Probability

SET THEORY

The collection of well defined things is called set. Well defined means a law by which we are able to find whether a given thing is contained in the given set or not.

Example : A = { x ; x , an odd number less than 15 } means

A = { 1, 3, 5, 7, 9 , 11, 13 }

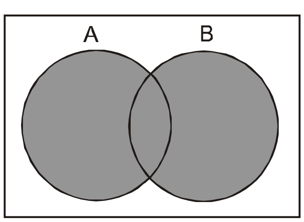
Example : W = { x ; x is a whole number } means

W = { 0 , 1, 2, 3, 4, 5, ... }

OPERATIONS OF SETS

The basic operations of sets and their related results are as follows.

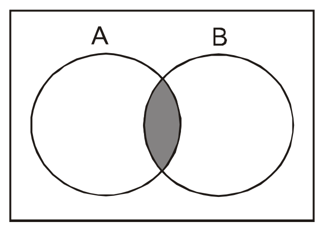
1. **Union of sets :** The union of two sets is represented by $A \cup B$ or $A + B$. This set contains those elements which are in A or in B or in A & B both. So



$A \cup B$

$A \cup B = \{ x ; x \in A \text{ or } x \in B \}$

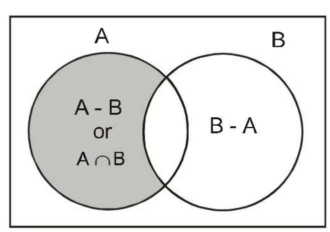
2. **Intersection of two sets :** The intersection of two sets is represented by $A \cap B$ or AB. It contains those all elements which are contained in both sets A & B both so



$A \cap B$

$A \cap B = \{ x ; x \in A \text{ and } x \in B \}$

3. **Difference of two sets :** If A and B are two sets then A–B represents the set of those elements which are in A and not in B. In the same manner B–A represents the set of those elements which are in B and not in A. So



$A - B = A \cap \bar{B} = \{ x ; x \in A \text{ and } x \notin B \}$

There are various phenomena in nature, leading to an outcome, which cannot be predicted apriori e.g. in tossing of a coin, a head or a tail may result. Probability theory aims at measuring the uncertainties of such outcomes.

(I) Important terminology:

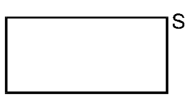
(i) *Random Experiment* :

It is a process which results in an outcome which is one of the various possible outcomes that are known to us before hand e.g. throwing of a die is a random experiment as it leads to fall of one of the outcome from $\{1, 2, 3, 4, 5, 6\}$. Similarly taking a card from a pack of 52 cards is also a random experiment.

(ii) *Sample Space :*

It is the set of all possible outcomes of a random experiment e.g. {H, T} is the sample space associated with tossing of a coin.

In set notation it can be interpreted as the universal set.



Solved Examples

Ex.1 Write the sample space of the experiment 'A coin is tossed and a die is thrown'.

Sol. The sample space S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.

Ex.2 Write the sample space of the experiment 'A coin is tossed, if it shows head a coin tossed again else a die is thrown.

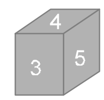
Sol. The sample space S = {HH, HT, T1, T2, T3, T4, T5, T6}

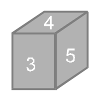
Ex.3 Find the sample space associated with the experiment of rolling a pair of dice (plural of die) once. Also find the number of elements of the sample space.

Sol. Let one die be blue and the other be grey. Suppose '1' appears on blue die and '2' appears on grey die. We denote this outcome by an ordered pair (1, 2). Similarly, if '3' appears on blue die and '5' appears on grey die, we denote this outcome by (3, 5) and so on. Thus, each outcome can be denoted by an ordered pair (x, y), where x is the number appeared on the first die (blue die) and y appeared on the second die (grey die). Thus, the sample space is given by

S = {(x, y) x is the number on blue die and y is the number on grey die}

We now list all the possible outcomes (figure)





	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Figure

Number of elements (outcomes) of the above sample space is 6×6 i.e., 36

Self Practice Problems :

1. A coin is tossed twice, if the second throw results in head, a die is thrown.

Answer {HT, TT, HH1, HH2, HH3, HH4, HH5, HH6, TH1, TH2, TH3, TH4, TH5, TH6}.

2. An urn contains 3 red balls and 2 blue balls. Write sample space of the experiment 'Selection of a ball from the urn at random'.

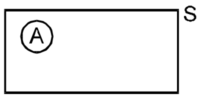
Answer $\{R_1, R_2, R_3, B_1, B_2\}$.

Note :

Here the balls are distinguished from one and other by naming red balls as R_1, R_2 and R_3 and the blue balls as B_1 and B_2.

(iii) ***Event :***

It is subset of sample space. e.g. getting a head in tossing a coin or getting a prime number is throwing a die. In general if a sample space consists 'n' elements, then a maximum of 2^n events can be associated with it.



(iv) ***Complement of event :***

The complement of an event 'A' with respect to a sample space S is the set of all elements of 'S' which are not in A. It is usually denoted by A', $\overline{A}$ or A^C.

(v) ***Simple Event :***

If an event covers only one point of sample space, then it is called a simple event e.g. getting a head followed by a tail in throwing of a coin 2 times is a simple event.

(vi) ***Compound Event :***

When two or more than two events occur simultaneously, the event is said to be a compound event. Symbolically $A \cap B$ or AB represent the occurrence of both A & B simultaneously.

Note :

"$A \cup B$" or $A + B$ represent the occurrence of either A or B.

Solved Examples

Ex.4 Write down all the events of the experiment 'tossing of a coin'.

Sol. S = {H, T}

the events are ϕ, {H}, {T}, {H, T}

Ex.5 A die is thrown. Let A be the event ' an odd number turns up' and B be the event 'a number divisible by 3 turns up'.

Write the events (a) A or B (b) A and B

Sol. A = {1, 3, 5}, B = {3, 6}

$\therefore$ A or B = $A \cup B$ = {1, 3, 5, 6}

A and B = $A \cap B$ = {3}

(vii) ***Equally likely Events :***

If events have same chance of occurrence, then they are said to be equally likely.

e.g

(i) In a single toss of a fair coin, the events {H} and {T} are equally likely.

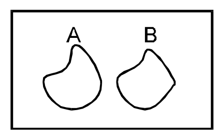
(ii) In a single throw of an unbiased die the events {1}, {2}, {3} and {4}, are equally likely.

(iii) In tossing a biased coin the events {H} and {T} are not equally likely.

(viii) ***Mutually Exclusive / Disjoint / Incompatible Events :***

Two events are said to be mutually exclusive if occurrence of one of them rejects the possibility of occurrence of the other i.e. both cannot occur simultaneously.

In the vein diagram the events A and B are mutually exclusive. Mathematically, we write $A \cap B = \phi$



Solved Examples

Ex.6 In a single toss of a coin find whether the events {H}, {T} are mutually exclusive or not.

Sol. Since $\{H\} \cap \{T\} = \phi$,

$\therefore$ the events are mutually exclusive.

Ex.7 In a single throw of a die, find whether the events {1, 2}, {2, 3} are mutually exclusive or not.

Sol. Since $\{1, 2\} \cap \{2, 3\} = \{2\} \neq \phi$

$\therefore$ the events are not mutually exclusive.

