AVERAGE

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Traditionally, average is calculated by dividing the sum of all the numbers by the number of numbers.

$$\text{Average} = \frac{\text{Sum of numbers}}{\text{Number of numbers}}$$

For example, the average of the four numbers 214, 215, 219, 224 will be

$$\text{Average} = \frac{214+215+219+224}{4} = 218$$

Central Value Meaning of Average

Average can also be seen as the central value of all the given values.

Applying this definition for the above example let us assume the central value of all the given numbers = 214

Now, find the deviations of all the numbers from 214

214 215 219 224

When assumed central value is (214), the sum of the deviations

= 0 + 1 + 5 + 10

Now, finding the average of deviations gives us

$$\frac{0+1+5+10}{4} = \frac{16}{4} = 4$$

So, average = assumed central value + average of deviations = 214 + 4 = 218

Thus, we can assume any value to be the assumed average and then find the average of all the deviations; and when we add all the numbers and divide it by number of numbers, 0 is assumed to be the central value.

Example 1 Average age of A, B and C is 84 years. When D joins them the average age of A, B, C and D becomes 80 years. A new person E, whose age is 4 years more than D, replaces A and the average age of B, C, D and E becomes 78 years. What is the age of A?

Solution Since the average age of A, B and C is 84 years so, we can assume that age of A, B and C is 84 years.

A = 84 years

B = 84 years

C = 84 years

After D has joined them,

	Initially	*Finally*
A	84 years	80 years
B	84 years	80 years
C	84 years	80 years
D	…..	80 years

Decrease in the age of A, B and C can be attributed to the increase in the age of D. So, after getting 12 years in total (4 years each from A, B and C) D is at 80 years. The original age of D = 80 − 12 = 68 years.

So, age of E = 72 years

Now, the average age of A, B, C and D = 80 years; A + B + C + D = 320

And average of B, C, D and E = 78 years; B + C + D + E = 312

(Since the average difference between the age of A and E is 2 years,)

Difference (A – E) = 2 × 4 = 8 years

Since E = 72 years, so A = 80 years

By using central value method of averages every question of average can be done by mental calculation only.

Example 2 Average of ten two digit numbers is S. However when we reverse one of the numbers AB as BA from the given 10 numbers, then the average becomes S + 1.8. What is the value of B – A?

Solution Average of 10 numbers is increasing by 1.8, so it can be assumed that 1.8 has been added to all the numbers.

So, BA is 1.8 × 10 = 18 more than AB.

There are so many two-digit numbers which satisfy above condition. Using hit and trial method, the number can be 13, 24, 35, 46, 57, 68, 79. In every case, difference between the digits = 2

Otherwise, we can use the formula

(BA – AB) = 9 × (B – A)

Where BA and AB are two-digit numbers.

Example 3 The average score of Rahul Dravid after 25 innings is 46 runs per innings. If after the 26th innings, his average runs increased by 2 runs, then what is his score in the 26th inning?

Solution Runs in 26th inning = Total runs after 26th innings – Total runs after 25th innings

= 26 × 48 − 25 × 46 = 98

Alternatively, this question can be done by the above given central value meaning of average. Since the average increases by 2 runs per innings, we can assume that 2 runs have been added to his score in each of the first 25 innings. Now, the total runs added in these innings have been contributed by the runs scored in the 26th inning, which must be equal to 25 × 2 = 50 runs.

And after contributing 50 runs, his score in the 26th inning is 48 runs.

Hence, runs scored in the 26th inning = new average + old innings × change in average

= 48 + 25 × 2 = 98.

To have a mental mapping, we can see the whole situation as:

Number of innings	*Average in the 1st 25 innings*	*Averate in the 1st 26 innings*	*Addition*
1	46	48	2
2	46	48	2
3	46	48	2
...	...	...	...
...	...	...	...
...	...	...	...
25	46	48	2
26		48	

Properties of Average

1. Average always lies in between the maximum and the minimum value. It can be equal to the maximum or minimum value if all the numbers are equal.
   For example—A1, A2, A3 and A4 are four numbers given where A1>A2>A3>A4.
   Average of these four numbers will always lie in between A1 and A4.
   However, if all the four numbers are equal (A1 = A2 = A3 = A4) then the average will be equal to each of these numbers.
   Average = A1 = A2 = A3 = A4.
2. Average is the resultant of net surplus and net deficit, as used in the central tendency method.
3. When weights of different quantities are same, then simple method is used to find the average. However, when different weights of different quantities are taken, then it is known as weighted average. Here the method of weighted average is used to find the average.
   For example, assume per capita income of India is USD 500 and per capita income of US is USD 200. Now if we merge India and US into one country then it is observed that per capita income of this new country will not be equal to $\frac{500+200}{2}$ = USD 350
4. If the value of each quantity is increased or decreased by the same value S, then the average will also increase or decrease respectively by S.
5. If the value of each quantity is multiplied by the same value S, then the average will also be multiplied by S.
6. If the value of each quantity is divided by the same value S ($S \neq 0$) then the average will also be divided by S.

