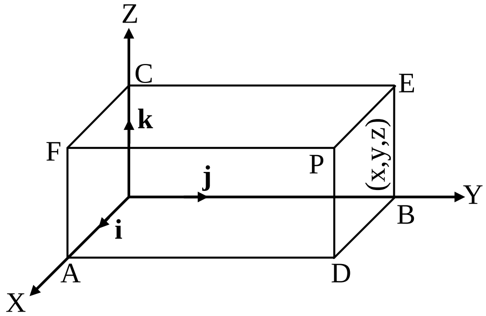
THREE DIMENSIONAL COORDINATE GEOMETRY

1. Coordinates of a point in space

Let O be a fixed point known as origin and let OX, OY and OZ be three mutually perpendicular lines, taken as x-axis, y-axis and z-axis respectively in such a way that they form a right - handed system.

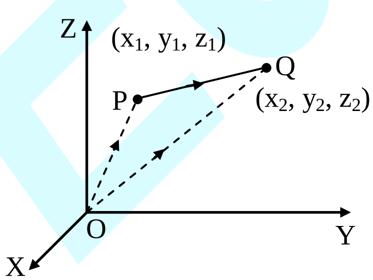


The planes XOY, YOZ and ZOX are known as xy-plane, yz-plane and zx-plane respectively.

Let P be a point in space and distances of P from yz, zx and xy-planes be x,y,z respectively (with proper signs), then we say that coordinates of P are (x, y, z). Also OA = x, OB = y, OC = z.

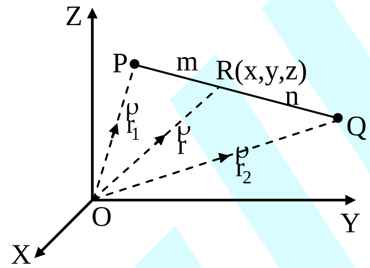
2. Distance between two points

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, then distance between them $PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ In particular distance of a point (x, y, z) from origin $= \sqrt{x^2 + y^2 + z^2}$.



3. Coordinates of division point

Coordinates of the point dividing the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m_1 : m_2$ are



(i) In case of internal division

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$$

(ii) In case of external division

$$\left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}, \frac{m_1z_2 - m_2z_1}{m_1 - m_2}\right)$$

Note :

(a) Coordinates of the Mid point :

When division point is the midpoint of PQ, then ratio will be 1 : 1; hence coordinates of the midpoint of PQ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

(b) Centroid of a Triangle :

If (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the vertices of a triangle, then the centroid of the triangle is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

(c) Division by Coordinate Planes :

The ratios in which the line segment PQ joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is divided by coordinate planes are as follows.

(i) by yz - plane: $-\frac{x_1}{x_2}$ ratio

(ii) by zx - plane: $-\frac{y_1}{y_2}$ ratio

(iii) by xy - plane : $-\frac{z_1}{z_2}$ ratio

(d) Centroid of a Tetrahedron :

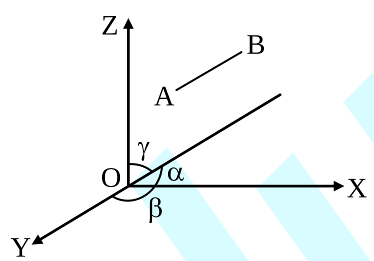
If (x_r, y_r, z_r), r = 1, 2, 3, 4 are vertices of a tetrahedron, then coordinates of its centroid are $\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$

4. Direction cosines & direction ratio's of a line

4.1 Direction cosines of a line [Dc's] :

The cosines of the angles made by a line with coordinate axes are called the direction cosines of that line.

Let α, β, γ be the angles made by a line AB with coordinate axes then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of AB which are generally denoted by λ, m, n. Hence $\lambda = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$



Note :

$-1 < \cos x < 1 \ \forall \ x \in R$, hence values of λ, m,n are such real numbers which are not less than – 1 and not greater than 1. Hence DC's $\in [-1, 1]$

Direction cosines of coordinate axes :

x-axis makes 0°, 90° and 90° angles with three coordinate axes, so its direction cosines are
cos 0°, cos 90°, cos 90°, i.e. 1, 0, 0.

