# Definite Integration

## 1. Definition

If $\frac{d}{dx}[f(x)] = \phi(x)$ and a and b, are two values independent of variable x, then

$$\int_a^b \phi(x)\,dx = [f(x)]_a^b = f(b) - f(a)$$

is called **Definite Integral** of $\phi$ (x) within limits a and b. Here **a** is called the **lower limit** and **b** is called the **upper limit** of the integral. The interval [a, b] is known as **range of integration**. It should be noted that every definite integral has a unique value.

## 2. Properties of Definite Integral

**[P-1]** $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

i.e. the value of a definite integral remains unchanged if its variable is placed by any other symbol.

**[P-2]** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

i.e. the interchange of limits of a definite integral changes only its sign.

**[P-3]** $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

where a < c < b.

or $\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + ....+$

$\int_{c_n}^b f(x)\,dx$ where $a < c_1 < c_2 < ....c_n < b$.

Generally this property is used when the integrand has two or more rules in the integration interval.

**[P-4]** $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ .

**Note :**

This property can be used only when lower limit is zero. It is generally used for those complicated integrals whose denominators are unchanged when x is replaced by a– x. With the help of above property following integrals can be obtained-

(i) $\int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f(\cos x)\,dx$

(ii) $\int_0^{\pi/2} f(\tan x)\,dx = \int_0^{\pi/2} f(\cot x)\,dx$

(iii) $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx$

(iv) $\int_0^1 f(\log x)\,dx = \int_0^1 f[\log(1-x)]\,dx$

(v) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx =$

$\int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}\,dx = \pi/4$

(vi) $\int_0^{\pi/2} \frac{\tan^n x}{1 + \tan^n x}\,dx =$

$\int_0^{\pi/2} \frac{\cot^n}{1 + \cot^n x}\,dx = \pi/4$

(vii) $\int_0^{\pi/2} \frac{1}{1 + \tan^n x}\,dx =$

$\int_0^{\pi/2} \frac{1}{1 + \cot^n x}\,dx = \pi/4$

(viii) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \mathrm{cosec}^n x}\,dx =$

$\int_0^{\pi/2} \frac{\mathrm{cosec}^n x}{\mathrm{cosec}^n x + \sec^n x}\,dx = \pi/4$

(ix) $\int_0^{\pi/4} \log(1 + \tan x)\,dx = (\pi/8)\log 2$

(x) $\int_0^{\pi/2} \log\cot x\,dx = \int_0^{\pi/2} \log\tan x\,dx = 0$

**[P-5]** $\int_{-a}^a f(x)\,dx$

$$= \begin{cases} 0, \text{ if } f(-x) = -f(x) \text{ i.e. if } f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx, \text{ if } f(-x) = f(x) \text{ i.e. if } f(x) \text{ even} \end{cases}$$

This property is generally used when integrand is either even or odd function of x.

**[P-6]**

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x)=-f(x) \end{cases}$$

It is generally used to make half the upper limit.

**[P-7]** If f(x) = f(x + a), then

$$\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx$$

**[P-8]** $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

**[P-9]** $\frac{d}{dt}\left[\int_{\phi(t)}^{\psi(t)} f(x)\,dx\right] = f\{\psi(t)\}\psi'(t) - f\{\phi(t)\}\phi'(t)$

## 3. Some Important Formulae

I. $\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx = -(\pi/2)\log 2.$

**II.** $\int_0^{\pi/2} \sin^m x\cos^n x\,dx$

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

Where Γ (n) is called **Gamma function** which satisfies the following properties

$\Gamma(n+1) = n\,\Gamma(n) = n!,\ \Gamma(1) = 1,\ \Gamma(1/2) = \sqrt{\pi}$

In place of gamma function, we can also use the following formula:

$\int_0^{\pi/2} \sin^m x\cos^n x\,dx$

$$= \frac{(m-1)(m-3)\ldots(2\text{ or }1)(n-1)(n-3)\ldots(2\text{ or }1)}{(m+n)(m+n-2)\ldots(2\text{ or }1)} \times (1\text{ or }\pi/2)$$

**It is important to note that we multiply by (π/2) when both m and n are even.**

**III. Walli's formula :**

**(i)** $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots\ldots\frac{2}{3}, & \text{when n is odd} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\ldots\ldots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, & \text{when n is even} \end{cases}$$

## 4. Summation of Series by Integration

For finding sum of an infinite series with the help of definite integration, following formula is used-

$$\lim_{n\to\infty}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\cdot\frac{1}{n} = \int_0^1 f(x)\,dx\,.$$

The following method is used to solve the questions on summation of series.

**(i)** After writing (r–1)th or rth term of the series, express it in the form $\frac{1}{n}f\left(\frac{r}{n}\right)$. Therefore the given series will take the form

$$\lim_{n\to\infty}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\cdot\frac{1}{n}$$

**(ii)** Now writing $\int$ in place of $\left(\lim_{n\to\infty}\sum\right)$, x in place of $\left(\frac{r}{n}\right)$ and dx in place of $\frac{1}{n}$, we get the integral $\int f(x)\,dx$ in place of above series.

**(iii)** The lower limit of this integral

$$= \lim_{n\to\infty}\left(\frac{r}{n}\right)_{r=0}$$

where r = 0 is taken corresponding to first term of the series and upper limit

$$= \lim_{n\to\infty}\left(\frac{r}{n}\right)_{r=n-1}$$

where r = n – 1 is taken corresponding to the last term.