Example 4 The average of 4 positive numbers is A and the average of all the possible triples formed out of these four positive numbers is B. Which of the following is true regarding A and B?

(a) A = B (b) A > B
(c) A < B (d) Cannot be determined

Solution Let us assume that the numbers are 1, 2, 3 and 4

Average of 1, 2, 3 and 4 : 1 + 2 + 3 + 4 = 10/4 = 2.5

The triplets are 1, 2 and 3; and the average = $\frac{6}{3}$

1, 2 and 4; and the average = $\frac{7}{3}$

1, 3 and 4; and the average = $\frac{8}{3}$

2, 3 and 4; and the average = $\frac{9}{3}$

Average of these four averages = $\frac{\frac{6}{3}+\frac{7}{3}+\frac{8}{3}+\frac{9}{3}}{4} = 2.5$

So, option a (A = B) is the answer.

Central value method

It should be observed here that when we find the average of all the possible triplets, all the numbers (1, 2, 3, 4) are added thrice. So effectively we are adding 12 numbers. Hence, the average should be equal to:

$$\frac{3(1+2+3+4)}{12} = 2.5$$

Thus, the average will be equal in all the cases.

Extension to this problem The average of four positive numbers is A and the average of all the possible pairs formed out of these four positive numbers is B. Which of the following is true?

(a) A = B (b) A > B
(c) A < B (d) Cannot be determined

Some Special Cases

1. Average involving time, speed and distance

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

However, while solving the questions involving time, speed and distance, we should assume some distance, preferably the LCM of all the given speeds.

Example 5 Lovely goes to Patna from New Delhi at a speed of 40 km/h and returns with a speed of 60 km/h. What is her average speed during the whole journey?

Solution Assuming that the total distance between Patna and New Delhi is 120 km (LCM of 40 and 60) the total time taken (Patna – New Delhi and New Delhi – Patna) = 3 + 2 = 5 h

So, average Speed = $\frac{240}{5}$ = 48 km/h

2. Average involving age

Average of a group of n persons given at any point of time can be calculated in the following way

5 years ago	*10 years ago*	*Now*	*10 years later*	*5 years later*
N – 5	N – 10	N	N + 10	N + 5

Example 6 The average age of the 5 members of a family is 20 years. The youngest member of the family is 4 years old. At the time of his birth the average age of the rest of the family was N years. What is the average age of the family (in terms of N) excluding the youngest member?

Solution Sum of ages of all the members of the family = 100

Sum of ages of all the members of the family excluding the youngest number = 100 – 4 = 96

So, average age of all the members of the family excluding the youngest number = 96/4 = 24 = N

What is the average age of the family(in terms of N) excluding the youngest member = N + 4

3. Average involving number system

1. Average of 1st n consecutive natural numbers $= \frac{n+1}{2}$
   For example, the average of 1st five natural numbers = 3
2. The average of 1st n consecutive even natural numbers $= n + 1$
   Sum of 1st n consecutive even natural numbers $= n(n + 1)$
   For example, the average of 1st five even natural numbers = 6
3. The average of 1st n consecutive odd natural numbers $= n$
   Sum of 1st n consecutive odd natural numbers $= n^2$
   For example, the average of 1st five odd natural numbers = 5

Weighted Average

It is observed that the average can be calculated only if the weights of all the factors, is same. So, the weighted average is a more generalized form of average. This can be further understood with the following illustration

	Class A	*Class B*
No. of students	10	10
Average age	12 years	16 years

Now, if we combine both these classes, then the average age of all the students $= \frac{12+16}{2} = \frac{28}{2} = 14$ years. This is one standard example of Average.

Let us see another example:

	Class A	*Class B*
No. of students	12	16
Average age	10 years	14 years

Now, if we combine these two classes, then the average can not be calculated by the above mentioned method, since the weights attached to different averages are different.