Similarly direction cosines of y-axis and z-axis are 0, 1, 0 and 0, 0, 1 respectively. Hence

dc's of x - axis = 1, 0, 0

dc's of y - axis = 0, 1, 0

dc's of z - axis = 0, 0, 1

Note : (i) The direction cosines of a line parallel to any coordinates axis are equal to the direction cosines of the corresponding axis.

(ii) Relation between dc's : $\lambda^2 + m^2 + n^2 = 1$

4.2 Direction ratios of a line [DR's]

Three numbers which are proportional to the direction cosines of a line are called the direction ratios of that line. If a, b, c are such numbers which are proportional to the direction cosines λ, m, n of a line then a, b, c are direction ratios of the line. Hence

a, b, c dr's $\Leftrightarrow \frac{a}{\lambda} = \frac{b}{m} = \frac{c}{n}$

Further we may observe that in above case

$$\frac{\lambda}{a} = \frac{m}{b} = \frac{n}{c} \pm \frac{\sqrt{\lambda^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}$$

$$= \pm \frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow \lambda = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}},$$

$$n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Note :

(i) Numbers of dr's are not unique whereas numbers of dc's are unique.

(ii) $a^2 + b^2 + c^2 \neq 1$.

4.3 Direction cosines of a line joining two points:

Let $P \equiv (x_1, y_1, z_1)$ and $Q \equiv (x_2, y_2, z_2)$; then

(i) dr's of PQ : $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$

(ii) dc's of PQ : $\dfrac{x_2 - x_1}{PQ}, \dfrac{y_2 - y_1}{PQ}, \dfrac{z_2 - z_1}{PQ}$ i.e.

$$\frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}}$$

5. Angle between to lines

Case-I : When dc's of the lines are given

If λ_1, m_1, n_1 and λ_2, m_2, n_2 are dc's of given two lines, then the angle θ between them is given by

* $\cos\theta = \lambda_1\lambda_2 + m_1m_2 + n_1n_2$

* $\sin\theta = \sqrt{(\lambda_1 m_2 - \lambda_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1\lambda_2 - n_2\lambda_1)^2}$

The value of sin θ can easily be obtained by the following form :

$$\sin\theta = \sqrt{\begin{vmatrix}\lambda_1 & m_1\\ \lambda_2 & m_2\end{vmatrix}^2 + \begin{vmatrix}m_1 & n_1\\ m_2 & n_2\end{vmatrix}^2 + \begin{vmatrix}n_1 & \lambda_1\\ n_2 & \lambda_2\end{vmatrix}^2}$$

Case-II : When dr's of the lines are given

If a_1, b_1, c_1 and a_2, b_2, c_2 are dr's of given two lines, then the angle θ between them is given by

* $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

* $\sin\theta = \dfrac{\sqrt{\Sigma(a_1b_2 - a_2b_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

Conditions of Parallelism and Perpendicularity of Two Lines :

If two lines are parallel then angle between them is 0° and if they are perpendicular then angle between them is 90°. In these cases using above formulae for sin θ and cos θ respectively, we shall get the following conditions.

Case-I : When dc's of two lines AB and CD, say λ_1, m_1, n_1 and λ_2, m_2, n_2 are known

$AB \parallel CD \Leftrightarrow \lambda_1 = \lambda_2, m_1 = m_2, n_1 = n_2$

$AB \perp CD \Leftrightarrow \lambda_1\lambda_2 + m_1m_2 + n_1n_2 = 0$

Case-II : When dr's of two lines AB and CD, say a_1, b_1, c_1 and a_2, b_2, c_2 are known

