CIRCULAR MOTION

If a particle moves in a plane such that its distance from a fixed (or moving) point remains constant then its motion is called as circular motion with respect to that fixed or moving point.

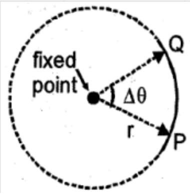
That fixed point is called the. centre and the corresponding distance is called the radius of circular path.

The vector joining the centre of the circle and the particle performing circular motion, directed towards the later is called the radius vector. It has constant magnitude but variable direction.

1. KINEMATICS OF CIRCULAR MOTION

Angular Displacement

Angle traced by the position vector of a particle moving w.r.t. some fixed point is called angular displacement.



$\Delta\theta$ = angular displacement $\because$ angle $= \dfrac{\text{arc}}{\text{radius}}$ $\therefore$ $\Delta\theta = \dfrac{\text{arc PQ}}{r}$

Frequency (n) : Number of revolutions described by particle per second is its frequency. its unit is revolutions per second (rps) or revolutions per minute (rpm).

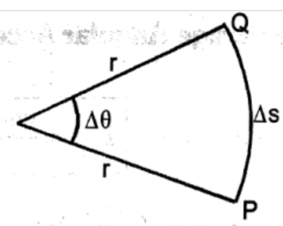
Note : 1 rps = 60 rpm

Time Period (T) : It is the time taken by particle to complete one revolution. i.e. $T = \dfrac{1}{n}$ **Angular Velocity (ω) :** It is defined as the rate of change of angular displacement of a moving particle, w.r.t. to time.

$$\omega = \frac{\text{angle traced}}{\text{time taken}} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

⇒ Its unit is rad/s and dimensions is $[T^{-1}]$

Relation between linear and Angular velocity



Angle $(\Delta\theta) = \dfrac{\text{arc}}{\text{radius}} = \dfrac{\Delta s}{r}$ ⇒ $\Delta s = r\Delta\theta$

$\therefore \dfrac{\Delta s}{\Delta t} = \dfrac{r\Delta\theta}{\Delta t}$ if $\Delta t \to 0$ then $\dfrac{ds}{dt} = r\dfrac{d\theta}{dt}$ ⇒ $\boxed{v = \omega r}$

In vector from $\vec{v} = \vec{\omega} \times \vec{r}$ (direction of $\vec{v}$ is according to right hand thumb rule)

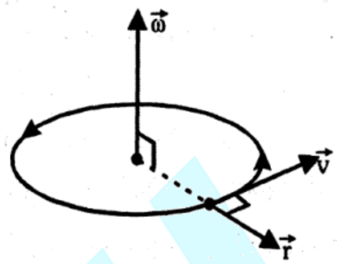
Here, $\vec{v}$ = Linear velocity [Tangential vector]

$\vec{\omega}$ = Angular velocity [Axial vector]

$\vec{r}$ = Radius vector or position vector

Note : Centrifugal means away from the centre and centripetal means towards the centre.

All the three vectors $\vec{v}, \vec{\omega}$ and $\vec{r}$ are mutually perpendicular to each other.



Here, $\vec{v} \perp \vec{\omega} \perp \vec{r}$ $\because$ $\vec{v} \perp \vec{\omega}$ $\therefore$ $\vec{v}.\vec{\omega} = 0$

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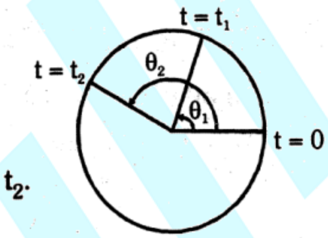
Average Angular Velocity (ω_{av})



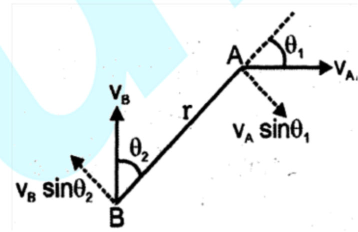
$$\omega_{av} = \frac{\text{total angle of rotation}}{\text{total time taken}} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

where θ_1 and θ_2 are the angular positions of the particle at instants t_1 and t_2.

Instantaneous Angular Velocity

The angular velocity at some particular instant $\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\vec{\theta}}{dt}$

Relative Angular Velocity



Relative angular velocity of a particle 'A' w.r.t. another moving particle B is the angular velocity of the position vector of A w.r.t. B. It means the rate at which the position vector of 'A' w.r. t. B rotates at that instant.

$$\omega_{AB} = \frac{(v_{AB})_\perp}{r_{AB}} = \frac{\text{relative velocity of A w.r.t. B perpendicular to line AB}}{\text{separation between A and B}}$$

Here $(v_{AB})_\perp = v_A \sin\theta_1 + v_B \sin\theta_2$ $\therefore$ $\omega_{AB} = \dfrac{v_A \sin\theta_1 + v_B \sin\theta_2}{r}$

Angular Acceleration (α)

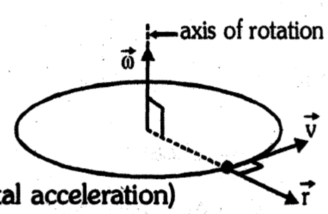
Rate of change of angular velocity is called angular acceleration. i.e. $\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt}$ -

Average Angular Acceleration :

$$\vec{\alpha}_{avg} = \frac{\text{change in angular velocity}}{\text{time taken}} = \frac{\Delta\vec{\omega}}{\Delta t}$$

Relation between Angular and Linear Accelerations

Velocity $\vec{v} = \vec{\omega} \times \vec{r}$



Acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$

$\Rightarrow \vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v} = \vec{a} = \vec{a}_T + \vec{a}_C$

($\vec{a}_T = \vec{\alpha} \times \vec{r}$ is tangential acceleration & $\vec{a}_C = \vec{\omega} \times \vec{v}$ is centripetal acceleration)

$\vec{a} = \vec{a}_T + \vec{a}_C$ ($\vec{a}_T$ and $\vec{a}_C$ are the two component of net linear acceleration)

