Separation of Real & Imaginary parts

1. Introduction

The separation of a complex quantity into real and imaginary parts, means to express the given complex quantity in the form of u + iv, where **u** is called the **real part** and **v** is called the **imaginary part**. On the basis of nature of functions, to separate the complex expressions into real and imaginary parts, we may divide complex expressions into following six categories-

(i) Algebraic expressions

(ii) Trigonometrical and hyperbolic expressions

(iii) Expressions of Inverse functions

(iv) Expressions of logarithmic functions

(v) Expressions of exponential functions

(vi) Expressions of the form $(\text{function})^{\text{function}}$

2. Separation of Algebraic Functions

If the given expression is the ratio of two complex quantities, then first we multiply the numerator and denominator by the complex conjugate of denominator to change denominator into real expression.

If the given expression is of the form $(x + iy)^n$, then for its separation into real and imaginary parts, we make the following substitutions in it :

$$x = r\cos\theta,\ y = r\sin\theta$$

Hence

$(x+ iy)^n = r^n (\cos\theta + i\sin\theta)^n$

$= r^n (\cos n\theta + i \sin n\theta)$

$= (x^2 + y^2)^{n/2} \cos$

$[n \tan^{-1} (y/x)] + i (x^2 + y^2)^{n/2} \sin\ [n \tan^{-1} (y/x)]$

Many times for the separation into real and imaginary parts of $(x + iy)^n$, it is expanded with the help of binomial theorem.

Note :

If the value of n is less then use binomial theorem and when n is more then use De-moivre's theorem.

3. Separation of Trigonometric and Hyperbolic Functions

If the given expressions involve trigonometric or Hyperbolic function, then we use the formulae of sin (x + iy), cos (x + iy) etc. which are given in the last chapter. For hyperbolic functions, we should first express them in the form of trigonometrical functions.

4. Separation of Inverse Trigonometric and Inverse Hyperbolic Functions

If $\sin(\alpha + i\beta) = x + iy$ then $(\alpha + i\beta)$, is called the **inverse sine of (x + iy).** We can write it as -

$$\sin^{-1}(x + iy) = \alpha + i\beta$$

Here the following results for inverse functions may be easily established.

(i) $\tan^{-1}(x + iy) = \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$

$+\ \tanh^{-1}\left(\frac{2y}{1+x^2+y^2}\right)$

or

$= \frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right) + \frac{1}{4}\log\left[\frac{x^2+(1+y)^2}{x^2+(1-y)^2}\right]$

(ii) $\sin^{-1}(\cos\theta + i\sin\theta)$

$= \cos^{-1}(\sqrt{\sin\theta}) + i\sinh^{-1}(\sqrt{\sin\theta})$

or

$= \cos^{-1}(\sqrt{\sin\theta}) + i\log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$

(iii) $\cos^{-1}(\cos\theta + i\sin\theta)$

$= \sin^{-1}(\sqrt{\sin\theta}) - i\sinh^{-1}(\sqrt{\sin\theta})$

or

$= \sin^{-1}(\sqrt{\sin\theta}) - i\log(\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$

(iv) $\tan^{-1}(\cos\theta + i\sin\theta) = \frac{\pi}{4}$

$+ \frac{i}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$, $(\cos\theta) > 0$

and

$\tan^{-1}(\cos\theta + i\sin\theta)$

$= \left(-\frac{\pi}{4}\right) + \frac{i}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$, $(\cos\theta) < 0$

since each inverse hyperbolic function can be expressed in terms of logarithmic function, therefore for separation into real and imaginary parts of inverse hyperbolic function of complex quantities use the method explained in Article (5).

Note :

Both inverse circular and inverse hyperbolic functions are many valued.

5. Separation of Logarithmic Functions

If $\alpha + i\beta = e^{x + iy}$, then **(x + iy)** is called the **logarithm** of $(\alpha + i\beta)$

5.1 Principal value of log z

If $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$, where θ **is the principal value of the argument of z,** then the principal value of log z is given by

$$\log z = \log r + i\theta$$

$\Rightarrow \log z = \log|z| + i\,\text{amp.}(z)$ ...(1)

If $|z|$ and amp.(z) are not directly known then putting $z = x + iy$, we get

$$\log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}(y/x) \quad ...(2)$$

5.2 General value of log z

Considering the general value of the amplitude of z, we shall get the general value of log z, which is denoted by Log z is given by-

$\text{Log } z = \log r + i(2n + \theta)$, n = 0, 1, 2...

$\text{Log } z = \log|z| + i(2n\pi + \text{amp } z)$ ...(3)

$\text{Log } z = \log z + (2n\pi)i$

Again if $z = x + iy$, then

$\text{Log}(x + iy) = 2n\pi i + \log(x + iy)$; when $n \in Z$

$$\text{Log}(x + iy) = \frac{1}{2}\log(x^2 + y^2)$$

$$+ i[2n\pi + \tan^{-1}(y/x)] \quad ...(4)$$

Note :

(i) $\log(\bar{z}) = \log|z| - i\,\text{amp.}(z)$

$= \overline{(\log z)}$

(ii) $\log(-z) = \log|z| + i(\pi + \text{amp } z)$

($\because |-z| = |z|$ and $\text{amp}(-z) = \pi + \text{amp.}z$)

(iii) $\log(-z) = \log z + \pi i$

(iv) $\log(-x) = \log x + \pi i$

(where x is a positive real number)

(v) from (3) & (4), the principal value of log z can be obtained by putting n = 0

(vi) In all the above formulas the base of logarithm is e only.

6. Separation of Exponential Functions

(i) From Euler formula

$e^{i\theta} = \cos\theta + i\sin\theta$

(ii) $e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

$\therefore$ Real part $= e^x \cos y$

and Imaginary part $= e^x \sin y$

Again, modulus of $e^{x+iy} = e^x$

and amplitude of $e^{x+iy} = y$

(iii) If $a \in R$, then

$a^{x+iy} = a^x \cdot a^{iy} = a^x \cdot e^{iy\log a}$

$\therefore a^{x+iy} = a^x[\cos(y\log a) + i\sin(y\log a)]$

7. Separation of Expressions of the Form (FUNCTION)$^{\text{function}}$

To find two parts of $(\alpha + i\beta)^{x + iy}$, first we change its base to e and reduce it to the form e^{p+iq}, Thus

$(\alpha + i\beta)^{x+iy} = e^{(x+iy)\,\text{Log}(\alpha + i\beta)}$

$= e^{(x+iy)\left[\frac{1}{2}\log(\alpha^2+\beta^2)+2n\pi i+i\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right]} = e^{p + iq}$

Where $p = x/2\log(\alpha^2 + \beta^2)$

$- y(2n\pi + \tan^{-1}\beta/\alpha)$ &

$q = y/2\log(\alpha^2 + \beta^2) + x[2n\pi + \tan^{-1}(\beta/\alpha)]$