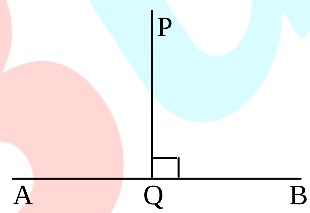
$AB \parallel CD \Leftrightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

$AB \perp CD \Leftrightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$

Projection of a point on a line

Let P be a point and AB be a given line. Draw perpendicular PQ from P on

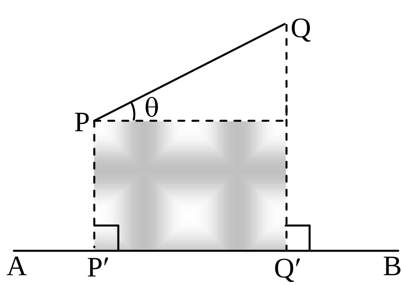
AB which meets it at Q. This point Q is called projection of P on the line AB.



Projection of a line segment joining two points on a line

Let PQ be a line segment where $P \equiv (x_1, y_1, z_1)$ and $Q \equiv (x_2, y_2, z_2)$; and AB be a given line with dc's as λ, m, n. If the line segment PQ makes θ angle with the line AB, then projection of PQ is P'Q' = PQ cos θ. On replacing the value of cos θ in this, we shall get the following value of P' Q'

$P'Q' = \lambda(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$



Note :

(i) For x-axis, λ = λ, m = 0, n = 0 hence

Projection of PQ on x-axis = $1.(x_2 - x_1) + 0 + 0$

$= (x_2 - x_1)$

Projection of PQ on y-axis = $y_2 - y_1$

Projection of PQ on z-axis = $z_2 - z_1$

(ii) $\because PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

= the sum of the squares of the projections of PQ on coordinate axes

$\therefore$ if a, b, c are the projections of a line segment on coordinate axes, then

length of the segment = $\sqrt{a^2 + b^2 + c^2}$

(iii) If a, b, c are projections of a line segment on coordinate axes then its dc's are

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

6. Cartesian equation of a line passing through a given point & given direction ratios

Cartesian equation of a straight line passing through a fixed point (x_1, y_1, z_1) and having direction ratios a, b, c is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

Note

(1) The parametric equations of the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ are $x = x_1 + a\lambda$, $y = y_1 + b\lambda$, $z = z_1 + c\lambda$, where λ is the parameter.

(2) The coordinates of any point on the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ are

$(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, where $\lambda \in R$.

(3) Since the direction cosines of a line are also direction ratios, therefore equation of a line passing through (x_1, y_1, z_1) and having direction cosines λ, m, n is $\frac{x - x_1}{\lambda} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$

(4) Since x, y and z- axis passes through the origin and have direction cosines 1, 0, 0; 0, 1, 0 and 0, 0, 1 respectively. Therefore their equations are x-axis

: $\frac{x - 0}{1} = \frac{y - 0}{0} = \frac{z - 0}{0}$ or y = 0 and z = 0

y- axis : $\frac{x - 0}{0} = \frac{y - 0}{1} = \frac{z - 0}{0}$ or x = 0 and z = 0

z- axis : $\frac{x - 0}{0} = \frac{y - 0}{0} = \frac{z - 0}{1}$ or x = 0 and y = 0

CARTESIAN EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS

The cartesian equation of a line passing through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Perpendicular distance of a point from a line

Cartesian Form : To find the perpendicular distance of a given point (α, β, γ) from a given line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Let L be the foot of the perpendicular drawn from P (α, β, γ) on the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

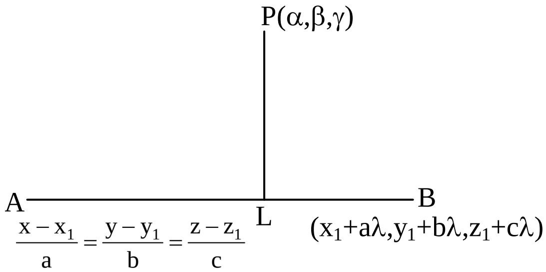
Let the coordinates of L be $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$. Then direction ratios of PL are $x_1 + a\lambda - \alpha$, $y_1 + b\lambda - \beta$, $z_1 + c\lambda - \gamma$.