As $\vec{a}_T \perp \vec{a}_C$ so $|\vec{a}| = \sqrt{a_T^2 + a_C^2}$

Tangential Acceleration

$\vec{a}_T = \vec{\alpha} \times \vec{r}$ its direction is parallel (or antiparallel) to velocity. $\vec{v} = \vec{\omega} \times \vec{r}$ as $\vec{\omega}$ and $\vec{\alpha}$ **both are parallel (or antiparallel) and along the axis.**

Magnitude of tangential acceleration in case of circular motion :

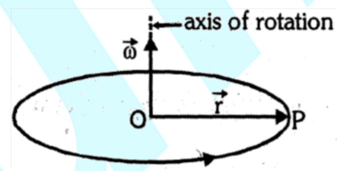
$a_T = \alpha\, r \sin 90° = \alpha r$ ($\vec{\alpha}$ is axial, $\vec{r}$ is radial so that $\vec{\alpha} \perp \vec{r}$)

As $\vec{a}_T$ is along the direction of motion (in the direction of $\vec{v}$ or opposite to $\vec{v}$) so $\vec{a}_T$ is responsible for change in speed of the particle. Its magnitude is the rate of change of speed of the particle.

Note : If a particle is moving on a circular path with constant speed then tangential acceleration is zero.

Centripetal acceleration

$\vec{a}_C = \vec{\omega} \times \vec{v} = \vec{\omega} \times (\vec{\omega} \times \vec{r})$ ($\because \vec{v} = \vec{\omega} \times \vec{r}$)



Let $\vec{r}$ be in $\hat{i}$ direction and $\vec{\omega}$ be in $\hat{j}$ direction then the direction of $\vec{a}_C$ is along $\hat{j} \times (\hat{j} \times \hat{i}) = \hat{j} \times (-\hat{k}) = -\hat{i}$, opposite to the direction of $\vec{r}$ i.e. from P to O and it is centripetal in direction. Magnitude of centripetal acceleration, $a_C = \omega v = \frac{v^2}{r} = \omega^2 r$ therefore $\vec{a}_C = \frac{v^2}{r}(-\hat{r})$

Note : Centripetal acceleration is always perpendicular to the velocity at each point.

2. UNIFORM AND NON-UNIFORM CIRCULAR MOTION

2.1 Uniform Circular Motion

If a particle moves with a constant speed in a circle, the motion is called uniform circular motion. In uniform circular motion a resultant non-zero force acts on the particle. The acceleration is due to the change in direction of the velocity vector. In uniform circular motion tangential acceleration (a_T) is zero. The acceleration of the particle is towards the centre and its magnitude is $\frac{v^2}{r}$. Here, v is the speed of the particle and r the radius of the circle.

The direction of the resultant force F is therefore, towards the centre and its F = $\frac{mv^2}{r}$ = $mr\omega^2$ (as $v = r\omega$)

Here, ω is the angular speed of the particle. This force F is called the **centripetal force**. Thus, a centripetal force of magnitude $\frac{mv^2}{r}$ is needed to keep the particle moving in a circle with constant speed. This force is provided by some external agent such as friction, magnetic force, coulomb force, gravitational force, tension. etc.

In this motion :

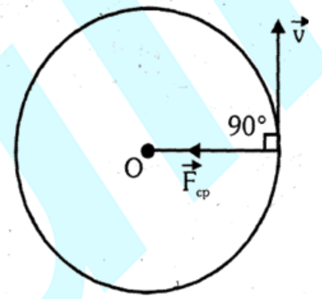
- Speed = constant
- |Velocity| = constant
- Velocity $\neq$ constant (because its direction continuously changes)

- K.E. $= \frac{1}{2}mv^2 =$ constant
- $\vec{\omega} =$ constant (because magnititude and direction, both are constants)
- $a_T = 0$ $\left[\because a_T = \frac{dv}{dt} = \frac{d(\text{constant})}{dt} = 0\right]$
- $\alpha = 0$ $\left[\because \alpha = \frac{a_T}{r} = \frac{0}{r} = 0\right]$ or $\left[\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(\text{constant}) = 0\right]$
- $|\vec{a}| |\vec{a}_{cp}| = \omega v = \omega^2 r = \frac{v^2}{r} =$ constant
- $\vec{a} = \vec{a}_{cp} \neq$ constant

  (because the direction of a_{cp} is toward the centre of circle which changes as the particle revolves)
- Total work done $= \vec{F}_{net}.\vec{s}$

  $= \vec{F}_{cp}.\vec{s}$ $\left[\because \vec{F}_{net} = \vec{F}_{cp}\right]$

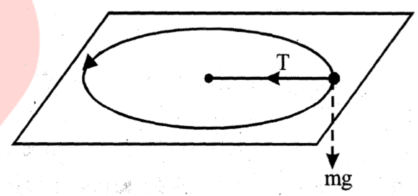
  $= F_{cp}\, s \cos 90° = 0$



- Power $= \frac{\text{Work}}{\text{time}} = \frac{0}{t} = 0$ or Power $= \vec{F}_{net}.\vec{v} = \vec{F}_{cp}.\vec{v} = F_{cp} v \cos 90° = 0$
- Uniform circular motion is usually executed in a horizontal plane.

  Example :

  A particle of mass 'm' is tied at one end of a string of length 'r' and it is made to revolve along a circular path in a horizontal plane with a constant speed means a (uniform circular motion) In this condition the required centripetal force is provided by the tension in the string.



$T = F_{cp}$ So, $T = \frac{mv^2}{r}$

2.2 Non-Uniform Circular Motion :

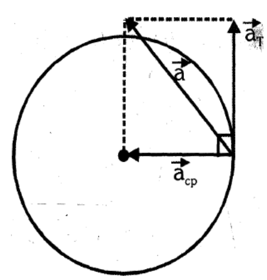
If a particle moves with variable speed in a circle, then the motion is called non-uniform circular motion.

In this motion :

Accleration (a) has two components :-

$\vec{a}_{cp}$ = responsible for change in direction only.

$\vec{a}_T$ = responsible for change in speed only.



