}^{30}C_2}$$

Since there are 15 pieces which are good apples of which 2 can be selected in ${}^{15}C_2$ ways, therefore
P $(A \cap B)$ = Probability of getting 2 pieces which are good apples = $\frac{15C_2}{{}^{30}C_2}$

From (i)
Required probability = $P(A) + P(B) - P(A \cap B)$

$$= \frac{{}^{20}C_2}{{}^{30}C_2} + \frac{{}^{22}C_2}{{}^{30}C_2} - \frac{{}^{15}C_2}{{}^{30}C_2} = \frac{316}{435}$$

CONDITIONAL PROBABILITY

Let A and B be the two events associated with a random experiment. Then, the probability of occurrence of A under the condition that B has already occurred and P (B) $\neq$ 0, is called the conditional probability and it is denoted by P(A/B).

Thus, P(A/B) = Probability of occurrence of A given that B has already happened.

Similarly, P (B/A) = Probability of occurrence of B given that A has already happened.

Sometimes, P (A/B) is also used to denote the probability occurrence of A when B occurs. Similarly, P(B/A) is used to denote the probability of occurrence of B when A occurs.

Following examples illustrate the various meanings of these notations:

Example 7 A bag contains 5 white and 4 red balls. Two balls are drawn form the bag one after the other without replacement. Consider the following events.

A = drawing a white ball in the first draw, B = drawing a red ball in the second draw.

Solution Now, P (B/A) = Probability of drawing a red ball in the second draw given that a white ball has already been drawn in the first draw.

Since 8 balls are left after drawing a white ball in the first draw and out of these 8 balls 4 balls are red, therefore

$$P(B/A) = \frac{4}{8} = \frac{1}{2}$$

Note that P (A/B) is not meaningful in this experiment because A cannot occur after the occurrence of B.

Example 8 One card is drawn from a pack of 52 cards, each of the 52 cards being equally likely to be drawn. Find the probability that the card drawn is

i. a king
ii. either a red or a king
iii. a red and a king

Solution Out of 52 cards, one card can be drawn in ${}^{52}C_1$ ways. Therefore, exhaustive number of cases = ${}^{52}C_1$ = 52

i. There are 4 kings in a pack of cards, out of which one can be drawn in ${}^{4}C_1$ ways. Therefore, the favourable number of cases = ${}^{4}C_1$ = 4, so the required probability $= \frac{4}{52} = \frac{1}{13}$
ii. There are 28 cards in a pack of cards which are either a red or a king. Therefore, one can be drawn in ${}^{28}C_1$ ways. Therefore, the favourable number of cases = ${}^{28}C_1$=28. So the required probability $= \frac{28}{52} = \frac{7}{13}$
iii. There are 2 cards which are red and king, i.e., red kings. Therefore, the favourable number of cases = ${}^{2}C_1$ = 2. so the required probability $= \frac{2}{52} = \frac{1}{26}$

Example 9 An urn contains 9 blue, 7 white and 4 black balls. If 2 balls are drawn at random, find the probability that
(i) Both the balls are blue
(ii) One ball is white

Solution There are 20 balls in the bag out of which 2 balls can be drawn in ${}^{20}C_2$ ways. So the total number of cases(sample space) = ${}^{20}C_2$ = 190.

i. There are 9 blue balls out of which 2 balls can be drawn in ${}^{9}C_2$ ways. Therefore, the favourable number of cases = ${}^{9}C_2$ = 36. So the required probability $= \frac{36}{190} = \frac{18}{95}$
ii. There are 7 white balls out of which one white can be drawn in ${}^{7}C_1$ ways. One ball from the remaining 13 balls can be drawn in ${}^{13}C_1$ ways. Therefore, one white and one other colour ball can be drawn in ${}^{7}C_1 \times {}^{13}C_1$ ways. So the favourable number of cases = ${}^{7}C_1 \times {}^{13}C_1$ = 91.

So, the required probability $= \frac{91}{190}$

Example 10 Three persons A, B and C are to speak at a function along with five others. If they all speak in random order, the probability that A speaks before B and B speaks before C is

(a) 3/8 (b) 1/6
(c) 3/5 (d) None of these

Solution The total number of ways in which 8 persons can speak is ${}^{8}P_8$ = 8!. The number of ways in which A, B and C can be arranged in the specified speaking order is ${}^{8}C_3$. There are 5! ways in which the other five can speak. So, the favourable number of ways is ${}^{8}C_3$ = 5!.

Hence, the required probability $= \frac{{}^{8}C_3 \times 5!}{8!} = \frac{1}{6}$