Some more cases of weighted average

1. As we have observed above in the case of average, if per capita income of India is USD 500 and per capita income of US is USD 200, and if we merge India and US into one country, then the per capita income of this new country (India + US) cannot be found by just adding the per capita income of both the countries and dividing it by 2.
   The weights, i.e., the population attached to the different averages, i.e., the per capita income would also have to the considered.
2. Average speed cannot be calculated by just adding the different speeds and then dividing it by 2. This can be understood by the following example:
   A person goes to A from B at a speed of 40 km/h and returns with a speed of 60 km/h, then the average speed for the whole journey can not be equal to 50 km/h.

   We know that average speed $= \frac{\text{Total distance}}{\text{Total time}}$

Finding expression for weighted average

Let us assume there are N groups with the following structure:

Group no.	*No. of members*	*Avg. age of the group*
G_1	N_1	A_1
G_2	N_2	A_2
G_3	N_3	A_3
…	…	…
…	…	…
G_N	N_N	A_N

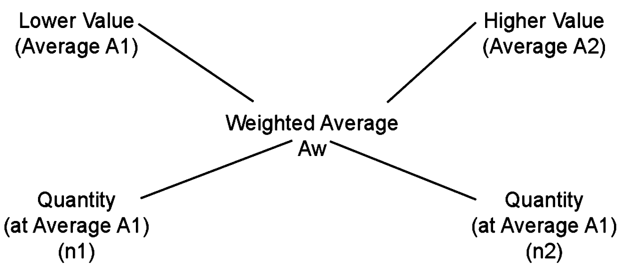
If we combine all these groups, then the average age of all the members = $(N_1 \times A_1 + N_2 \times A_2 \ldots + N_N \times A_N)/(N_1 + N_2 + N_3 + \ldots + N_N)$

$$= \frac{\sum AN}{\sum N}$$

Group no.	*No. of members*	*Avg. age of the group*
G_1	N_1	A_1
G_2	N_2	A_2

Considering that there are only two groups and both the groups are combined then the average age of all the members $= (N_1 \times A_1 + N_2 \times A_2)/(N_1 + N_2) = A_w$

Simplifying the above written expression, we get the conventional criss-cross method as given below



And we write this as: $\frac{n_1}{n_2} = \frac{A_2 - A_w}{A_w - A_1}$

i.e.,

$$\frac{\text{Quantity (Lower Priced)}}{\text{Quantity (Higher Priced)}} = \frac{\text{Higher Price} - \text{Averge Price}}{\text{Average Price} - \text{Lower Price}}$$

It is quite obvious that the ratio of the number of persons/items in different groups is proportionate to the deviations of their average from the average of all the people combined.

This average of all the members combined is known as weighted average, and is denoted by A_w. This process of mixing the two groups is also referred as alligation.

Elements of weighted average

As we can see from the above derivation there are five quantities

i. Number of members in 1st group (n_1)
ii. Number of members in 2nd group (n_1)
iii. Average of 1st group (A_1)
iv. Average of 2nd group (A_2)
v. Weighted average (A_w)

Normally, in the case of weighted average, we get questions in which one of these five elements is missing, and with the help of the remaining four quantities, the value of that missing quantity is found. Different possibility (situations) are given below (Y represents – data given, N represents – data not given):

Situations	n_1	n_2	A_1	A_2	A_w
First	Y	Y	Y	Y	N
Second	Y	Y	Y	N	Y
Third	Y	Y	N	Y	Y
Fourth	Y	N	Y	Y	Y
Fifth	N	Y	Y	Y	Y
Sixth	N	N	Y	Y	Y

First Situation

Example 7 10 kg of rice priced at Rs 12 per kg is mixed with 6 kg of rice priced at Rs 16 per kg. What is the average price of the whole mixture?

Solution Lower priced value = Rs 12 per kg and its quantity = 10 kg

Higher priced value = Rs 16 per kg and its quantity = 6 kg

Using Alligation,

$\frac{10}{6} = \frac{16 - A_w}{A_w - 12}$, or, A_w = Rs 13.5/kg

However, in my opinion, in this situation it is better to use the normal method rather than using the weighted average method of finding A_w.

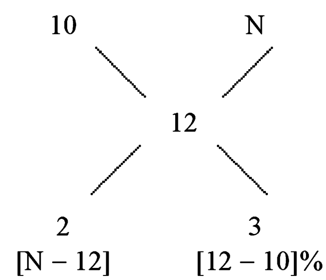
Normal method – Total value = $12 \times 10 + 16 \times 6 = 216$

So, average price = $\frac{216}{16}$ = Rs 13.5/kg

Second/Third Situation

Example 8 Two varieties of rice are mixed in the ratio 2 : 3. The price of the mixture is Rs 12 per kg and the price of the variety having lower weight is Rs 10 per kg. Find the price of the other variety.

