Vectors & 3D
(Three Dimensional Geometry)

SCALAR QUANTITY & VECTOR QUANTITY

Scalar Quantity :

A quantity which has only magnitude and not related to any direction is called a scalar quantity.

For example, Mass, Length, Time, Temperature, Area, Volume, Speed, Density, Work etc.

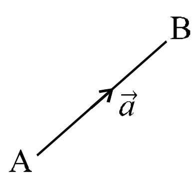
Vector Quantity :

A quantity which has magnitude and also a direction in space is called a vector quantity.

For example, Displacement, Velocity, Acceleration, Force, Torque, etc.

REPRESENTATION OF VECTORS

Vectors are represented by directed line segments. A vector $\vec{a}$ is represented by the directed line segment $\overrightarrow{AB}$. The magnitude of vector $\vec{a}$ is equal is equal to AB and the direction of vector $\vec{a}$ is along the line form A to B.



TYPE OF VECTORS

Zero vector:

A vector of Zero magnitude i.e. which has the same initial and terminal point, is called a **Zero vector.** It is denoted by **O**. **The direction of Zero vector is indeterminate.**

Unit vector:

A vector of unit magnitude in the direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$, symbolically $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

Note :

(a) $|\hat{a}| = 1$.

(b) Two unit vectors may not be equal unless they have the same direction.

(c) Unit vectors parallel to x-axis, y-axis and Z-axis are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively.

Solved Examples

Ex.1 Find unit vector of $\hat{i} - 2\hat{j} + 3\hat{k}$

Sol. $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

if $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

then $|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

$\therefore$ $|\vec{a}| = \sqrt{14}$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{14}}\hat{i} - \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$$

Reciprocal vector :

A vector whose direction is same as that of a given vector $\vec{a}$ but its magnitude is the reciprocal of the magnitude of the given vector $\vec{a}$ is called the reciprocal of $\vec{a}$ and is denoted by a^{-1}.

Thus if $\vec{a} = a\,.\,\hat{a}$ then $a^{-1} = \frac{1}{a}\hat{a}$

Equal vector :

Two non-Zero vectors are said to be equal vectors if their magnitude are equal and directions are same i.e. they act parallel to each other in the same direction.

Negative vector :

The negative of a vector is defined as the vector having the same magnitude but opposite direction.

For example, if $\vec{a} = \overrightarrow{PQ}$, then the negative of $\vec{a}$ is the vector $\overrightarrow{QP}$ and is denoted as $-\vec{a}$.

Collinear vectors:

Two vectors are said to be collinear if their directed line segments are parallel irrespective of their directions. Collinear vectors are also called **parallel vectors**. If they have the same direction they are named as **like vectors** otherwise **unlike vectors**.

Symbolically, two non-Zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = \lambda\vec{b}$, where $\lambda \in R$

$\vec{a} = \lambda\vec{b} \Leftrightarrow \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) = \lambda\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$

$\Leftrightarrow a_1 = \lambda b_1,\ a_2 = \lambda b_2,\ a_3 = \lambda b_3$

$\Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}\ (=\lambda)$

Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Solved Examples

Ex.2 Find values of x & y for which the vectors

$\vec{a} = (x+2)\hat{i} - (x-y)\hat{j} + \hat{k}$

$\vec{b} = (x-1)\hat{i} + (2x+y)\hat{j} + 2\hat{k}$ are parallel.

Sol. $\vec{a}$ and $\vec{b}$ are parallel if $\frac{x+2}{x-1} = \frac{y-x}{2x+y} = \frac{1}{2}$

$x = -5,\ y = -20$

Coplanar vector :

Two or more non-Zero vectors are said to be coplanar vectors if these are parallel to the same plane.

Localised vector and free vector :

A vector drawn parallel to a given vector through a specified point as the initial point, is known as a localised vector. If the initial point of a vector is not specified it is said to be a free vector.

Position vector : Let O be the origin and let A be a point such that $\overrightarrow{OA} = \vec{a}$ then, we say that the position vector of A is $\vec{a}$.