(ix) ***Exhaustive System of Events :***

If each outcome of an experiment is associated with at least one of the events $E_1, E_2, E_3, ...E_n$, then collectively the events are said to be exhaustive. Mathematically we write

$E_1 \cup E_2 \cup E_3 ....E_n = S$. (Sample space)

Solved Examples

Ex.8 In throwing of a die, let A be the event 'even number turns up', B be the event 'an odd prime turns up' and C be the event 'a numbers less than 4 turns up'. Find whether the events A, B and C form an exhaustive system or not.

Sol. $A \equiv \{2, 4, 6\}$, $B \equiv \{3, 5\}$ and $C \equiv \{1, 2, 3\}$.

Clearly $A \cup B \cup C = \{1, 2, 3, 4, 5, 6\} = S$.

Hence the system of events is exhaustive.

Ex.9 Three coins are tossed. Describe

(i) two events A and B which are mutually exclusive

(ii) three events A, B and C which are mutually exclusive and exhaustive.

(iii) two events A and B which are not mutually exclusive.

(iv) two events A and B which are mutually exclusive but not exhaustive.

(v) three events A, B and C which are mutually exclusive but not exhaustive.

Sol. (i) A : "getting at least two heads"

B : "getting at least two tails"

(ii) A : "getting at most one heads"

B : "getting exactly two heads"

C : "getting exactly three heads"

(iii) A : "getting at most two tails"

B : "getting exactly two heads"

(iv) A : "getting exactly one head"

B : "getting exactly two heads"

(v) A : " getting exactly one tail"

B : "getting exactly two tails"

C : "getting exactly three tails"

[Note : There may be other cases also]

MATHEMATICAL DEFINITION OF PROBABILITY

Let the outcomes of an experiment consists of n exhaustive mutually exclusive and equally likely cases. Then the sample spaces S has n sample points. If an event A consists of m sample points, $(0 \le m \le n)$, then the probability of event A, denoted by P(A) is defined to be m/n i.e. P(A) = m/n.

Let $S = a_1, a_2, \ldots\ldots, a_n$ be the sample space

* $P(S) = \frac{n}{n} = 1$ corresponding to the certain event.
* $P(\phi) = \frac{0}{n} = 0$ corresponding to the null event ϕ or impossible event.
* If $A_i = \{a_i\}$, i = 1, ..., n then A_i is the event corresponding to a single sample point a_i. Then $P(A_i) = \frac{1}{n}$.
* $0 \le P(A) \le 1$

  $\because\ 0 \le m \le n \Rightarrow\ 0 \le \frac{m}{n} \le 1\ \Rightarrow 0 \le P(A) \le 1$
* $P(A') = 1 - P(A)$

  If the event A has m elements, then A' has (n – m) elements in S.

  $\therefore\ P(A') = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$

Solved Examples

Ex.10 If three cards are drawn from a pack of 52 cards, what is the chance that all will be queen ?

Sol. If the sample space be s then n(s) = the total number of ways of drawing 3 out of 52 cards = ${}^{52}C_3$ Now, if A = the event of drawing three queens then $n(A) = {}^4c_3$

$$\therefore\ P(E) = \frac{n(A)}{n(s)} = \frac{{}^4C_3}{{}^{52}C_3} = \frac{4}{\frac{52 \times 51 \times 50}{3 \times 2}} = \frac{1}{5525}$$

Ex.11 Words are formed with the letters of the word PEACE. Find the probability that 2 E's come together

Sol. Total number of words which can be formed with the letters P, E, A, C, E $= \frac{\underline{|5}}{\underline{|2}} = 60$

Number of words in which 2 E's come together

$= \underline{|4} = 24$ $\quad \therefore$ reqd. prob. $= \frac{24}{60} = \frac{2}{5}$

Ex.12 A bag contains 5 red and 4 green balls. For balls are drawn at random then find the probability that two balls are of red and two balls are of green colour.

Sol. n(s) = the total number of ways of drawing 4 balls out of total 9 balls : ${}^9C_4$

If A_1 = the event of drawing 2 red balls out of 5 red balls then $n(A_1) = {}^5C_2$

A_2 = the event of drawing 2 green balls out of 4 green balls then $n(A_2) = {}^4C_2$

$\therefore n(A) = n(A_1) \cdot n(A_2) = {}^5C_2 \times {}^4C_2$

$$\therefore P(A) = \frac{n(A)}{n(s)} = \frac{{}^5C_2 \times {}^4C_2}{{}^9C_4} = \frac{\frac{5\times4\times4\times3}{2\times2}}{\frac{9\times8\times7\times6}{4\times3\times2}} = \frac{10}{21}$$

Ex.13 Two dice are thrown at a time. Find the probability of the following

(i) these numbers shown are equal

(ii) the difference of numbers shown is 1

Sol. The sample space in a throw of two dice

s = { 1 , 2 , 3, 4 , 5 , 6 } × { 1 , 2, 3, 4, 5, 6 }

$\therefore$ total no. of cases n (s) = 6 × 6 = 36.

(i) Here E_1 = the event of showing equal number on both dice

= { (1 , 1) (2 , 2) (3 , 3) (4 , 4) (5 , 5) (6 , 6) }

$$\therefore n(E_1) = 6 \quad \therefore P(E_1) = \frac{n(E_1)}{n(s)} = \frac{6}{36} = \frac{1}{6}$$

(ii) Here E_2 = the event of showing numbers whose difference is 1.

= { (1, 2) (2 , 1) (2 , 3) (3, 2) (3 , 4) (4 , 3) (4, 5) (5, 4) (5 , 6) (6 , 5) }

$$\therefore n(E_2) = 10 \quad \therefore p(E_2) = \frac{n(E_2)}{n(s)} = \frac{10}{36} = \frac{5}{18}$$

Ex.14 In throwing of a fair die find the probability of the event ' a number ≤ 4 turns up'.

Sol. Sample space S = {1, 2, 3, 4, 5, 6} ;

event A = {1, 2, 3, 4}

$\therefore$ n(A) = 4 and n(S) = 6

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{4}{6} = \frac{2}{3}.$$

Ex.15 In throwing of a fair die, find the probability of turning up of an odd number ≥ 4.

Sol. S = {1, 2, 3, 4, 5, 6}

Let E be the event 'turning up of an odd number ≥ 4'

then E = {5}

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}.$$

Ex.16 In throwing a pair of fair dice, find the probability of getting a total of 8.

Sol. When a pair of dice is thrown the sample space consists

{(1, 1) (1, 2) .... (1, 6)

(2, 1,) (2, 2,) .... (2, 6)

....

....

(6, 1), (6, 2) .... (6, 6)}

Note that (1, 2) and (2, 1) are considered as separate points to make each outcome as equally likely.

To get a total of '8', favourable outcomes are, (2, 6) (3, 5) (4, 4) (5, 3) and (6, 2).

Hence required probability = $\frac{5}{36}$

Ex.17 A four digit number is formed using the digits 0, 1, 2, 3, 4 without repetition. Find the probability that it is divisible by 4

Sol. Total 4 digit numbers formed

$4 \times 4 \times 3 \times 2 = 96$

Each of these 96 numbers are equally likely & mutually exclusive of each other.