Direction ratio of AB are a, b, c. Since PL is perpendicular to AB, therefore

$(x_1 + a\lambda - \alpha)\, a + (y_1 + b\lambda - \beta)$

$$b + (z_1 + c\lambda - \gamma)\, c = 0$$

$$\Rightarrow \lambda = \frac{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)}{a^2 + b^2 + c^2}$$



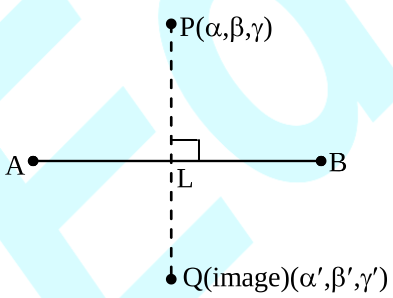
Putting this value of λ in $(x_1 + a\lambda,\ y_1 + b\lambda,\ z_1 + c\lambda)$, we obtain coordinates of L. Now, using distance formula we can obtain the length PL.

➢ Reflection or image of a point in a straight line

Cartesian form : To find the reflection or image of a point in a straight line in certesian form.

Let P(α, β, γ) be the point and $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ be the given line

Let L be the foot of perpendicular from P to AB and let Q be the image of the point in the given line, where, PL = LQ



Let the co-ordinate of L be ;

$(x_1 + a\lambda,\ y_1 + b\lambda,\ z_1 + c\lambda)$

Then direction ratios of PL are ;

$(x_1 + a\lambda - \alpha,\ y_1 + b\lambda - \beta,\ z_1 + c\lambda - \gamma)$

since PL is perpendicular to the given line, whose direction ratios are a, b, c

$\therefore\ (x_1 + a\lambda - \alpha) \cdot a + (y_1 + b\lambda - \beta) \cdot b$

$$+ (z_1 + c\lambda - \gamma) \cdot c = 0$$

$$\Rightarrow \lambda = \frac{\{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)\}}{a^2 + b^2 + c^2}$$

substituting λ we get L. (foot of perpendicular).

let co-ordicates of Q(α', β', γ') be image

$\therefore$ mid point of PQ is L.

$\therefore\ \frac{\alpha + \alpha'}{2} = x_1 + a\lambda,\ \frac{\beta + \beta'}{2} = y_1 + b\lambda,\ \frac{\gamma + \gamma'}{2} = z_1 + c\lambda,$

$\therefore\ \alpha' = 2(x_1 + a\lambda) - \alpha,\quad \beta' = 2(y_1 + b\lambda) - \beta,$

$\gamma' = 2(z_1 + c\lambda) - \gamma$

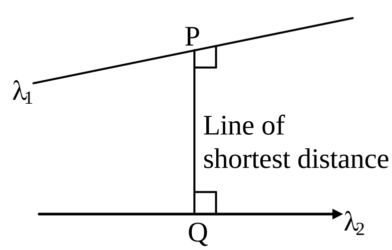
7. Skew lines

Two straight lines in a space which are neither parallel nor intersecting are called skew-lines.

Thus, the skew lines are those lines which do not lie in the same plane.

(i) Shortest distance between two skew straight lines : If λ_1 and λ_2 are two skew lines, then there is one and only one line perpendicular to each of lines λ_1 and λ_2 which is known as three line of shortest distance.

Here, distance PQ is called to be shortest distance.



