Linear Programming

INTRODUCTION

In any firm, the manufacturer is always interested in using his resources so that the cost of production is minimised and the profit is maximised. The methods of linear programming help us in making the best possible use of limited resources to meet the desired results. The programming is termed linear, because we shall be considering only linear inequations in the variables of the problem under consideration. The resources may be in the form of men, material, machines etc.

Linear programming is defined as that branch of mathematics which deals with the optimisation (maximisation or minimisation) of a linear function of a number of variables subject to a number of conditions on the variables, in the form of linear inequations in the variables of the optimisation funnciton.

DEFINITIONS

(i) Objective fucntion : A function of certain variables, which is to be maximised or minimised, subject to given conditions on the variables of the function, is called the objective function of the problem, under consideration. In economic applications, if an objective function is a profit function then it is to be maximised and if an objective function is a cost function, then it is to be minimised.

(ii) Constraints : The conditions on the variables of an objective function are called the contraints of the problem, under consideration. The constraints $x \geq 0$, $y \geq 0$, are called non-negativity constraints.

(iii) Feasible region : The region which is common to all constraints of a linear programming problem (L.P.P.) is callled the feasible region of the given L.P.P.

(iv) Feasible solution : Every point in the feasible region of linear programming problem (L.P.P.) is called a feasible solution of the given L.P.P.

(v) Optimal feasible solution : A point in the feasible region of a linear programming problem is called an optimal feasible solution if the objective function of the given L.P.P. is maximised or minimised at that point.

GRAPHICAL METHOD OF SOLVING LINEAR PROGRAMMING PROBLEM

We shall use graphical method for solving linear programming problems. This method is also known as corner point method.

Let x and y be two non-negative variables and $Z = ax + by$ be the function of x and y which is to be optimised, subject to finitely many constraints.

$a_1x + b_1y \leq (\text{or} \geq)\ c_1$, $a_2x + by_2 \leq (\text{or} \geq)\ c_2$,.............and $x \geq 0$, $y \geq 0$.

In this method the half-plane of each linear constraint in the given L.P. problem is drawn. The intersection of these half-planes gives the set of all feasible solutions of the L.P. problem.

STEPS FOR SOLVING LINEAR PROGRAMMING PROBLEM

1. Find the feasible region of the linear programming problem and determine its corner points (vertices) either by inspection or by solving the two equations of the lines intersecting at that point.

2. Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m, respectively denote the largest and smallest values of these points.

3. (i) When the feasible region is bounded, M and m are the maximum and minimum values of Z.

(ii) In case, the feasible region is unbounded, we have :

(a) M is the maximum value of Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.

(b) Similarly, m is the minimum value of Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

Solved Examples

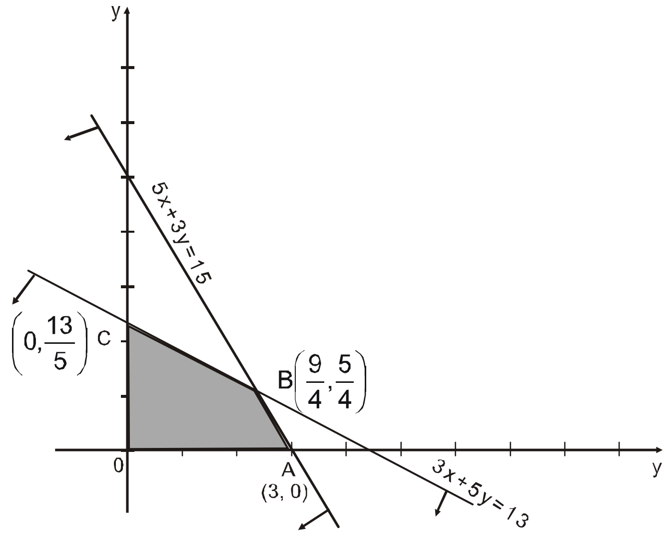
Ex.1 Solve the problem

Maximise	$z = 6x + 10y$
Subject to	$3x + 5y \le 13$
	$5x + 3y \le 15$
and	$x, y \ge 0$

Ans. Max. = 26

Sol.