Now, A number is divisible by 4, if last two digits of the number is divisible by 4

Hence we can have

		0	4	$\rightarrow$	first two places can be filled in $3 \times 2 = 6$ ways
		1	2	$\rightarrow$	first two places can be filled in $2 \times 2 = 4$ ways
		2	0	$\rightarrow$	6 ways
		2	4	$\rightarrow$	4 ways
		3	2	$\rightarrow$	4 ways
		4	0	$\rightarrow$	6 ways

Total number of ways 30 ways

$$\text{probability} = \frac{\text{favorable outcomes}}{\text{Total outcomes}}$$

$$= \frac{30}{96} \qquad = \frac{5}{16} \quad \text{Ans.}$$

ODDS AGAINST AND ODDS IN FAVOUR OF AN EVENT

Let there be $m + n$ equally likely, mutually exclusive and exhaustive cases out of which an event A can occur in m cases and does not occur in n cases. Then by definition of probability of occurrences

$$= \frac{m}{m+n}$$

The probability of non-occurrence $= \dfrac{n}{m+n}$

$\therefore$ P(A) : P(A') = m : n

Thus the odd in favour of occurrences of the event A are defined by m : n i.e. P(A) : P(A'); and the odds against the occurrence of the event A are defined by n : m i.e. P(A') : P(A).

Solved Examples

Ex.18 Find the odds in favors of getting a king when a card is drawn from a well shuffled pack of 52 cards

Sol. $\dfrac{{}^4C_1}{{}^{48}C_1} = \dfrac{4}{48} = \dfrac{1}{12}$

Ex.19 In a single cast with two dice find the odds against drawing 7

Sol. E = {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}.

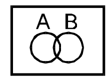
$\therefore$ $P(E) = \dfrac{6}{6 \times 6} = \dfrac{1}{6}$. So, the odds against drawing

$$7 = \frac{P(\bar{E})}{P(E)} = \frac{1 - \frac{1}{6}}{\frac{1}{6}} = \frac{5}{1} = 5 : 1.$$

Addition theorem of probability :

If 'A' and 'B' are any two events associated with an experiment, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



De Morgan's Laws : If A & B are two subsets of a universal set U, then

(a) $(A \cup B)^c = A^c \cap B^c$

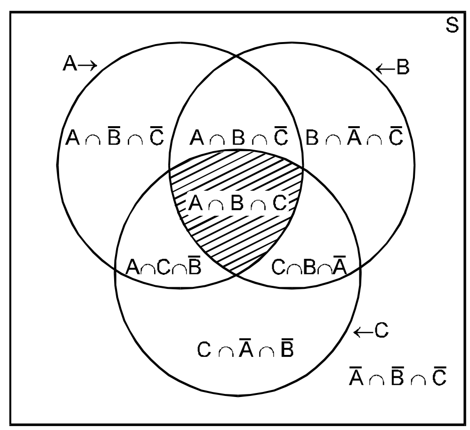
(b) $(A \cap B)^c = A^c \cup B^c$

Distributive Laws :

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

For any three events A, B and C we have the figure



(i) P(A or B or C) = $P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(ii) P (at least two of A, B, C occur) = $P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(iii) P(exactly two of A, B, C occur) = $P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(iv) P(exactly one of A, B, C occur)

$= P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

Note :

If three events A, B and C are pair wise mutually exclusive then they must be mutually exclusive, i.e. $P(A \cap B) = P(B \cap C) = P(C \cap A) = 0$
$\Rightarrow P(A \cap B \cap C) = 0$.
However the converse of this is not true.

Solved Examples

Ex.20 A bag contains 4 white, 3red and 4 green balls. A ball is drawn at random. Find the probability of the event 'the ball drawn is white or green'.

Sol. Let A be the event 'the ball drawn is white' and B be the event 'the ball drawn is green'.

P(The ball drawn is white or green) = $P(A \cup B)$

$= P(A) + P(B) - P(A \cap B) = \frac{8}{11}$

Ex.21 In throwing of a die, let A be the event 'an odd number turns up', B be the event 'a number divisible by 3 turns up' and C be the event 'a number ≤ 4 turns up'. Then find the probability that exactly two of A, B and C occur.

Sol. Event $A = \{1, 3, 5\}$, event $B = \{3, 6\}$ and event $C = \{1, 2, 3, 4\}$

$\therefore$ $A \cap B = \{3\}, B \cap C = \{3\}, A \cap C = \{1, 3\}$ and $A \cap B \cap C = \{3\}$.

Thus P(exactly two of A, B and C occur)

$= P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C)$

$= \frac{1}{6} + \frac{1}{6} + \frac{2}{6} - 3 \times \frac{1}{6} = \frac{1}{6}$

Ex.22 One digit is selected from 20 positive integers. What is the probability that it is divisible by 3 or 4.

Sol. Let A = event that selected number is divisible by 3

B = event that selected number is divisible by 4.

Here, the events are not mutually exclusive then

$$P(A) = \frac{6}{20},\ P(B) = \frac{5}{20}, P(AB) = \frac{1}{20}$$

$$\therefore P(A+B) = P(A) + P(B) - P(AB)$$

$$= \frac{6}{20} + \frac{5}{20} - \frac{1}{20} = \frac{10}{20} = \frac{1}{2}$$

Ex.23 A bag contains 6 white, 5 black and 4 red balls. Find the probability of getting either a white or a black ball in a single draw.

Sol. Let A = event that we get a white ball,

B = event that we get a black ball

So, the events are mutually exclusive

$$P(A) = \frac{^6C_1}{^{15}C_1}, P(B) = \frac{^5C_1}{^{15}C_1}$$

So, $P(A+B) = P(A) + P(B) = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}$

CONDITIONAL PROBABILITY

Conditional Probability : If A and B are any events in S then the conditional probability of B relative to A is given by

$$P(B/A) = \frac{P(B \cap A)}{P(A)}, \quad \text{If } P(A) \ne 0$$

Independent Events : If the occurrence of the event A does not depend on the occurrence or the non-occurrence of the event B then A and B are said to be independent events.

Clearly $P(B/A) = P(B)$ and $P(A/B) = P(A)$.

Ex.24 A coin is tossed thrice. If E be the event of showing atleast two heads and F the event of showing head in the first throw, then find $P\left(\frac{E}{F}\right)$

Sol. There are following 8 outcomes of three throws: HHH, HHT, HTH, HTT THH, THT, TTH, TTT

Also $P(E \cap F) = \frac{3}{8}$ and $P(F) = \frac{4}{8}$

$$\therefore \text{ reqd. prob. } = P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{3}{8}}{\frac{4}{8}} = \frac{3}{4}$$

Ex.25 Two dice are thrown. Find the probability that the numbers appeared has a sum of 8 if it is known that the second dice always exhibits 4

Sol. Let A be the event of occurence of 4 always on the second die

$= \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\}$;

$\therefore n(A) = 6$

and B be the event of occurence of such numbers on both dice whose sum is $8 = \{(4,4)\}$.

Thus, $A \cap B = A \cap \{(4,4)\} = \{(4,4)\}$

$\therefore n(A \cap B) = 1$

$\therefore P\left(\frac{B}{A}\right) = \frac{n(A \cap B)}{n(A)} = \frac{1}{6}$

Solved Examples

Ex.26 If P(A/B) = 0.2 and P(B) = 0.5 and P(A) = 0.2. Find $P(A \cap \overline{B})$.