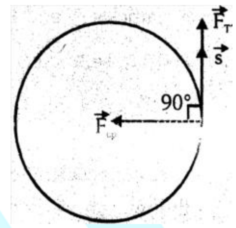
Hence due to a_T

Speed = |velocity| is variable,

As speed variable hence given physical quantities are also variable.

- K.E. $= \frac{1}{2}mv^2$
- $|\vec{\omega}| = \frac{v}{r}$
- $a \neq 0$
- $a_T \neq 0$
- $|\vec{a}_{cp}| = \omega v = w^2 r = \frac{v^2}{r}$
- $\vec{a} = \vec{a}_T + \vec{a}_{cp}$
- $|\vec{a}| = \sqrt{a_{T^2} + a_{cp^2}} = \sqrt{(\alpha r)^2 + \left(\frac{v^2}{r}\right)^2}$

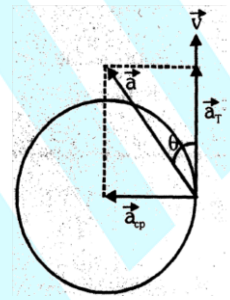
- $\vec{F} = \vec{F}_T + \vec{F}_{cp}$ • $|\vec{F}| = \sqrt{F_T^2 + F_{cp}^2}$



- Work done by centripetal force will be zero but work done by tangential force is not zero.
- Total work done

  $W = \vec{F}_T.\vec{s} -= F_T \, s \cos 0^o = F_T s$

  ⇒ (where s is the distance travelled by the particle)
- Power $= \dfrac{\text{work}}{\text{time}} = \dfrac{\vec{F}_T.\vec{s}}{t} = \vec{F}_T.\vec{v} = F_T v \cos 0^o = F_T v$



- Angle between velocity and acceleration is given by :

  $\tan\theta = \dfrac{a_{cp}}{a_T} = \dfrac{F_{cp}}{F_T}$
- **Example**

  Circular motion in vertical plane is an example of non-uniform circular motion.

2.3 Equations of circular motion

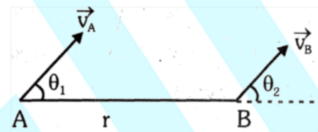
Translatory / Linear Motion	Rotational Motion
• Initial velocity (u)	Initial angular velocity (ω_0)
• Final velocity (v)	Final angular velocity (ω)
• Displacement (s)	Angular displacement (θ)
• Acceleration (a)	Angular Acceleration (α)
If a = constant, then $v = u + at$ $s = ut + \frac{1}{2}at^2$ $v^2 = u^2 + 2as$ $s_{n^{th}} = u + \frac{a}{2}(2n-1)$ $s = \left(\frac{u+v}{2}\right)t$	If α = constant, then $\omega = \omega_0 + \alpha t$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega^2 = \omega_0^2 + 2\alpha\theta$ $\theta_{n^{th}} = \omega_0 + \frac{\alpha}{2}(2n-1)$ $\theta = \left(\frac{\omega_0+\omega}{2}\right)t$

GOLDEN KEY POINTS

- Small angular displacement $d\vec{\theta}$ is a vector quantity, but large angular displacement q is not a vector quantity.

  $d\vec{\theta}_1 + d\vec{\theta}_2 = d\vec{\theta}_2 + d\vec{\theta}_1$ But $\vec{\theta}_1 + \vec{\theta}_2 \neq \vec{\theta}_2 + \vec{\theta}_1$

- Direction of angular displacement is perpendicular to the plane of rotation and is given by right hand thumb rule.
- Angular displacement is dimensionless and its S.I. unit is radian while other unit are degree and revolution.
  2π radian = 360° = 1 revolution
- Instantaneous angular velocity is an axial vector quantity.
- Direction of angular velocity is same as that of angular displacement i.e. perpendicular to the plane of rotation and along the axis according to right hand screw rule or right hand thumb rule.
- If particles A and B are moving with a velocity $\vec{v}_A$ and $\vec{v}_B$ are separated by a distance r at a given instant then



  (i) $\frac{dr}{dt} = v_B\cos\theta_2 - v_A\cos\theta_1$ (ii) $\frac{d\theta_{AB}}{dt} = \omega_{BA} = \frac{v_A \sin\theta_2 - v_A \sin\theta_1}{r}$
- Angular acceleration is an axial vector quantity. It's direction is along the axis according to the right hand thumb rule or right hand screw rule.
- **Important difference between projectile motion and uniform circular motion :**
  In projectile motion, both the magnitude and the direction of acceleration (g) remain constant, while in uniform circular motion the magnitude remains constant but the direction continuously changes.

Illustrations

Illustration 1.

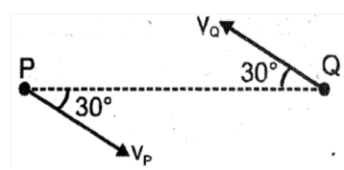
A particle revolving in a circular path completes first one third of the circumference in 2 s, while next one third in 1s. Calculate its average angular velocity.

Solution:

$\theta_1 = \frac{2\pi}{3}$ and $\theta_2 = \frac{2\pi}{3}$ total time T = 2 + 1 = 3 s $\therefore$ $\langle\omega_{av}\rangle = \frac{\theta_1 + \theta_2}{T} = \frac{\frac{2\pi}{3}+\frac{2\pi}{3}}{3} = \frac{\frac{4\pi}{3}}{3} = \frac{4\pi}{9}$ rad/s

Illustration 2.

Two moving particles P & Q are 10 m apart at any instant. Velocity of P is 8 m/s and that of Q is 6 m/s at 30° angle with the line joining P & Q. Calculate the angular velocity of P w.r.t. Q

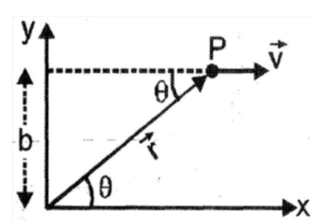


Solution:

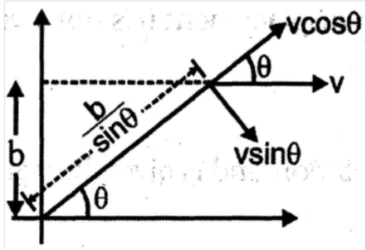
$\omega_{PQ} = \frac{8\sin 30^\circ - (-6\sin 30^\circ)}{10} = 0.7$ rad/s.