At A (3, 0) ;	$z = 6 \times 3 + 10(0) = 18$
At B (9/4, 5/4) ;	$z = 6 \times \frac{9}{4} + 10\frac{5}{4} = 26$
At C (0, 13/ 5) ;	$z = 6(0) + 10\left(\frac{13}{5}\right) = 26$



Here we find that z is maximum at B $\left(\frac{9}{4},\frac{5}{4}\right)$ as well as at C $\left(0,\frac{13}{5}\right)$ and its maximum value is 26.

Every point on the line segment BC gives this maximum value because in this case the line of objective function lies along one boundary line of the feasible region. In such a case the problem has infinite number of optimal solutions.

Ex.2 Minimise $Z = 3x_1 + 5x_2$ subject to the constraints: $x_1 + 3x_2 \ge 3$, $x_1 + x_2 \ge 2$, $x_1, x_2 \ge 0$.

Ans. 7

Sol. The objective function is $Z = 3x_1 + 5x_2$.

The constraints are :

$x_1 + 3x_2 \ge 3$ ...(1)

$x_1 + x_2 \ge 2$ ...(2)

$x_1 \ge 0$ ...(3)

$x_2 \ge 0$ ....(4)

The line corresponding to (1) is $x_1 + 3x_2 = 3$

(0, 1) and (3, 0) lie on (5). The origin does not lie on (5) and it lies in the half-plane of (1) if $0 + 3(0) \ge 3$, which is not true.

$\therefore$ The closed half-plane not containing the origin is the graph of (1).

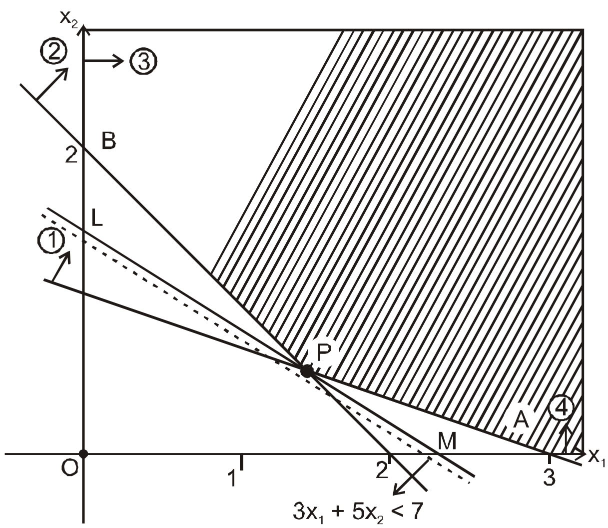
The line corresponding to (2) is $x_1 + x_2 = 2$

(0, 1) and (2, 0) lie on (6). The origin does not lie on (6) and it lies in the half-plane of (2) if $0 + 0 \ge 2$, which is not true.

∴ The closed half-plane not containing the origin is the graph of (2).

$x_2 \geq 0$, $x_2 \geq 0$ respresent the closed half-planes on the right of x_2-axis and above x_1-axis respectively.

The shaded unbounded region is the feasible region of the given L.P.P. The vertices of the unbounded feasible region are A(3, 0), $P\left(\frac{3}{2},\frac{1}{2}\right)$ and B(0, 2).



At A(3, 0), Z = 3(3) + 5(0) = 9

At $P\left(\frac{3}{2},\frac{1}{2}\right)$, $Z = 3\left(\frac{3}{2}\right) + 5\left(\frac{1}{2}\right) = 7$

At B(0, 2), Z = 3(0) + 5(2) = 10

The minimum of 9, 7, 10 is 7. We draw the graph of the inequality $3x_1 + 5x_2 < 7$. The corresponding equation is $3x_1 + 5x_2 = 7$ and this passes through $L\left(0,\frac{7}{5}\right)$ and $M\left(\frac{7}{3},0\right)$. The open half-plane of $3x_1 + 5x_2 < 7$ is shown in the figure. This half-plane has no point in common with the feasible region.

∴ Minimum value of Z = 7 and occurs when $x_1 = \frac{3}{2}$ and $x_2 = \frac{1}{2}$.

DIFFERENT TYPES OF LINEAR PROGRAMMING PROBLEMS

A few linear programming problems are listed below:

1. **Manufacturing problems :**

   In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hour per unit product, warehouse space per unit of the output etc. in order to make maximum profit.