Sol. $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

Also $\quad P(A/B) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow \quad P(A \cap B) = 0.1$

From given data, $\quad P(A \cap \overline{B}) = 0.1$

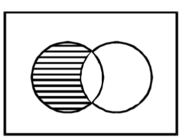
Ex.27 If P(A) = 0.25, P(B) = 0.5 and $P(A \cap B) = 0.14$, find probability that neither 'A' nor 'B' occurs. Also find $P(A \cap \overline{B})$

Sol. We have to find $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$

(by De-Morgan's law)

Also, $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

putting data we get, $P(\overline{A} \cap \overline{B}) = 0.39$



The shaded region denotes the simultaneous occurrence of A and $\overline{B}$

Hence $P(A \cap \overline{B}) = P(A) - P(A \cap B) = 0.11$

MALTIPLICATION THEOREM

Independent and dependent events

If two events are such that occurence or non-occurence of one does not affect the chances of occurence or non-occurence of the other event, then the events are said to be independent. Mathematically : if $P(A \cap B) = P(A)\,P(B)$, then A and B are independent.

Note:

(i) If A and B are independent, then (a) A′ and B′ are independent, (b) A and B′ are independent and (c) A′ and B are independent.

(ii) If A and B are independent, then P(A / B) = P(A).
If events are not independent then they are said to be dependent.

Independency of three or more events

Three events A, B & C are independent if & only if all the following conditions hold :

$P(A \cap B) = P(A) . P(B)$;

$P(B \cap C) = P(B) . P(C)$

$P(C \cap A) = P(C) . P(A)$;

$P(A \cap B \cap C) = P(A) . P(B) . P(C)$

i.e. they must be independent in pairs as well as mutually independent.

Similarly for n events $A_1, A_2, A_3, \ldots\ldots\ldots A_n$ to be independent, the number of these conditions is equal to ${}^nC_2 + {}^nC_3 + \ldots\ldots + {}^cC_n = 2^n - n - 1$.

Solved Examples

Ex.28 In drawing two balls from a box containing 6 red and 4 white balls without replacement, which of the following pairs is independent ?

(a) Red on first draw and red on second draw

(b) Red on first draw and white on second draw

Sol. Let E be the event 'Red on first draw', F be the event 'Red on second draw' and G be the event 'white on second draw'.

$$P(E) = \frac{6}{10},\ P(F) = \frac{6}{10},\ P(G) = \frac{4}{10}$$

(a) $P(E \cap F) = \frac{{}^6P_2}{{}^{10}P_2} = \frac{1}{3}$

$P(E) . P(F) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25} \neq \frac{1}{3}$

$\therefore$ E and F are not independent

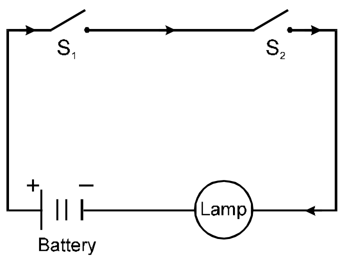
(b) $P(E) . P(G) = \frac{6}{10} \times \frac{4}{10} = \frac{6}{25}$

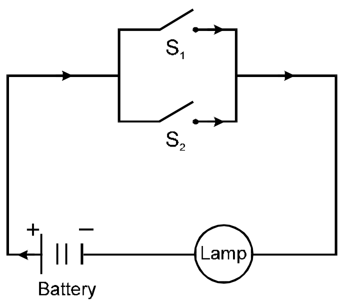
$P(E \cap G) = \frac{{}^6P_1 \times {}^4P_1}{{}^{10}P_2} = \frac{4}{15}$

$\therefore$ $P(E) . P(G) \neq P(E \cap G)$

$\therefore$ E and G are not independent

Ex.29 If two switches S_1 and S_2 have respectively 90% and 80% chances of working. Find the probabilities that each of the following circuits will work.





Sol. Consider the following events :

A = Switch S_1 works,

B = Switch S_2 works,

We have,

$$P(A) = \frac{90}{100} = \frac{9}{10} \text{ and } P(B) = \frac{80}{100} = \frac{8}{10}$$

(i) The circuit will work if the current flows in the circuit. This is possible only when both the switches work together. Therefore,

Required probability

$= P(A \cap B) = P(A)\,P(B)$

[$\because$ A and B are independent events]

$= \frac{9}{10} \times \frac{8}{10} = \frac{72}{100} = \frac{18}{25}$

(ii) The circuit will work if the current flows in the circuit. This is possible only when at least one of the two switches S_1, S_2 works. Therefore,

Required Probability

$= P(A \cup B) = 1 - P(\bar{A})\,P(\bar{B})$

[$\because$ A, Bare independent events]

$= 1 - \left(1 - \frac{9}{10}\right)\left(1 - \frac{8}{10}\right)$

$= 1 - \frac{1}{10} \times \frac{2}{10} = \frac{49}{50}$

Ex.30 A speaks truth in 60% of the cases and b in 90% of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact?

Sol. Let E be the event that A speaks truth and F be the event that B speaks truth. Then E and F are independent events such that

$$P(E) = \frac{60}{100} = \frac{3}{5} \text{ and } P(F) = \frac{90}{100} = \frac{9}{10}$$

A and B will contradict each other in narrating the same fact in the following mutually exclusive ways :

(i) A speaks truth and B tells a lie i.e. $E \cap \bar{F}$

(ii) A tells a lie and B speaks truth lie i.e. $\bar{E} \cap F$

$\therefore$ P(A and B contradict each other)

$= P(\text{I or II}) = (\text{I} \cup \text{II})$

$= P[(E \cap \bar{F}) \cup (\bar{E} \cap F)]$

$= P(E \cap \bar{F}) + P(\bar{E} \cap F)$

[$\because E \cap \bar{F}$ and $\bar{E} \cap F$ are mutually exclusive]

$= P(E)\,P(\bar{F}) + P(\bar{E})\,P(F)$

[$\because$ E and F are in dep.]

$= \frac{3}{5} \times \left(1 - \frac{9}{10}\right) + \left(1 - \frac{3}{5}\right) \times \frac{9}{10}$

$= \frac{3}{5} \times \frac{1}{10} + \frac{2}{5} \times \frac{9}{10} = \frac{21}{50}$

Ex.31 An urn contains 7 red and 4 blue balls. Two balls are drawn at random with replacement. Find the probability of getting

(i) 2 red balls

(ii) 2 blue balls

(iii) one red and one blue ball

Sol. (i) $\frac{49}{121}$ (ii) $\frac{16}{121}$ (iii) $\frac{56}{121}$

Ex.32 Probabilities of solving a specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(i) the problem is solved

(ii) exactly one of them solves the problem.

Sol. (i) $\frac{2}{3}$ (ii) $\frac{1}{2}$

Ex.33 A box contains 5 bulbs of which two are defective. Test is carried on bulbs one by one untill the two defective bulbs are found out. Find the probability that the process stops after

(i) Second test (ii) Third test

Sol. (i) Process will stop after second test. Only if the first and second bulb are both found to be defective

probability = $\frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$ (Obviously the bulbs drawn are not kept back.)

(ii) Process will stop after third test when either

(a) DND $\rightarrow$ $\frac{2}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{10}$

Here 'D' stands for defective

or (b) NDD $\rightarrow$ $\frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} = \frac{1}{10}$

and 'N' is for not defective.

or (c) NNN $\rightarrow$ $\frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} = \frac{1}{10}$

hence required probability $= \frac{3}{10}$

Ex.34 If E_1 and E_2 are two events such that $P(E_1) = \frac{1}{4}$; $P(E_2) = \frac{1}{2}$; $P\left(\frac{E_1}{E_2}\right) = \frac{1}{4}$, then choose the correct options.