Illustration 3.

A particle is moving parallel to x-axis as shown in fig. such that they component of its position vector is constant at all instants and is equal to 'b'. Find the angular velocity of the particle about the origin when its radius vector makes an angle θ with the x-axis.



Solution:



$$\therefore \quad \omega_{PO} = \frac{v\sin\theta}{\frac{b}{\sin\theta}} = \frac{v}{b}\sin^2\theta$$

Illustration 4.

The angular velocity of a particle is given by $\omega = 1.5t - 3t^2 + 2$. (where t is in seconds). Find the instant when its angular acceleration becomes zero.

Solution:

$$\alpha = \frac{d\omega}{dt} = 1.5 - 6t = 0 \Rightarrow t = 0.25 \text{ s.}$$

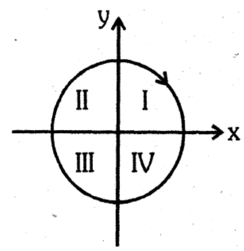
Illustration 5.

A disc start from rest and gains an angular acceleration given by $\alpha = 3t - t^2$ (where t is in seconds) upon the application of a torque. Calculate its angular velocity after 2 s.

Solution:

$$\alpha = \frac{d\omega}{dt} = 3t - t^2 \Rightarrow \int_0^{\omega} d\omega = \int_0^{t}(3t - t^2)dt \Rightarrow \omega = \frac{3t^2}{2} - \frac{t^3}{3} \Rightarrow \text{at } t = 2 \text{ s, } \omega = \frac{10}{3} \text{ rad/s}$$

Illustration 6.



A particle is moving in clockwise direction in a circular path as shown in figure. The instantaneous velocity of particle at a certain instant is $\vec{v} = (3\hat{i} + 3\hat{j})$ m/s. Then in which quardant does the particle lie at that instant? Explain your answer.

Solution:

II quardant. According to following figure x & y components of velocity are positive when the particle is in II quardant.

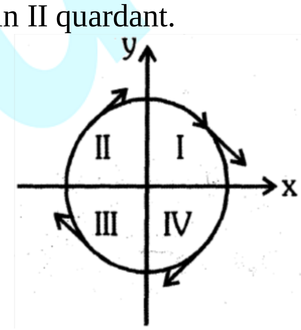


Illustration 7.

A particle is performing circular motion of radius 1 m. Its speed is $v = (2t^2)$ m/s. What will be the magnitude of its acceleration at t = 1s ?

Solution:

Tangential acceleration $= a_T = \frac{dv}{dt} = 4t$, at t = 1s, $a_T = 4 \text{ m/s}^2$

Centripetal acceleration $a_c = \frac{v^2}{r} = \frac{4t^4}{1} = 4t^4$, at t = 1 s, $a_c = 4 \text{ m/s}^2$

Net acceleration (a) = $\sqrt{a_T^2 + a_c^2} = \sqrt{4^2 + 4^2} = 4\sqrt{2}$ m/s^2.

Illustration 8.

A cyclist is riding with a speed of 18 km/h. As he approaches a circular turn on the road of radius $25\sqrt{2}$ m, he applies brakes which reduces his speed at a constant rate of 0.5 m/s every second. Determine the magnitude and direction of his net acceleration on the circular turn.

Solution :

$v = 18 \times \frac{5}{18} = 5$ m/s and $a_{cp} = \frac{v^2}{R} = \frac{25}{25\sqrt{2}} = \frac{1}{\sqrt{2}}$ m/s^2, $a_T = -\frac{dv}{dt} = -\frac{1}{2}$ m/s^2

$a_{net} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2} = 0.86$ m/s^2, $\tan\theta = \frac{a_{CP}}{a_T} = \frac{1/\sqrt{2}}{1/2} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\theta = \tan^{-1}(\sqrt{2})$ from tangential direction

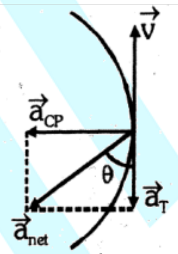
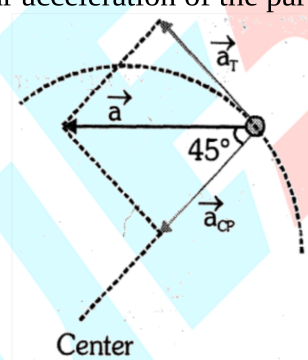


Illustration 9.

A particle is moving in a circular orbit with a constant tangential acceleration starting from rest. After 2 s of the beginning of its motion, angle between the acceleration vector and the radius becomes 45°. What is the angular acceleration of the particle ?



Solution:

In the adjoining figure the total acceleration vector $\vec{a}$ and its components the tangential acceleration a_T and normal acceleration $\vec{a}_{CP}$ are shown. These two components are always mutually perpendicular to each other and act along the tangent to the circle and radius respectively. Therefore, if the total acceleration vector makes an angle of 45° with the radius, both the tangential and the normal components must be equal in magnitude.

$a_T = a_{cp} \Rightarrow \alpha R = \omega^2 R \Rightarrow \alpha = \omega^2$ ... (i)

Since angular acceleration is uniform, we have $\omega = \omega_0 + \alpha t$

Substituting $\omega_0 = 0$ and t = 2 s, we have $\omega = 2\alpha$ ... (ii)

From equations (i) and (ii), we have $\alpha = 0.25$ rad/s^2

BEGINNER'S BOX - 1

1. If angular velocity of a particle depends on the angle rotated θ as $\omega = \theta^2 + 2\theta$, then its angular acceleration α at θ = 1 rad is :

(A) 8 rad/s^2 (B) 10 rad/s^2 (C) 12 rad/s^2 (D) None of these

2. The second's hand of a watch has 6 cm length. The speed of its tip and magnitude of difference in velocities of its at any two perpendicular positions will be respectively :

(A) 2π & 0 mm/s
(B) $2\sqrt{2}\,\pi$ & 4.44 mm/s
(C) $2\sqrt{2}\,\pi$ & 2π mm/s
(D) 2π & $2\sqrt{2}\,\pi$ mm/s

3. A particle is moving on a circular path of radius 6 m. Its linear speed is v = 2t, here t is time in second and v is in m/s. Calculate its centripetal acceleration at t = 3 s.