Solution



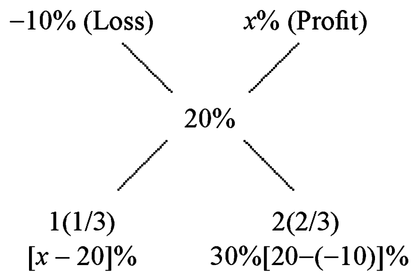
Now, $\frac{2}{3} = \frac{(N-12)}{(12-10)} = \frac{N-12}{2}$

So, N = Rs 13.33 per kg

Fourth/Fifth Situation

Example 9 Some articles are purchased for Rs 450. 1/3rd of the articles are sold at a loss of 10%. At what percentage profit should the remaining articles be sold to obtain a net profit of 20% on the whole transaction?

Solution



Now, = $[20-(-10)]/[x-20] = \frac{2}{1}$

So, $x = 35\%$

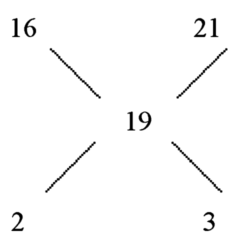
It is seen that the quantities are in the ratio of 1 : 2, so the deviation from mean percentage profit in the loss percentage and profit percentage will also be same.

2 corresponds to 30%, so 1 will correspond to 15%. So, $x = 35\%$

Sixth Situation

Example 10 Two different qualities of sugar are mixed in some ratio. The price of one quality of sugar is Rs 16/kg and that of another quality is Rs 21/kg. In what ratio have the sugar of two qualities been mixed if the price of the mixture is Rs 19/kg?

Solution



So, the ratio of quantity of sugar of different qualities = 2 : 3

MIXTURES

When two or more than two pure substances/mixtures are mixed in a certain ratio, they create a mixture. Here we shall confine ourselves to mostly homogenous mixtures in view of the questions commonly asked in CAT.

Mixing without Replacement

In this particular type of mixing, two or more than two substances are mixed without any part of any mixture being replaced.

Example 11 In a mixture of 420 litres the ratio of milk and water is 6 : 1. Now, 120 litres of the water is added to the mixture. What is the ratio of milk and water in the final mixture?

Solution Volume of milk = 360 litres and volume of water = 60 litres.

When 120 litres of water is added, volume of water = 180 litres

So, the ratio of milk water = 2 : 1

Example 12 A milkman mixes 20 litres of water with 80 litres of milk. After selling one-fourth of this mixture, he adds water to replenish the quantity that he had sold. What is the current proportion of water to milk?

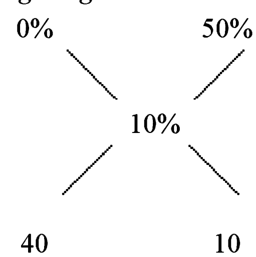
Solution Ratio of milk and water = 20 : 80

When 1/4th of this mixture is sold, total volume of mixture will be reduced by 25%, so 25% of milk and water both will reduce. So, volume of milk and water after selling out 1/4th of mixture = 60 litres and 15 litres respectively. Addition of 25 litres of water will finally give us the following: volume of milk = 60 litres and volume of water = 40 litres. Hence, the ratio of water and milk = 40 : 60 = 2:3.

Example 13 How many litres of fresh water should be mixed with 30 litres of 50% milk solution so that resultant solution is a 10% milk solution?

Solution

Method 1 *Using Alligation*



So, the ratio of fresh water added: milk solution = 4 : 1

Hence, 120 litres of fresh water should be added.

Method 2 *Principle of constant volume of one component*

Since we add fresh water, the volume of milk will be constant.

Now volume of milk = 15 litres = 10% of the new mixture.

So, 100% of the new mixture = 150 litres

So, volume of fresh water added = 150 – 30 = 120 litres.

Method 3 *Principle of inverse proportion*

We know that concentration is inversely proportional to the volume of solute added.

So, in this case $30 \times 50\% = 10\% \times (30 + x)$, where x is the volume of water added.

So, $x = 120$ litres

Method 4 *Using equation*

In the final mixture, $\frac{\text{Milk}}{\text{Total}} = 10\% = \frac{15}{30+x}$

So, x = 120 litres

Mixing with replacement In this particular type of mixing, two or more than two substances are mixed by replacing some part of a mixture. In these types of questions, total volume may or may not be the same and information regarding the same can be obtained from the question.

Case 1 When the quantity withdrawn and the quantity replaced are of the same volume.

Initially there are 40 litres of milk, and 4 litres of milk is replaced with 4 litres of water

Obviously, there will be 36 litres of milk and 4 litres of water.