(i) E_1 and E_2 are independent

(ii) E_1 and E_2 are exhaustive

(iii) E_1 and E_2 are mutually exclusive

(iv) E_1 & E_2 are dependent

Also find $P\left(\frac{\overline{E}_1}{E_2}\right)$ and $\left(\frac{E_2}{\overline{E}_1}\right)$

Sol. Since $\left(\frac{E_2}{E_1}\right) = P(E_1)$

$\Rightarrow$ E_1 and E_2 are independent of each other

Also since $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1) \cdot P(E_2) \neq 1$

Hence events are not exhaustive. Independent events can't be mutually exclusive.

Hence only (i) is correct

Further since E_1 & E_2 are independent; E1 and $\overline{E}_2$ or $\overline{E}_1, E_2$ are $\overline{E}_1, \overline{E}_2$ are also independent.

Hence $P\left(\frac{\overline{E}_1}{E_2}\right) = P(\overline{E}_1) = \frac{3}{4}$ and

$P\left(\frac{E_2}{\overline{E}_1}\right) = P(E_2) = \frac{1}{2}$

Ex.35 If cards are drawn one by one from a well shuffled pack of 52 cards without replacement, until an ace appears, find the probability that the fourth card is the first ace to appear.

Sol. Probability of selecting 3 non-Ace and 1 Ace out of 52 cards is equal to $\frac{{}^{48}C_3 \times {}^4C_1}{{}^{52}C_4}$

Since we want 4th card to be first ace, we will also have to consider the arrangement, Now 4 cards in sample space can be arranged in 4! ways and, favorable they can be arranged in 3 ! ways as we want 4th position to be occupied by ace

Hence required probability $= \frac{{}^{48}C_3 \times {}^4C_1}{{}^{52}C_4} \times \frac{3!}{4!}$

Aliter :

'NNNA' is the arrangement than we desire in taking out cards, one by one

Hence required chance is $\frac{48}{52} \times \frac{47}{51} \times \frac{46}{50} \times \frac{4}{49}$

Ex.36 Let A, B, C be 3 independent events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{4}$. Then find probability of exactly 2 events occuring out of 3 events-

Sol. P (exactly two of A, B, C occur)

$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

$= P(B) \cdot P(C) + P(C) \cdot P(A) + P(A) \cdot P(B) - 3P(A) \cdot P(B) \cdot P(C)$

$= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{4}$

Ex.37 A bag contains 3 red, 6 white and 7 blue balls. Two balls are drawn one by one. What is the probability that first ball is white and second ball is blue when first drawn ball is not replaced in the bag?

Sol. Let A be the event of drawing first ball white and B be the event of drawing second ball blue.

Here A and B are dependent events.

$$P(A)=\frac{6}{16},\ P\left(\frac{B}{A}\right)=\frac{7}{15}$$

$$P(AB)=P(A).P\left(\frac{B}{A}\right)=\frac{6}{16}\times\frac{7}{15}=\frac{7}{40}$$

Ex.38 Three coins are tossed together. What is the probability that first shows head, second shows tail and third shows head?

Sol. Let A, B, C denote three given component events which are mutually independent.

So, $P(ABC)=P(A).P(B).P(C)=\frac{1}{2}.\frac{1}{2}.\frac{1}{2}=\frac{1}{8}$

Ex.39 A bag contains 4 red and 4 blue balls. Four balls are drawn one by one from the bag, then find the probability that the drawn balls are in alternate colour.

Sol. E_1 : Event that first drawn ball is red, second is blue and so on.

E_2 : Event that first drawn ball is blue, second is red and so on.

$\therefore P(E_1)=\frac{4}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{3}{5}$ and $P(E_2)=\frac{4}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{3}{5}$

$$P(E)=P(E_1)+P(E_2)=2\times\frac{4}{8}.\frac{4}{7}.\frac{3}{6}.\frac{3}{5}=\frac{6}{35}$$

Probability of at least one of the n Independent events

If p_1 , p_2 , p_3 , ... p_n are the probabilites of n independent events A_1 , A_2, A_3 ..... A_n then the probability of happening of at least one of these event is

$1-[(1-p_1)(1-p_2)\ .....\ (1-p_n)]$

$P(A_1+A_2+A_3+\ ....\ +A_n)=1-P(\bar{A}_1)P(\bar{A}_2)$
$P(\bar{A}_3\ ....\ P(\bar{A}_n)$

Solved Examples

Ex.40 A probalem of mathematic is given to three students A, B and C. Whose chances of solving it are 1/2, 1/3 , 1/4 respectively. Then probability that the problem is solved is

Sol. Obviously the events of solving the problem by A, B and C are independent. Therefore required probability

$$=1-\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\right]=1-\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{2}=\frac{3}{4}$$

Total Probability Theorem

If an event A can occur with one of the n mutually exclusive and exhaustive events B_1, B_2, ..., B_n and the probabilities $P(A/B_1)$, $P(A/B_2)$ .... $P(A/B_n)$ are known, then

$$P(A)\ =\ \sum_{i=1}^{n}P(B_i)\,.\,P(A/B_i)$$

Solved Examples

Ex.41 Box - I contains 5 red and 4 white balls whilst box - II contains 4 red and 2 white balls. A fair die is thrown. If it turns up a multiple of 3, a ball is drawn from box - I else a ball is drawn from box - II. Find the probability that the ball drawn is white.

Sol. Let A be the event 'a multiple of 3 turns up on the die' and R be the event 'the ball drawn is white' then P (ball drawn is white)

$= P(A)\ .\ P(R / A) + P(\bar{A})\ P(R / \bar{A})$

$$=\frac{2}{6}\times\frac{4}{9}+\left(1-\frac{2}{6}\right)\frac{2}{6}=\frac{10}{27}$$

Ex.42 Cards of an ordinary deck of playing cards are placed into two heaps. Heap - I consists of all the red cards and heap - II consists of all the black cards. A heap is chosen at random and a card is drawn, find the probability that the card drawn is a king.

Sol. Let I and II be the events that heap - I and heap - II are choosen respectively. Then P(I) = P(II) = $\frac{1}{2}$

Let K be the event 'the card drawn is a king'

$\therefore$ P (K / I) = $\frac{2}{26}$ and P(K / II) = $\frac{2}{26}$

$\therefore$ P(K) = P (I) P(K / I) + P(II) P(K / II)

$= \frac{1}{2} \times \frac{2}{26} + \frac{1}{2} \times \frac{2}{26} = \frac{1}{13}.$

BAYE'S THEOREM

Let $A_1, A_2, \ldots, A_n$ be n mutually exclusive and exhaustive events of the sample space S and A is event which can occur with any of the events then

$$P\left(\frac{A_i}{A}\right) = \frac{P(A_i)\ P(A/A_i)}{\sum_{i=1}^{n} P(A_i)\ P(A/A_i)}$$

Solved Examples

Ex.43 A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from the bag B.

Sol. Let E_1 = The event of ball being drawn from bag A

E_2 = The event of ball being drawn from bag B.

E = The event of ball being red.