Vector form :

Let λ_1 and λ_2 be two lines whose equations are: $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ respectively clearly λ_1 and λ_2 pass through the points A and B with position vectors $\vec{a}_1$ and $\vec{a}_2$

respectively and are parallel to the vectors $\vec{b}_1$ and $\vec{b}_2$ respectively

$$\text{Distance } \overrightarrow{PQ} = \left|\frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

Condition for lines to intersect

The two lines are intersecting if ;

$$\left|\frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| = 0$$

$$\Rightarrow \quad (\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1) = 0$$

$$\Rightarrow \quad [\vec{b}_1\ \vec{b}_2\ (\vec{a}_2 - \vec{a}_1)] = 0$$

Cartesian form :

Let the two skew lines be :

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

$$\text{and } \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

$$\text{shortest distance} = \left|\frac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1\lambda_2 - n_2\lambda_1)^2 + (\lambda_1 m_2 - \lambda_2 m_1)^2}}$$

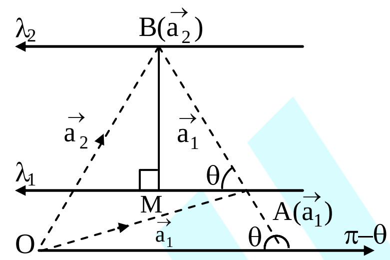
Conditions for lines to intersect

The lines are intersecting, if shortest distance = 0

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

(ii) Distance between parallel lines : Let λ_1 and λ_2 are two parallel lines whose equations are

$\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ respetively



Clearly, λ_1 and λ_2 Pass through the points A and B with position vectors $\vec{a}_1$ and $\vec{a}_2$ respectively and both are parallel to the vector $\vec{b}$, where BM is the shortest distance between λ_1 and λ_2

shortest distance between parallel lines :

$\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ is :

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$$

8. Plane

A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface. In other words, every point on the line segment joining any two points lies on the plane.

Theorem : Every first degree equation in x, y and z represents a plane i.e. ax+by+cz+ d=0 is the general equation of a plane.

Equation of a plane passing through a givenpoint

The general equation of a plane passing through a point (x_1, y_1, z_1) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$, where a, b and c are constants.

Intercept form of a plane :

The equation of a plane intercepting lengths a, b and c with x- axis , y-axis and z-axis respectively is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Cartesian Form : If λ, m, n are direction cosines of the normal to a given plane which is at a distance p from the origin, then the equation of the plane is $\lambda x + my + nz = p$.

9. Normal to a plane

A line perpendicular to a plane is called a normal to the plane. Clearly, every line lying in a plane is perpendicular to the normal to the plane.

For example : The direction ratios of a vector normal to the plane $3x + 2y + 5z - 6 = 0$ are 3, 2, 5 and hence a vector normal to the plane is $3\hat{i}+2\hat{j}+5\hat{k}$.

(i) Vector equation of plane passing through a point and normal to a given vector

The vector equation of a plane passing through a point having position vector $\vec{n}$ is

$(\vec{r}-\vec{a})\ \vec{n}=0$

Reduction to cartesian form :

If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{n}=a\hat{i}+b\hat{j}+c\hat{k}$

Then ; $(\vec{r}-\vec{a})=(x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k}$

Then (i) can be written as

$\{(x-x_1)\hat{i}+(y-y_1)\hat{j}+(z-z_1)\hat{k}\}.$

$\{(a\hat{i}+b\hat{j}+c\hat{k})\}=0$

$\Rightarrow a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

Thus, the coefficient of x, y, z in the cartesian equation of a plane are the direction ratios of normal to the plane.

(ii) Equation of plane in normal form

vector form

The vector equation of a plane normal to unit vector $\hat{n}$ and at a distance d from the origin is $\vec{r}.\hat{n}=d$

Cartesian form

If λ, m, n, be the direction cosines of the normal to a given plane and p be the length of perpendicular from origin to the plane, then the equation of the plane is $\lambda x + my + nz = p$.

10. Angle between two planes

(i) **Vector form** - The angle between the two planes is defined as the angle between normals.

Let θ be the angle between planes;

$\vec{r}.\vec{n}_1=d_1$ and $\vec{r}.\vec{n}_2=d_2$ is given by

$$\cos\theta=\frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

(ii) **Cartesian form** - The angle θ between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

11. Intersection of two planes

The equation of a plane passing through the intersection of $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$, where λ is a constant.