Solved Examples

Ex.3 A producer has 30 and 17 units of labour and capital respectively which he can use to produce two types of goods X and Y. To produce one unit of X, 2 units of labour and 3 units of capital are required. similarly, 3 units of labour and 1 unit of capital is required to produce one unit of Y. If X and Y are priced at Rs. 100 and Rs. 120 per unit respectively, how should the producer use his resources to maximise the total revenue ? Solve the problem, graphically.

Sol. Let x be the no. of units of X and y be the number of units of Y produced. We can arrange the information in the problem as :

	X	Y	Available Units
Labour	2	3	30
Capital	3	1	17
Revenue per unit	Rs. 100	Rs. 120	

∴ We can formulate this problem as

Maximize : z = 100 x + 120 y

subject to the constraints

$2x + 3y \leq 30$

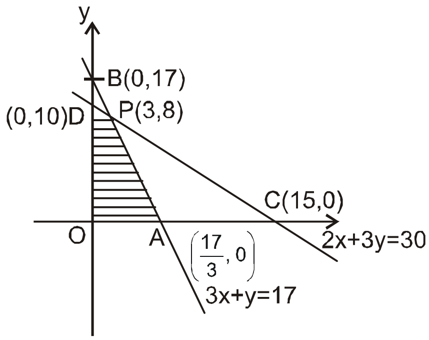
$3x + y \leq 17$

$x \geq 0$; $y > 0$

We first plot the lines

$2x + 3y = 30$

$3x + y = 17$



These lines meet at P(3, 8). Feasible region has been shaded. It extreme points are O. A, P and D. we now calculate the value of the objective function:

$z = 100x + 120y$

at these points.

Extreme Point	Value of Z
O(0, 0)	0
$A\left(\frac{17}{3}, 0\right)$	$\frac{1700}{3}$
P(3, 8)	1260
D(0, 10)	1200

$\therefore$ Z has maximum value at P(3, 8) and this is equal to Rs. 1260.

Solved Examples

Ex.4 A manufacturer produces two types of steel trunks. He has two machines A and B. The first type of trunk requires 3 hours on machine A and 3 hours on machine B. The second type requires 3 hours on machine A and 2 hours on machine B. Machines A and B can work at most for 18 hours and 15 hours per day respectively. He earns a profit of Rs. 30 and Rs. 25 per trunk of the first type and second type respectively. How many trunks of each type must he make each day to make maximum profit ?

Sol. Let x and y be the number of two types of steel trunks to be manufactured

Machine	Time(in hr.) Trunk – I (x)	Time(in hr.) Trunk – II (y)	Available Time (in hr.)
A	3	3	18
B	3	2	15
Profit per unit	Rs.30	Rs.25	

Then Maximize : $z = 30x + 25y$

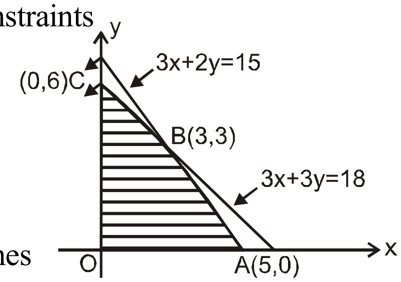
subject to the constraints

$3x + 3y \le 18$

$3x + 2y \le 15$

$x \ge 0, y \ge 0$

Now draw the lines



$3x + 3y = 18$, $3x + 2y = 15$

Their point of intersection is B(3, 3)

The feasible region, OABC is shaded in the adjoining diagram.

Now value of $z = 30x + 25y$

at A(5, 0) is $30 \times 5 + 25 \times 0 = 150$

at B(3, 3) is $30 \times 3 + 25 \times 3 = 165$

at C(0, 6) is $30 \times 0 + 25 \times 6 = 150$

$\therefore$ For getting maximum profit of Rs. 165,

3 trunks of each type should be manufactured.

2. Diet Problems :

In these problems, we determine the amount of different kinds of constituents/nutrients which should be included in a diet so as to minimise the cost of the desired diet such that it contains a certain minimum amount of each constituent/nutrients.