Since, both the bags are equally likely to be selected, therefore

$P(E_1) = P(E_2) = \frac{1}{2}$ and $P\left(\frac{E}{E_1}\right) = \frac{3}{5}$ and $P\left(\frac{E}{E_2}\right) = \frac{5}{9}$

$\therefore$ Required probability

$$P\left(\frac{E_2}{E}\right) = \frac{P(E_2)P\left(\frac{E}{E_2}\right)}{P(E_1).P\left(\frac{E}{E_1}\right) + P(E_2)P\left(\frac{E}{E_2}\right)}$$

$$= \frac{\frac{1}{2} \times \frac{5}{9}}{\frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{5}{9}} = \frac{25}{52}$$

Ex.44 Pal's gardener is not dependable, the probability that he will forget to water the rose bush is $\frac{2}{3}$. The rose bush is in questionable condition any how, if watered the probability of its withering is $\frac{1}{2}$, if not watered, the probability of its withering is $\frac{3}{4}$. Pal went out of station and upon returning, he finds that the rose bush has withered, what is the probability that the gardener did not water the bush. [Here result is known that the rose bush has withered, therefore. Bayes's theorem should be used]

Sol. Let A = the event that the rose bush has withered

Let A_1 = the event that the gardener did not water.

A_2 = the event that the gardener watered.

By Bayes's theorem required probability,

$$P(A_1/A) = \frac{P(A_1).P(A/A_1)}{P(A_1).P(A/A_1) + P(A_2).P(A/A_2)} \quad \text{.....(i)}$$

Given, $P(A_1) = \frac{2}{3}$ $\therefore$ $P(A_2) = \frac{1}{3}$

$P(A/A_1) = \frac{3}{4}$, $P(A/A_2) = \frac{1}{2}$

From (1), $P(A_1/A) = \dfrac{\frac{2}{3}.\frac{3}{4}}{\frac{2}{3}.\frac{3}{4} + \frac{1}{3}.\frac{1}{2}} = \frac{6}{6+2} = \frac{3}{4}$

Ex.45 There are 5 brilliant students in class XI and 8 brilliant students in class XII. Each class has 50 students. The odds in favour of choosing the class XI are 2 : 3. If the class XI is not chosen then the class XII is chosen. A student is a chosen and is found to be brilliant, find the probability that the chosen student is from class XI.

Sol. Let E and F be the events 'Class XI is chosen' and 'Class XII is chosen' respectively.

Then P(E) = $\frac{2}{5}$, P(F) = $\frac{3}{5}$

Let A be the event 'Student chosen is brilliant'.

Then $P(A / E) = \frac{5}{50}$ and $P(A / F) = \frac{8}{50}$.

$\therefore$ $P(A) = P(E) . P(A / E) + P(F) . P(A / F)$

$= \frac{2}{5} \cdot \frac{5}{50} + \frac{3}{5} \cdot \frac{8}{50} = \frac{34}{250}$.

$\therefore$ $P(E / A) = \frac{P(E) . P(A/E)}{P(E) . P(A/E) + P(F) . P(A/F)} = \frac{5}{17}$.

Ex.46 A pack of cards is counted with face downwards. It is found that one card is missing. One card is drawn and is found to be red. Find the probability that the missing card is red.

Sol. Let A be the event of drawing a red card when one card is drawn out of 51 cards (excluding missing card.) Let A_1 be the event that the missing card is red and A_2 be the event that the missing card is black.

Now by Bayes's theorem, required probability,

$$P(A_1/A) = \frac{P(A_1) . (P(A/A_1)}{P(A_1) . P(A/A_1) + P(A_2) . P(A/A_2)} \quad .....(i)$$

In a pack of 52 cards 26 are red and 26 are black.

Now $P(A_1)$ = probability that the missing card is red $= \frac{^{26}C_1}{^{52}C_1} = \frac{26}{52} = \frac{1}{2}$

$P(A_2)$ = probability that the missing card is black

$= \frac{26}{52} = \frac{1}{2}$

$P(A/A_1)$ = probability of drawing a red card when the missing card is red.

$= \frac{25}{51}$

[$\because$ Total number of cards left is 51 out of which 25 are red and 26 are black as the missing card is red]

Again $P(A/A_2)$ = Probability of drawing a red card when the missing card is black $= \frac{26}{51}$

Now from (i), required probability,

$$P(A_1/A) = \frac{\frac{1}{2} \cdot \frac{25}{51}}{\frac{1}{2} \cdot \frac{25}{51} + \frac{1}{2} \cdot \frac{26}{51}} = \frac{25}{51}$$

Ex.47 A bag contains 6 white and an unknown number of black balls (≤ 3). Balls are drawn one by one with replacement from this bag twice and is found to be white on both occassion. Find the probability that the bag had exactly '3' Black balls.

Sol. Apriori, we can think of the following possibilies

(i) E_1	6W	,	0 B
(ii) E_2	6W	,	1 B
(iii) E_3	6W	,	2 B
(iv) E_4	6W	,	3 B

Clearly $P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}$

Let 'A' be the event that two balls drawn one by one with replacement are both white therefore we have to find $P\left(\frac{E_4}{A}\right)$

By Baye's theorem $P\left(\frac{E_4}{A}\right)$

$$= \frac{P\left(\frac{A}{E_4}\right) \times P(E_4)}{P\left(\frac{A}{E_1}\right) \times P(E_1) + P\left(\frac{A}{E_2}\right) . P(E_2) + P\left(\frac{A}{E_3}\right) . P(E_3) + P\left(\frac{A}{E_4}\right) . P(E_4)}$$

$P\left(\frac{A}{E_4}\right) = \frac{6}{9} \times \frac{6}{9}$; $\quad P\left(\frac{A}{E_3}\right) = \frac{6}{8} \times \frac{6}{8}$;

$P\left(\frac{A}{E_2}\right) = \frac{6}{7} \times \frac{6}{7}$; $\quad P\left(\frac{A}{E_1}\right) = \frac{6}{6} \times \frac{6}{6}$;

Putting values $P\left(\frac{E_4}{A}\right) = \frac{\frac{1}{81}}{\frac{1}{81} + \frac{1}{64} + \frac{1}{49} + \frac{1}{36}}$

Binomial Distribution for repeated trials

Binomial Experiment : Any experiment which has only two outcomes is known as binomial experiment.

Outcomes of such an experiment are known as success and failure.

Probability of success is denoted by p and probability of failure by q.

$\therefore$ $p + q = 1$

If binomial experiment is repeated n times, then

$(p + q)^n = {}^nC_0\, q^n + {}^nC_1\, p\, q^{n-1} + {}^nC_2\, p^2\, q^{n-2} + \ldots.. + {}^nC_r\, p^r\, q^{n-r} + \ldots\ldots.. + {}^nC_n\, p^n = 1$

Probability of exactly r successes in n trials

$= {}^nC_r p^r q^{n-r}$

Probability of at most r successes in n trails

$$= \sum_{\lambda=0}^{r} {}^nC_\lambda p^\lambda q^{n-\lambda}$$

Probability of atleast r successes in n trails

$$= \sum_{\lambda=r}^{n} {}^nC_\lambda p^\lambda q^{n-\lambda}$$

Probability of having Ist success at the r^{th} trials $= p\, q^{r-1}$.

The mean the variance and the standard deviation of binomial distribution are np, npq, $\sqrt{npq}$.