4. Two particles move in concentric circles of radii r_1 and r_2 such that they maintain a straight line through the centre. Find the ratio of their angular velocities.

5. If the radii of circular paths of two particles are in the ratio of 1 : 2, then in order to have same centripetal acceleration, their speeds should be in the ratio of :

(A) 1 : 4
(B) 4 : 1
(C) 1: $\sqrt{2}$
(D) $\sqrt{2}$: 1

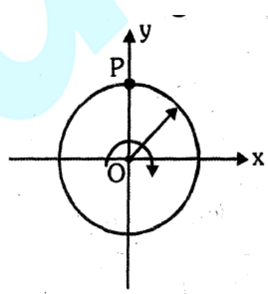
6. A stone tied to the end of a 80 cm long string is whirled in a horizontal circle with a constant speed. If the stone makes 14 revolutions in 25 s, the magnitude of its acceleration is :

(A) 20 m/s^2
(B) 12 m/s^2
(C) 9.9 m/s^2
(D) 8 m/s^2

7. For a body in a circular motion with a constant angular velocity, the magnitude of the average acceleration over a period of half a revolution is .... times the magnitude of its instantaneous acceleration.

(A) $\frac{2}{\pi}$
(B) $\frac{\pi}{2}$
(C) π
(D) 2

8. A ring rotates about z axis as shown in figure. The plane of rotation is xy. At a certain instant the acceleration of a particle P (shown in figure) on the ring is $(6\hat{i} - 8\hat{j})$ m/s^2. Find the angular acceleration of the ring and its angular velocity at that instant. Radius of the ring is 2 m.



3. DYNAMICS OF CIRCULAR MOTION

3.1 Circular Turning on Roads

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which provides the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by the following three ways:

- By friction only.
- By banking of roads only.
- By friction and banking of roads both.

In real life the necessary centripetal force is provided by friction and. banking of roads both.

- **By Friction only**

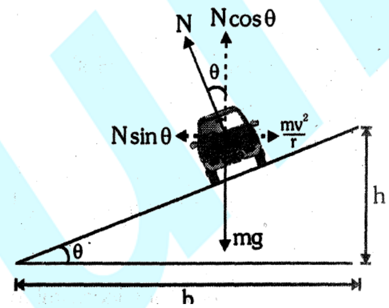
Suppose a car of mass m is moving with a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force will be provided to the car by the force of friction f acting towards centre of the circular path.

Thus, $f = \frac{mv^2}{r}$ $\qquad \because f_{max} = \mu N = \mu mg$

Therefore, for a safe turn without skidding $\frac{mv^2}{r} \le f_{max} \Rightarrow \frac{mv^2}{r} \le \mu mg \Rightarrow v \le \sqrt{\mu rg}$.

- **By Banking of Roads only**

Friction is not always reliable at turns particularly when high speeds and sharp turns are involved. To avoid dependence on friction, the roads are banked at the turn in the sense that the outer part of the road is some what lifted compared to the inner part.



$N \sin\theta = \frac{mv^2}{r}$ and $N \cos\theta = mg$

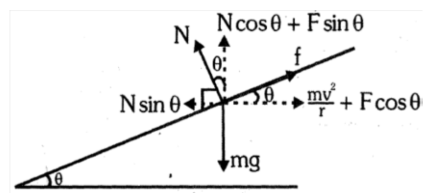
$\Rightarrow \tan\theta = \frac{v^2}{rg} \Rightarrow v = \sqrt{rg\tan\theta}$ $\qquad \because \tan\theta = \frac{h}{b}$

Note : $\tan\theta = \frac{v^2}{rg} = \frac{h}{b}$

- **Friction and Banking of Road both**

If a vehicle is moving on a circular road which is rough and banked also, then three forces may act on the vehicle, of these the first force, i.e., weight (mg) is fixed both in magnitude and direction. The direction of second force, i.e., normal reaction N is also fixed (perpendicular to road) while the direction of the third force, i.e., friction f can be either inwards or outwards while its magnitude can be varied upto a maximum limit ($f_{max} = \mu N$).

So, direction and the magnitude of friction f are so adjusted that the resultant of the three forces mentioned above is $\frac{mv^2}{r}$ towards the centre.



(a) If speed of the vehicle is small then friction acts outward

In this case, $N \cos\theta + f \sin\theta = mg$ …..(i)

and $N \sin\theta - f \cos\theta = \frac{mv^2}{R}$ …..(ii)

For minimum speed $f = \mu N$ so by dividing Eqn. (1) by Eqn. (2)

$$\frac{N\cos\theta - \mu N\sin\theta}{N\sin\theta - \mu N\cos\theta} = \frac{mg}{mv_{min}^2 / R}$$

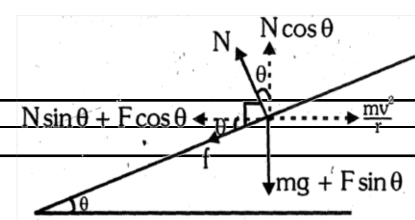
Therefore $v_{min} = \sqrt{Rg\left(\frac{\tan\theta - \mu}{1 + \mu\tan\theta}\right)}$

If we assume $\mu = \tan\phi$, then

$$V_{min} = \sqrt{Rg\left(\frac{\tan\theta - \phi}{1 + \tan\phi\tan\theta}\right)} = \sqrt{Rg\tan(\theta - \phi)}$$

(b) If speed of vehicle is high then friction force act inwards.

in this case for maximum speed



$$N\cos\theta - \mu N \sin\theta = mg$$

and $$N\sin\theta + \mu N\cos\theta = \frac{mv_{max}^2}{R}$$

which gives $v_{max} = \sqrt{Rg\left(\frac{\tan\theta+\mu}{1-\mu\tan\theta}\right)} = \sqrt{Rg\tan(\theta+\phi)}$

Hence for successful tuning on a rough banked road, velocity of vehicle must satisfy following relation

$$\sqrt{Rg\tan(\theta-\phi)} \le v \le \sqrt{Rg\tan(\theta-\phi)}$$

where θ = banking angle and $\phi = \tan^{-1}(\mu)$.