➢ Distance of a point from a plane

(i) Vector form : - The length of the perpendicular from a point having position vector $\vec{a}$ to, the plane $\vec{r}\,.\,\vec{n} = d$ is given by P = $\dfrac{|\vec{a}.\vec{n}-d|}{|\vec{n}|}$

(ii) Cartesian Form : The length of the perpendicular from a point $P(x_1, y_1, z_1)$ to the plane ax + by + cz + d = 0 is given by

$$\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Distance between the parallel planes

(i) Vector form : The distance between two parallel plane $\vec{r}\,.\,\vec{n} = d_1$ and $\vec{r}\,.\,\vec{n} = d_2$ is given by

$$d = \frac{|d_1-d_2|}{|\vec{n}|}$$

(ii) Cartesian form

The distance between two parallel planes

$ax + by + cz + d_1 = 0$ and

$ax + by + cz + d_2 = 0$ is given by

$$d = \left|\frac{(d_2-d_1)}{\sqrt{a^2+b^2+c^2}}\right|$$

Equation of planes bisecting the angles between two given planes

Cartesian Form :

The equation of the planes bisecting the angles between the planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are

$$\frac{(a_1x+b_1y+c_1z+d_1)}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{(a_2x+b_2y+c_2z+d_2)}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

12. Lines and plane

12.1 Angle between a line and a plane

The angle between a line and a plane is the complement of the angle between the line and the normal to the plane

If α, β, γ be the direction ratios of the line and ax + by + cz + d = 0 be the equation of plane and θ be the angle between the line and the plane.

$$\Rightarrow \cos(90°-\theta) = \frac{a\alpha+b\beta+c\gamma}{\sqrt{a^2+b^2+c^2}\,\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

$$\text{or } \sin\theta = \frac{a\alpha+b\beta+c\gamma}{\sqrt{a^2+b^2+c^2}\,\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

Vector form : If θ is the angle between the line;

$\vec{r} = \vec{a} + \lambda\vec{b}$ and plane $\vec{r}.\vec{n} = d$

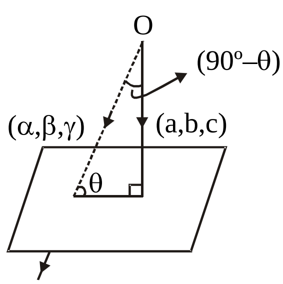
$$\Rightarrow \sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}$$

12.2 Condition for a line to be parallel to a plane

Let line $\dfrac{x-x_1}{\lambda} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$ be parallel to plane ax + by + cz + d = 0 iff;

θ = 0 or π or sin θ = 0

or aλ + bm + cn = 0



12.3 Condition for a line to lie in the plane

Condition for $\frac{x-x_1}{\lambda} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ to lie in the plane

$ax + by + cz + d = 0$ are

$a\lambda + bm + cn = 0$ and

$ax_1 + by_1 + cz_1 + d = 0$

Note : A line will be in a plane iff.

(i) the normal to the plane is perpendicular to the line

(ii) a point on the line lies in the plane.

13. Condition of coplanarity of two lines & equation of the plane containing them

Cartesian form :

If the line $\frac{x-x_1}{\lambda_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and

$\frac{x-x_2}{\lambda_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar, then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ \lambda_1 & m_1 & n_1 \\ \lambda_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing them is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \lambda_1 & m_1 & n_1 \\ \lambda_2 & m_2 & n_2 \end{vmatrix} = 0$$

or $$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ \lambda_1 & m_1 & n_1 \\ \lambda_2 & m_2 & n_2 \end{vmatrix} = 0$$

14. Miscellaneous point

14.1 Volume of a Tetrahedron :

Volume of a tetrahedron with vertices A (x_1,y_1,z_1), B (x_2,y_2,z_2) C (x_3,y_3,z_3) and D (x_4,y_4,z_4) is given by

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$