Solved Examples

Ex.48 Two dice are tossed four times find the probability of getting

(i) equal digits exactly two times

(ii) equal digits at least two times

(iii) equal digits at the most two times

Sol. Let A be the event of getting equal digits on the dice. Since number of exhaustive cases is 36 and favourable cases is 6.

$$\therefore P(A) = p = \frac{6}{36} = \frac{1}{6},\ P(\bar{A}) = q = 1 - \frac{1}{6} = \frac{5}{6}$$

Hence by Binomial theorem, we have

$$\left(\frac{5}{6}+\frac{1}{6}\right)^4 = \left(\frac{5}{6}\right)^4 + {}^4C_1\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + {}^4C_2\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 + {}^4C_3\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^3 + \left(\frac{1}{6}\right)^4$$

Thus from above result, we have

(i) Probability of getting equal digits exactly two times

$$= {}^4C_2\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 = \frac{25}{216}$$

(ii) Probability of getting equal digits at least two times

$$= {}^4C_2\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2 + {}^4C_3\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^3 + \left(\frac{1}{6}\right)^4$$

$$= \frac{25}{216} + \frac{20}{1296} + \frac{1}{1296} = \frac{171}{1296}$$

(iii) Probability of getting equal digits at the most two times

$$= \left(\frac{5}{6}\right)^4 + {}^4C_1\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right) + {}^4C_2\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2$$

$$= \frac{625}{1296} + \frac{500}{1296} + \frac{150}{1296} = \frac{1275}{1296}$$

Solved Examples

Ex.49 A pair of dice is thrown 5 times. Find the probability of getting a doublet twice.

Sol. In a single throw of a pair of dice probability of getting a doublet is $\frac{1}{6}$

con sidering it to be a success, $p = \frac{1}{6}$

$\therefore\ q = 1 - \frac{1}{6} = \frac{5}{6}$

number of success r = 2

$$\therefore\ P(r = 2) = {}^5C_2\, p^2\, q^3 = 10 \cdot \left(\frac{1}{6}\right)^2 \cdot \left(\frac{5}{6}\right)^3 = \frac{625}{3888}$$

Ex.50 A pair of dice is thrown 4 times. If getting 'a total of 9' in a single throw is considered as a success then find the probability of getting 'a total of 9' thrice.

Sol. p = probability of getting 'a total of 9' $= \frac{4}{36} = \frac{1}{9}$

$\therefore\ q = 1 - \frac{1}{9} = \frac{8}{9}$

r = 3, n = 4

$$\therefore\ P(r = 3) = {}^4C_3\, p^3\, q = 4 \times \left(\frac{1}{9}\right)^3 \cdot \frac{8}{9} = \frac{32}{6561}$$

Ex.51 In an examination of 10 multiple choice questions (1 or more can be correct out of 4 options). A student decides to mark the answers at random. Find the probability that he gets exactly two questions correct.

Sol. A student can mark 15 different answers to a MCQ with 4 option i.e. ${}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15$

Hence if he marks the answer at random, chance that his answer is correct $= \frac{1}{15}$ and being incorrecting $\frac{14}{15}$. Thus $p = \frac{1}{15}$ $q = \frac{14}{15}$.

P (2 success) $= {}^{10}C_2 \times \left(\frac{1}{15}\right)^2 \times \left(\frac{14}{15}\right)^8$

Ex.52 A family has three children. Event 'A' is that family has at most one boy, Event 'B' is that family has at least one boy and one girl, Event 'C' is that the family has at most one girl. Find whether events 'A' and 'B' are independent. Also find whether A, B, C are independent or not.

Sol. A family of three children can have

(i) All 3 boys (ii) 2 boys + 1 girl

(iii) 1 boy + 2 girls (iv) 3 girls

(i) P (3 boys) $= {}^3C_0 \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ (Since each child is equally likely to be a boy or a girl)

(ii) P (2 boys +1girl) $= {}^3C_1 \times \left(\frac{1}{2}\right)^2 \times \frac{1}{2} = \frac{3}{8}$

(Note that there are three cases BBG, BGB, GBB)

(iii) P (1 boy + 2 girls) $= {}^3C_2 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^2 = \frac{3}{8}$

(iv) P (3 girls) $= \frac{1}{8}$

Event 'A' is associated with (iii) & (iv). Hence P(A) $= \frac{1}{2}$

Event 'B' is associated with (ii) & (iii). Hence P(B) $= \frac{3}{4}$

Event 'C' is associated with (i) & (ii). Hence P(C) $= \frac{1}{2}$

$P(A \cap B) = P(iii) = \frac{3}{8} = P(A) \,.\, P(B)$. Hence A and B are independent of each other

$P(A \cap C) = 0 \neq P(A) \,.\, P(C)$. Hence A, B, C are not independent

Ex.53 There are 100 tickets in a raffle (Lottery). There is 1 prize each of Rs. 1000/-, Rs. 500/- and Rs. 200/-. Remaining tickets are blank. Find the expected price of one such ticket.

Sol. Expectation $= \Sigma p_i M_i$ outcome of a ticket can be

	p_i	M_i	$p_i M_i$
(i) I prize	$\frac{1}{100}$	1000	10
(ii) II prize	$\frac{1}{100}$	500	5
(iii) III prize	$\frac{1}{100}$	200	2
(iv) Blank	$\frac{97}{100}$	0	0
			$\Sigma p_i M_i = 17$

Hence expected price of one such ticket Rs. 17

Ex.54 A purse contains four coins each of which is either a rupee or two rupees coin. Find the expected value of a coin in that purse.

Sol. Various possibilities of coins in the purse can be

	p_i	M_i	$p_i M_i$
(i) 4 1 rupee coins	$\frac{1}{16}$	4	$\frac{4}{16}$
(ii) 3 one Rs. + 1 two Rs.	$\frac{4}{16}$	5	$\frac{20}{16}$
(iii) 2 one Rs. + 2 two Rs.	$\frac{6}{16}$	6	$\frac{36}{16}$
(iv) 1 one Rs. + 3 two Rs.	$\frac{4}{16}$	7	$\frac{28}{16}$
(iv) 4 two Rs.	$\frac{1}{16}$	8	$\frac{8}{16}$
			6 / -

Note that since each coin is equally likely to be one Rs. or two Rs. coin, the probability is determined using Binomial probability; unlike the case when the purse contained the coins with all possibility being equally likely, where we take $p_i = \frac{1}{5}$ for each. Hence expected value is Rs. 6/-

Ex.55 A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

Determine

(i) k (ii) $P(X < 3)$

(iii) $P(X > 6)$ (iv) $P(0 < X < 3)$

[**Hint :** Use $\Sigma P(X) = 1$ to determine k,

$P(X < 3) = P(0) + P(1) + P(2)$,

$P(X > 6) = P(7)$ etc.]

Ex.56 A pair of dice is thrown 5 times. If getting a doublet is considered as a success, then find the mean and variance of successes.

Sol. In a single throw of a pair of dice, probability of getting a doublet $= \frac{1}{6}$

con sidering it to be a success, $p = \frac{1}{6}$

$\therefore\ q = 1 - \frac{1}{6} = \frac{5}{6}$

mean $= 5 \times \frac{1}{6} = \frac{5}{6}$

variance $= 5 \times \frac{1}{6} \cdot \frac{5}{6} = \frac{25}{36}$

Ex.57 A pair of dice is thrown 4 times. If getting a total of 9 in a single throw is considered as a success then find the mean and variance of successes.