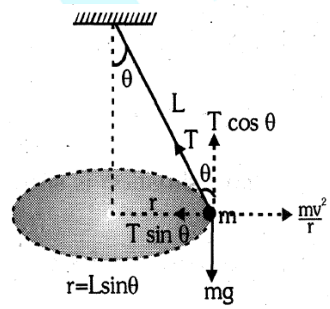
3.2 Conical Pendulum

If a small particle of mass m tied to a string is whirled along a horizontal circle, as shown in figure then the arrangement is called a 'conical pendulum'. In case of conical pendulum the vertical component of tension balances the weight while its horizontal component provides the necessary centripetal force. Thus,

$$T\sin\theta = \frac{mv^2}{r} \quad \text{and} \quad T\cos\theta = mg \Rightarrow v = \sqrt{rg\tan\theta}$$

$\therefore$ Angular speed $\quad \omega = \frac{v}{r} = \sqrt{\frac{g\tan\theta}{r}}$

So, the time period of pendulum is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{g\tan\theta}} = 2\pi\sqrt{\frac{L\cos\theta}{g}}$

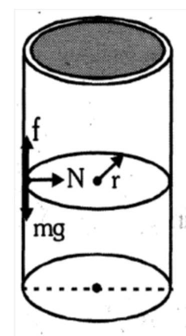


3.3 'Death Well' or Rotor

In case of 'death well' a person drives a motorcycle on the vertical surface of a large wooden well while in case of a rotor a person hangs resting again the wall without any support from the bottom at a certain angular speed of rotor. In death well walls are at rest and person revolves while in case of rotor person is at rest and the walls rotate.

In both cases friction balances the weight of person while reaction provides the centripetal force for circular motion, i.e.,

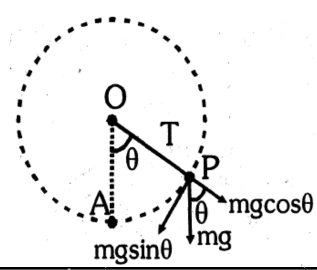
$$f = mg \text{ and } N = \frac{mv^2}{r} = mr\omega^2$$



Death well

4. VERTICAL CIRCULAR MOTION

Suppose a particle of mass m is attached to a light inextensible string of length R. The particle is moving in a vertical circle of radius R about a fixed point O. It is imparted a velocity u in the horizontal direction at lowest point A Let v be its velocity at point P of the circle as shown in the figure .



When a particle is whirled in a vertical circle then three cases are possible –

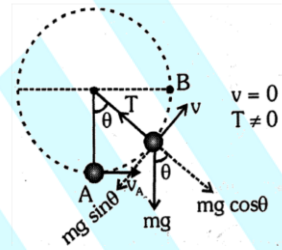
Case I: Particle oscillates in lower half circle.

Case II: Particle moves to upper half circle but not able to complete loop.

Case III: Particle completes loop

Condition of Oscillation $\left(0 < u \le \sqrt{2gR}\right)$

The particle will oscillate if velocity of the particle becomes zero but tension in the string is not zero. (In lower half circle (A to B))



Here, $T - mg\cos\theta = \frac{mv_A^2}{R}$

$$T = \frac{mv_A^2}{R} + mg\cos\theta$$

In the lower part of circle when velocity become zero and tension is non zero means when v = 0, but T ≠ 0. So, to make the particle oscillate in lower half cycle, maximum possible velocity at A can be given by

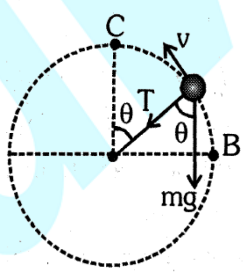
$$\frac{1}{2}mv_A^2 + 0 = mvR + 0 \qquad \text{(by COME between A and B)}$$

$$v_A = \sqrt{2gR} \qquad \text{.... (i)}$$

Thus, for $0 < u \le \sqrt{2gR}$, particle oscillates in lower half of the circle ($0° < \theta \le 90$)

This situation is shown in the figure. $0 < u \le \sqrt{2gR}$ or $0° < \theta \le 90°$

Condition of Leaving the Circle : $\left(\sqrt{2gR} < u < \sqrt{5gR}\right)$

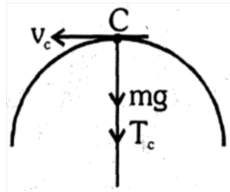


In upper half cycle (B to C)

Here, $T + mg\cos\theta = \frac{mv^2}{R}$

$$T = \left(\frac{mv^2}{R} - mg\cos\theta\right) \qquad \text{.....(ii)}$$

In this part of circle tension force can be zero without having zero velocity mean when T = 0, v ≠ 0 form equation (ii) it is clear that tension decreases if velocity decreases. So to complete the loop tension force should not be zero, in between B to C. Tension will be minimum at C i.e., $T_c \ge 0$ is the required condition.



At Top $\quad T_c + mg = \frac{mv_c^2}{R}$

if $T_c = 0$

Then $\quad mg = \frac{mv_c^2}{R}$

$v_c^2 = gR \quad \Rightarrow \quad v_c = \sqrt{gR}$

By COME (Between A and C)

$$\frac{1}{2}mv_A^2 + 0 = \frac{1}{2}mv_c^2 + mg(2R)$$

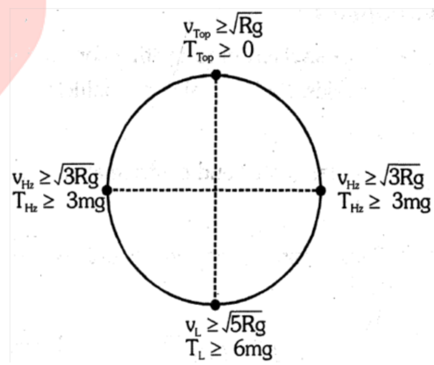
$$v_A^2 = v_c^2 + 4gR \quad \Rightarrow \quad v_A^2 = 5gR \Rightarrow v_A = \sqrt{5gR}$$

Therefore, if $\sqrt{2gR} < u < \sqrt{5gR}$, the particle leaves the circle.