Solved Examples

Ex.5 A housewife wishes to mix up two kinds of foods X and Y in such a way that mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of 1 kg of food X and 1 kg of food Y are as given in the following table :

Food	Vitamin A	Vitamin B	Vitamin C
X	1	2	3
Y	2	2	1

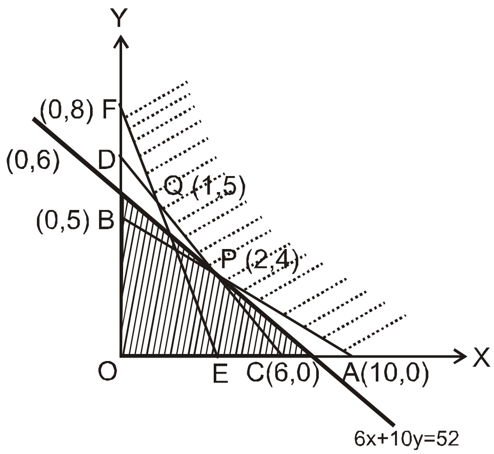
If one kg of food X costs Rs. 6 and one kg of food Y costs Rs. 10, find the least cost of the mixture which will produce the desired diet.

Sol Let x kg of food X and y kg of Food Y be mixed to get the desired diet. Then the L.P.P. is

Minimise : $z = 6x + 10y$

Subject to the constraints

(Vitamin A) $x + 2y \geq 10$



(Vitamin B) $2x + 2y \geq 12$

(Vitamin C) $3x + y \geq 8$

$x \geq 0$, $y \geq 0$

Now draw the lines

AB : $x + 2y = 10$

CD : $2x + 2y = 12$

EF : $3x + y = 8$

AB and CD meet at P(2, 4)

CD and EF meet at Q(1, 5).

The shaded region APQF, is the feasible region.

The value of $z = 6x + 10y$

at A is $z = 6 \times 10 + 10 \times 0 = 60$

at P is $z = 6 \times 2 + 10 \times 4 = 52$

at Q is $z = 6 \times 1 + 10 \times 5 = 56$

at F is $z = 6 \times 0 + 10 \times 8 = 80$

The smallest value of Z is 52 at the point P(2, 4), but as the feasible region is unbounded therefore we have to draw the graph of inequality $6x + 10y < 52$

Now from the figure it is clear that $6x + 10y < 52$ has no point common with the feasible region.

Thus the minimum value of Z is 52 i.e., when 2 kg of Food X is mixed with 4 kg of Food Y.

3. Transportation problems :

In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants/factories situated at different locations to different markets.

Solved Examples

Ex.6 (Transportation Problem)

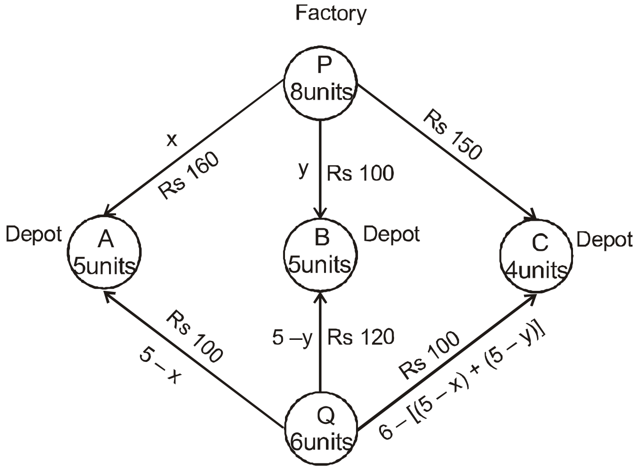
There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situted at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the facories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below :

FROM / TO	COST (in Rs)		
	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

Sol. The problem can be explained diagrammatically as follows

Let x units and y units of the commodity be transported from the factory at P to the depots at A and B respectively. Then $(8 - x - y)$ units will be transported to depot at C(why ?)



Hence we have $x \geq 0$, $y \geq 0$ and $8 - x - y \geq 0$

i.e. $x \geq 0$, $y \geq 0$ and $x + y \leq 8$

Now, the weekly requirement of the depot at A is 5 units of the commodity. Since x units are transported from the factory at P, the remaining $(5 - x)$ units need to be transported from the factory at Q. obviously, $5 - x \geq 0$, i.e. $x \leq 5$.

Similarly $(5 - y)$ and $6 - (5 - x + 5 - y) = x + y - 4$ units are to be transported from the factory at Q to the depots at B and C respectively.