Sol. p = probability of getting a total of 9 $= \frac{4}{36} = \frac{1}{9}$

$\therefore\ q = 1 - \frac{1}{9} = \frac{8}{9}$

$\therefore$ mean $= np = 4 \times \frac{1}{9} = \frac{4}{9}$

variance $= npq = 4 \times \frac{1}{9} \times \frac{8}{9} = \frac{32}{81}$

Ex.58 Difference between mean and variance of a Binomial variate is '1' and difference between their squares is '11'. Find the probability of getting exactly three success

Sol. Mean = np & variance = npq

therefore, $np - npq = 1$ ....(i)

$n^2p^2 - n^2p^2q^2 = 11$ ....(ii)

Also, we know that $p + q = 1$ ....(iii)

Divide equation (ii) by square of (i) and solve, we get, $q = \frac{5}{6}$, $p = \frac{1}{6}$ & $n = 36$

Hence probability of '3' success

$= {}^{36}C_3 \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^{33}$ Ans.

GEOMETRICAL APPLICATIONS

The following statements are axiomatic :

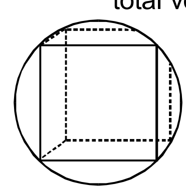
(i) If a point is taken at random on a given straight line segment AB, the chance that it falls on a particular segment PQ of the line segment is PQ/AB.

(ii) If a point is taken at random on the area S which includes an area σ, the chance that the point falls on σ is σ/S.

Solved Examples

Ex.59 A sphere is circumscribed over a cube. Find the probability that a point lies inside the sphere, lies outside the cube.

Sol. Required probability $= \frac{\text{favorable volume}}{\text{total volume}}$



Clearly if edge length of cube is a radius of sphere will be $\frac{a\sqrt{3}}{2}$

Thus, volume of sphere $= \frac{4}{3}\pi\left(\frac{a\sqrt{3}}{2}\right)^3 = \frac{\pi a^3\sqrt{3}}{2}$

Hence $P = 1 - \frac{1}{\pi\frac{\sqrt{3}}{2}} = 1 - \frac{2}{\pi\sqrt{3}}$

Ex.60 A given line segment is divided at random into three parts. What is the probability that they form sides of a possible triangle ?

Sol. Let AB be the line segment of length ℓ.

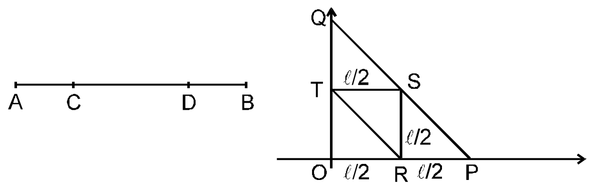
Let C and D be the points which divide AB into three parts.

Let AC = x, CD = y. Then DB = $\ell - x - y$.

Clearly $x + y < \ell$

$\therefore$ the sample space is given by

the region enclosed by Δ OPQ, where OP = OQ = ℓ



Area of ΔOPQ = $\dfrac{\ell^2}{2}$

Now if the parts AC, CD and DB form a triangle, then

$x + y > \ell - x - y$ i.e. $x + y > \dfrac{\ell}{2}$ ....(i)

$x + \ell - x - y > y$ i.e. $y < \dfrac{\ell}{2}$ ....(ii)

$y + \ell - x - y > x$ i.e. $x < \dfrac{\ell}{2}$ ....(iii)

from (i), (ii) and (iii), we get

the event is given by the region closed in ΔRST

$\therefore$ Probability of the event = $\dfrac{\text{ar }(\Delta RST)}{\text{ar }(\Delta OPQ)}$

$$= \frac{\frac{1}{2}\cdot\frac{\ell}{2}\cdot\frac{\ell}{2}}{\frac{\ell^2}{2}} = \frac{1}{4}$$

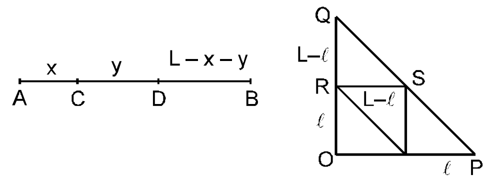
Ex.61 On a line segment of length L two points are taken at random, find the probability that the distance between them is ℓ, where $\ell < 1$

Sol. Let AB be the line segment

Let C and D be any two points on AB so that AC = x and CD = y. Then $x + y < L$, $y > \ell$

$\therefore$ sample space is represented by the region enclosed by ΔOPQ.

Area of ΔOPQ = $\dfrac{1}{2}L^2$



The event is represented by the region, bounded by the ΔRSQ

Area of ΔRSQ = $\dfrac{1}{2}(L - \ell)^2$

$\therefore$ probability of the event = $\left(\dfrac{L-\ell}{L}\right)^2$

SOME IMPORTANT RESULTS

(i) Let A and B be two events, then

* $P(A) + P(\bar{A}) = 1$
* $P(A + B) = 1 - (\bar{A}\,\bar{B})$
* $P(A/B) = \dfrac{P(AB)}{P(B)}$
* $P(A + B) = P(AB) + P(\bar{A}\ B) + P(A\ \bar{B})$
* $A \subset B \Rightarrow P(A) \le P(B)$
* $P(\bar{A}\ B) = P(B) - P(AB)$
* $P(AB) \le P(A)\,P(B) \le P(A + B) \le P(A) + P(B)$
* $P(AB) = P(A) + P(B) - P(A + B)$
* P(Exactly one event) = $P(A\ \bar{B}) + P(\bar{A}\ B)$

  $= P(A) + P(B) - 2P(AB)$

  $= P(A + B) - P(AB)$
* P(neither A nor B) = $P(\bar{A}\ \bar{B}) = 1 - P(A + B)$
* $P(\bar{A} + \bar{B}) = 1 - P(AB)$

(ii) Number of exhaustive cases of tossing n coins simultaneously (or of tossing a coin n times) = 2^n

(iii) Number of exhaustive cases of throwing n dice simultaneously (or throwing one dice n times) = 6^n

(iv) Playing Cards :

* Total Cards : 52(26 red, 26 black)
* Four suits : Heart, Diamond, Spade, Club - 13 cards each
* Court Cards : 12 (4 Kings, 4 queens, 4 jacks)
* Honour Cards : 16 (4 aces, 4 kings, queens, 4 jacks)

(v) Probability regarding n letters and their envelopes :

If n letters corresponding to n envelopes are placed in the envelopes at random, then

* Probability that all letters are in right envelopes $= \frac{1}{n!}$
* Probability that all letters are not in right envelopes $= 1 - \frac{1}{n!}$
* Probability that no letters is in right envelopes $= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - ..... + (-1)^n \frac{1}{n!}$
* Probability that exactly r letters are in right envelopes $= \frac{1}{r!}\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - ..... + (-1)^{n-r} \frac{1}{(n-r)!}\right]$

Solved Examples

Ex.62 There are four letters and four envelopes, the letters are placed into the envelopes at random, find the probability that all letters are placed in the wrong envelope.

Sol. We know from the above given formula that probability that no letter is in right envelope out of n letters and n envelopes is given by

$$= \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} ..... + (-1)^n \frac{1}{n!}\right]$$

Since all 4 letters are to be placed in wrong envelopes then required probability

$$= \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{3}{8}$$