Note : After leaving the circle, the particle will follow a parabolic path.

Condition of Looping the Loop $\left(\sqrt{2gR} < u < \sqrt{5gR}\right)$

The particle will complete the circle if the string does not slack even at the highest point ($\theta = \pi$) Thus, tension in the string should be greater than or equal to zero ($T \geq 0$) at $\theta = \pi$. In critical case substituting T = 0.



Thus, if $u \geq \sqrt{5gR}$, the particle will complete the circle.

Note : In case of light rod tension at top most point can never be zero so velocity will become zero.

$\therefore$ For completing the loop $\quad v_L \geq \sqrt{4gR}$

Illustrations

Illustration 10.

Find the maximum speed at which a car can turn round a curve of 30 m radius on a levelled road if the coefficient of friction between the tyres and the road is 0.4 [acceleration due to gravity = 10 m/s^2]

Solution

Here centripetal force is provided by friction so

$$\frac{mv^2}{r} \leq \mu mg \Rightarrow v_{max} = \sqrt{\mu rg} = \sqrt{120} \approx 11 \text{ m/s}$$

Illustration 11.

For traffic moving at 60 km/h, if the radius of the curve is 0.1 km, what is the correct banking angle of the road? ($g = 10\ m/s^2$)

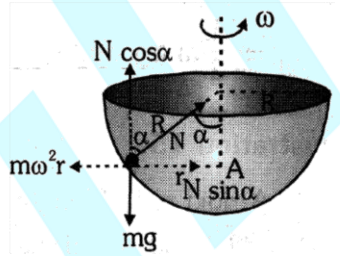
Solution:

In case of banking $\tan\theta = \dfrac{v^2}{rg}$. Here $v = 60\ km/h = 60 \times \dfrac{5}{18}\ m/s = \dfrac{50}{3}\ m/s$ $r = 0.1\ km = 100\ m$

So $\tan\theta = \dfrac{50/3 \times 50/3}{100 \times 10} = \dfrac{5}{18} \Rightarrow \theta = \tan^{-1}\left(\dfrac{5}{18}\right)$.

Illustration 12.

A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small ball kept in the bowl rotates with the bowl without slipping on its surface. If the surface of the bowl is smooth and the angle made by the radius through the ball with the vertical is α. Find the angular speed at which the bowl is rotating.



Solution:

$N\cos\alpha = mg$ .....(1)

$N\sin\alpha = mr\omega^2$ .....(2)

$r = R\sin\alpha$ ......(3)

From equations (2) & (3)

$N\sin\alpha = m\omega^2 R\sin\alpha$

$N = mR\omega^2$ .....(4)

$\Rightarrow (mR\omega^2)\cos\alpha = mg \Rightarrow \omega = \sqrt{\dfrac{g}{R\cos\alpha}}$

Illustration 13.

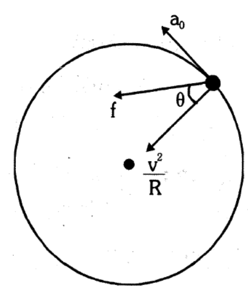
A car starts from rest with a constant tangential acceleration a_0 in a circular path of radius r. At time t_0, the car skids, find the value of coefficient of friction.

Solution:

The tangential and centripetal acceleration is provided only by the frictional force.

Thus, $f\sin\theta = ma_0$ and $f\cos\theta = \dfrac{mv^2}{r} = \dfrac{m(a_0t_0)^2}{r}$

$\Rightarrow f = m\sqrt{a_0^2 + \dfrac{(a_0t_0)^4}{r^2}} \Rightarrow m = \dfrac{a_0}{g}\sqrt{1 + \dfrac{a_0^2t_0^4}{r^2}}$

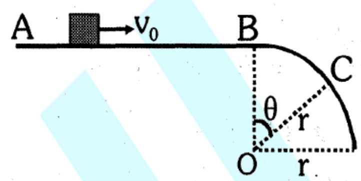


$\mu mg = ma_0 \sqrt{1+\frac{a_0^2 t_0^4}{r^2}} \Rightarrow \mu = \frac{a_0}{g}\sqrt{1+\frac{a_0^2 t_0^4}{r^2}}$

(since the car skids beyond this speed)

Illustration 14.

A small block slides with a velocity $0.5\sqrt{gr}$ on a horizontal frictionless surface as shown in the figure. The block leaves. the surface at point C. Calculate the angle e shown in the figure.



Solution:

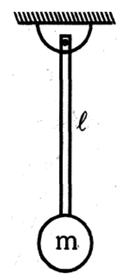
As the block leaves the surface at C so there, normal reaction = 0 $\Rightarrow mg\cos\theta = \frac{mv_C^2}{r}$

By energy conservation at points B & C, $\frac{1}{2}mv_C^2 - \frac{1}{2}mv_0^2 = mgr\,(1-\cos\theta)$

$\Rightarrow \frac{1}{2}m\,(rg\cos\theta) - \frac{1}{2}m\left(0.5\sqrt{gr}\right)^2 = mgr\,(1-\cos\theta) \Rightarrow \cos\theta = \frac{3}{4} \Rightarrow \theta = \cos^{-1}\left(\frac{3}{4}\right)$

Illustration 15.

A rigid rod of length £ and negligible mass has a ball of mass m attached to one end with its other end fixed, to form a pendulum as shown in figure. The pendulum is inverted, with the rod vertically up, and then released. Find the speed of the ball and the tension in the rod at the lowest point of the trajectory of ball.