THus, $5 - y \geq 0$, $x + y - 4 \geq 0$ i.e. $y \leq 5$, $x + y \geq 4$

Therefore, the problem reduces to Minimise $Z = 10(x - 7y + 190)$

subject to constraints :

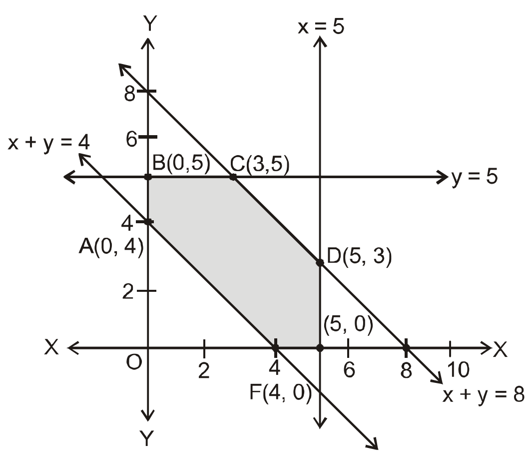
$x \geq 0, y \geq 0$ ...(i)

$x + y \leq 8$ ...(ii)

$x \leq 5$ ...(iii)

$y \leq 5$ ...(iv) and

$x + y \geq 4$ ...(v)



The shaded region ABCDEF represented by the constraints (i) to (v) is the feasible region. Observe that the feasible region is bounded. The coordinates of the corner points of the feasible region are (0, 4), (0, 5), (3, 5), (5, 3), (5, 0) and (4, 0).

Let us evaluate Z at these points

Corner Point	$Z = 10(x - 7y + 190)$
(0, 4)	1620
(0, 5)	1550
(3, 5)	1580
(5, 3)	1740
(5, 0)	1950
(5, 0)	1940

From the table, we see that the minimum value of Z is 1550 at the point (0, 5).

Hence, the optimal transportation strategy will be to deliver 0, 5 and 3 units from the factory at P and 5, 0 and 1 units from the factory at Q to the depots at A, B and C respectively. Corresponding to this strategy, the transportation cost would be minimum, i.e. Rs 1550.

4. Allocation Problem :

Solved Examples

Ex.7 A cooperative society of farmers has 50 hectare of land to grow two crops X and Y. The profit from cropsX and Y per hectare are estimated as Rs 10,500 and Rs 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 liters and 10 liters per hectare. Further, no more than 800 liters of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so as to maximise the total profit of the society ?

Sol. Let x hectare of land be allocated to crop X and y hectare to crop Y. Obviously $x \geq 0$, $y \geq 0$.

Profit per hectare on crop X = Rs 10500

Profit per hectare on crop Y = Rs 9000

Therefore, total profit = Rs (10500x + 9000y)

The mathematical formulation of the problem is as follows :

Maximise $Z = 10500x + 9000y$

subject to the constraints :

$x + y \leq 50$ (constraint related to land) ....(1)

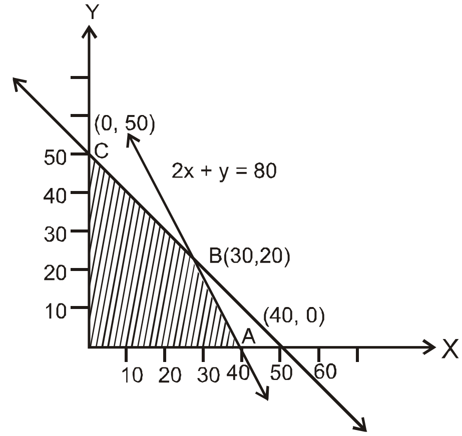
$20x + 10y \leq 800$ (constraint related to use of herbicide)

i.e. $2x + y \leq 80$ ....(2)

$x \geq 0, y \geq 0$ (non negative constraint) .....(3)

Let us draw the graph of the system of inequalities (1) to (3). The feasible region OABC is shown (shaded) in the figure. Observe that the feasible region is bounded.

The coordinates of the corner points O, A, B and C are (0, 0), (40, 0), (30, 20) and (0, 50) respectively. Let us evaluate the objective function Z = 10500x + 9000y at these vertices to find which one gives the maximum profit.



Corner Point	$Z = 10500x + 9000y$
O(0,0)	0
A(40,0)	420000
B(30,20)	495000
C(0,50)	450000

Hence, the society will get the maximum profit of Rs 4,95,000 by allocating 30 hectares for crop X and 20 hectares for crop Y.