Now, 4 litres of mixture is replaced with 4 litres of water

The quantity of milk and water being withdrawn here will be in the ratio of 9 : 1 (36:4). So, quantity of milk withdrawn = 9/10 × 4 = 3.6 l.

So, the volume of milk = 32.4

And the volume of water = 7.6

Now, again 4 litres of mixture is replaced with 4 litres of water

The quantity of milk and water being withdrawn here will be in the ratio of 81 : 19 (32.4:7.6). So, the quantity of milk withdrawn = (81/100) × 4 = 3.24 l

So, the volume of milk = 29.16

And the volume of water = 10.84

If we summarize the above values, then it looks like

	1st operation		*2nd operation*		*3rd operation*	
	Taken out	*Left*	*Taken out*	*Left*	*Taken out*	*Left*
Milk	4	36	3.6	32.4	3.24	29.16
Water	0	4	0.4	7.6	0.76	10.84

It can be seen that the quantity of water or milk withdrawn is 10% of the existing volume of milk or water because only 10% of the total volume of 40 litres taken out.

With this we can deduce a standard formula for these kinds of calculations.

If V is the initial volume of milk (or any liquid), and x litres of milk is always replaced by water, then quantity of milk left after n such operations = $V\left(1-\frac{x}{v}\right)^n$

This formula is very similar to the standard formula we have seen in the case of Compound Interest $\left[P(1+\frac{R}{100})n\right]$.

The only difference between the two formulae is that while the interest is being added every year (or for the given time-period), volume of milk gets reduced after every operation.

Using the values of the above example, quantity of milk left after 3 operations $= 40\times\frac{36}{40}\times\frac{36}{40}\times\frac{36}{40} = 19.16$ litres

The same problem can be solved with straight-line approach of percentage also Since 10% of existing volume is taken out every time, the percentage of milk in the final mixture after the third operation = 72.9%

$\left(100\% \xrightarrow{10\%\downarrow} 90\% \xrightarrow{10\%\downarrow} 81\% \xrightarrow{10\%\downarrow} 72.9\%\right)$

Since 100% = 40, so 72.9% = 29.16 litres

Case 2 When the quantity withdrawn and the quantity replaced are of the same volume, but the total volume before replacement does not remain the same.

Initially, there are 40 litres of milk, and 4 litres of milk is taken out and 4 litres of water is poured in

So, there will be 36 litres of milk and 4 litres of water.

Now, 5 litres of mixture is taken out and 5 litres of water is poured in.

The quantity of milk and water being withdrawn here will be in the ratio of 36 : 4. So, the quantity of milk withdrawn $= \frac{36}{40}\times 5$

Milk left = $40\times\frac{36}{40}\times\frac{35}{40}$

Again, if now 6 litres of mixture is taken out and 6 litres of water is poured in

Milk left = $40\times\frac{36}{40}\times\frac{35}{40}\times\frac{34}{40}$

Case 3 When the quantity withdrawn and the quantity replaced are not of the same volume.

Initially there are 40 litres of milk, and 4 litres of milk is taken out and 5 litres of water is poured in

Obviously, there will be 36 litres of milk and 5 litres of water.

Now, 5 litres of mixture is taken out and 6 litres of water is poured in then the quantity of milk and water being withdrawn will be in the ratio of 36 : 5. So, the quantity of milk withdrawn = $\frac{36}{41}\times 5$

Milk left = $40\times\frac{36}{40}\times\frac{36}{41}$

Again 6 litres of mixture is taken out and 7 litres of water is poured in.

Thus, the volume of milk in the final mixture

$= 40\times\frac{36}{40}\times\frac{36}{41}\times\frac{36}{42}$

Example 14 Two vessels A and B of equal capacities contain mixtures of milk and water in the ratio 4 : 1 and 3 : 1, respectively. 25% of the mixture from A is taken out and added to B. After mixing it thoroughly, an equal amount is taken out from B and added back to A. The ratio of milk to water in vessel A after the second operation is

(a) 79:21 (b) 83:17

(c) 77:23 (d) 81:19

Solution Assume there is 20 litres of the mixture in both the vessels.

In vessel A, milk = 16 litres and water = 4 litres

25% from A to B = milk in B = 15 + 4 = 19 litres

= water in B = 5 + 1 = 6 litres

ratio = 19:6

Equal amount from vessel B to vessel A

$$= \text{milk in } A = 12 + \frac{19}{5} = \frac{79}{5}$$

$$= \text{water in } A = 3 + \frac{6}{5} = \frac{21}{5}$$

Hence, the ratio is 79:21