Solution:

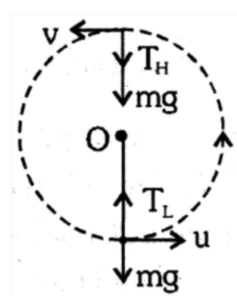
From COME : $2mg\lambda = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{4gl} = 2\sqrt{gl}$

At the lowest point, laws of circular dynamics yield, $T - mg = \frac{mv2}{l} \Rightarrow T = mg + \frac{m}{l}(4g\lambda) = 5mg.$

Illustration 16.

A particle of mass m tied to a string of length λ and given a circular motion in the vertical plane. If it performs the complete loop motion then prove that difference in tensions at the lowest and the highest point is 6 mg.

Solution:



Let the speeds at the lowest and highest point be u and v respectively.

At the lowest point, tension $= T_L = mg + \frac{mu^2}{l}$ .....(i)

At the highest point, tension $= T_H = \frac{mv^2}{l} - mg.$ .....(ii)

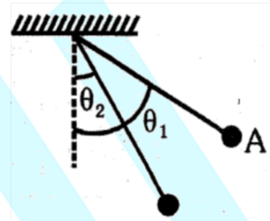
At the conservation of mechanical energy, $\frac{mu^2}{2} - \frac{mv^2}{2} = mg(2\lambda) \Rightarrow u^2 = v^2 + 4g\lambda$

Substituting this in eqn. (i) $T_L = mg + \frac{m[v^2 + 4gl]}{l}$ .....(iii)

$\therefore$ From eq^n. (ii) & (iii) $T_L - T_H = 6mg$

Illustration 17.

A particle of mass m is connected to a light inextensible string of length λ such that it behaves as a simple pendulum. Now the string is pulled to point A making an angle θ_1 with the vertical and is released then obtain expressions for the : **[AIPMT (Mains) 2008]**



(i) Speed of the particle and

(ii) the tension in the string when it makes an angle θ_2 with the vertical.

Solution:

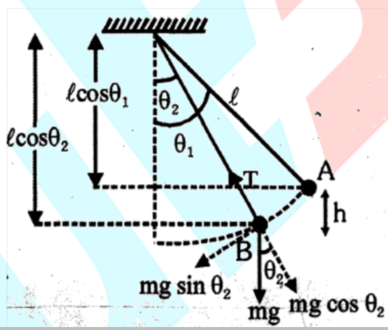
(i) $h = \lambda(\cos\theta_2 - \cos\theta_1)$

Applying conservation of mechanical energy between points A & B

$$\frac{1}{2}mv^2 = mgh \Rightarrow v = \sqrt{2gh} = \sqrt{2gl(\cos\theta_2 - \cos\theta_1)}$$

(ii) At position B, $T - mg\cos\theta_2 = \frac{mv^2}{l}$ where $v = \sqrt{2gl(\cos\theta_2 - \cos\theta_1)}$

$$\Rightarrow T = mg\cos\theta_2 + \frac{mv^2}{l}[2g\lambda(\cos\theta_2 - \cos\theta_1)] = mg(3\cos\theta_2 - 2\cos\theta_1).$$



BEGINNER'S BOX - 2

1. A particle of mass m_1 is fastened to one end of a string and another one of mass m_2 to the middle point; the other end of the string being fastened to a fixed point on a smooth horizontal table. The particles are then projected, so that the two portions of the string are always in the same straight line and describe horizontal circles. Find the ratio of the tensions in the two parts of the string.

2. A road is 8 m wide. Its average radius of curvature is 40 m. The outer edge is above the lower edge by a distance of 1.28 m. Find the velocity of vehicle for which the road is most suited ? ($g = 10\ m/s^2$)

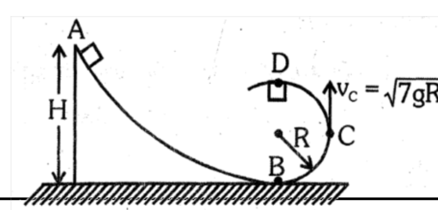
3. A stone of mass 1 kg tied to a light string of length λ = 10 m is whirling in a circular path in the vertical plane. If the ratio of the maximum to minimum tensions in the· string is 3, find the speeds of the stone at the lowest and highest points.

4. Calculate the following for the situation shown :-

(a) Speed at D

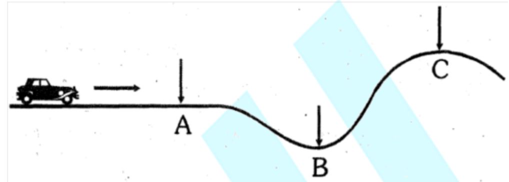
(b) Normal reaction at D

(c) Height H



5. A car is moving along a hilly road as shown (side view). The coefficient of static friction between the tyres and the pavement is constant and the car maintains a steady speed. If at one of the points shown the driver applies brakes as hard as possible without making the tyres slip, the magnitude of the frictional force immediately after the brakes are applied will be maximum if the car was at :-

(A) point A
(B) point B
(C) point C
(D) friction force same for positions A, B and C



6. A stone weighing 0.5 kg tied to a rope of length 0.5 m revolves along a circular path in a vertical plane. The tension of the rope at the bottom point of the circle is 45 newton. To what height will the stone rise if the rope breaks at the moment when the velocity is directed upwards? ($g = 10\ m/s^2$)

ANSWERS KEYS

BEGINNER'S BOX - 1

1.	(C)	**2.**	(D)	**3.**	$6\ m/s^2$ toward the centre.		
4.	1 : 1	**5.**	(C)	**6.**	(C)	**7.**	(A)
8.	$-3\hat{k}\ rad/s^2, -2\hat{k}\ rad/s^2$						

BEGINNER'S BOX - 2

1.	$\frac{2m_1}{m_2+2m_1}$	**2.**	$8\ m/s^2$	**3.**	$v_{lowest} = 20\sqrt{2}\ m/s$; $v_{highest} = 20\ m/s$		
4.	(a) $\sqrt{5gR}$ (b) 4 mg (c) $\frac{9}{2}R$			**5.**	(B)	**6.**	1.5 m