MULTIPLICATION OF A VECTOR BY A SCALAR

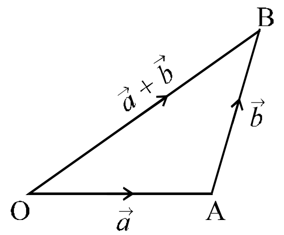
If $\vec{a}$ is a vector and m is a scalar, then $m\vec{a}$ is a vector parallel to $\vec{a}$ whose magnitude is $|m|$ times that of $\vec{a}$. This multiplication is called **scalar multiplication**. If $\vec{a}$ and $\vec{b}$ are vectors and m, n are scalars, then :

$m(\vec{a}) = (\vec{a})m = m\vec{a}$, $\quad m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

$(m+n)\vec{a} = m\vec{a} + n\vec{a}$, $\quad m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

ADDITION OF VECTORS

* Let $\vec{a}$ and $\vec{b}$ be any two vectors. From the terminal point of $\vec{a}$, vector $\vec{b}$ is drawn. Then, the vector from the initial point O of $\vec{a}$ to the terminal point B of $\vec{b}$ is called the sum of vectors $\vec{a}$ and $\vec{b}$ and is denoted by $\vec{a} + \vec{b}$. This is called the triangle law of addition of vectors.



* The vectors are also added by using the following method. Let $\vec{a}$ and $\vec{b}$ be any two vectors. From the initial point of $\vec{a}$, vector $\vec{b}$ is drawn. let O be their common initial point. If A and B be respectively the terminal points of $\vec{a}$ and $\vec{b}$, then parallelogram OACB is completed with OA and OB as adjacent sides. The vector $\overrightarrow{OC}$ is defined as the sum of $\vec{a}$ and $\vec{b}$. This is called the parallelogram law of addition of vectors.

(a) Properties of Vector Addition :

(i) Vector addition is commutative, *i.e.* $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.

(ii) Vector addition is associative, *i.e.* $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$

(iii) $\vec{O}+\vec{a}=\vec{a}+\vec{O}=\vec{a}$ .So, the Zero vector is additive identity.

(iv) $\vec{a}+(-\vec{a})=\vec{O}=(-\vec{a})+\vec{a}$. So, the additive inverse of $\vec{a}$ is $-\vec{a}$.

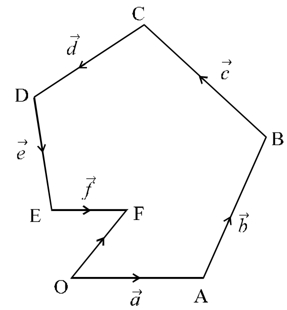
(b) Addition of any Number of Vectors :

* To find the sum of any number of vectors we represent the vectors by directed line segment with the terminal point of the previous vector as the initial point of the next vector. Then the line segment joining the initial point of the first vector to the terminal point of the last vector will represent the sum of the vectors:

Thus if,

$\overrightarrow{OA}=\vec{a}, \overrightarrow{AB}=\vec{b}, \overrightarrow{BC}=\vec{c}, \overrightarrow{CD}=\vec{d}, \overrightarrow{DE}=\vec{e}$ and $\overrightarrow{EF}=\vec{f}$ then $\vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}+\vec{f}$

$= \overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EF}=\overrightarrow{OF}$



* If the terminal point F of the last vector coincide with initial point of the first vector then

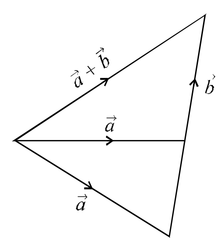
$\vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}+\vec{f}$

$= \overrightarrow{OA}+\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{DE}+\overrightarrow{EO}=\vec{O}$,

i.e. the sum of vectors is Zero or null vector in this case.

DIFFERENCE OF VECTORS

If $\vec{a}$ and $\vec{b}$ be any two vectors, then their difference $\vec{a}-\vec{b}$ is defined as $\vec{a}+(-\vec{b})$.



Solved Examples

Ex.3 If $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.

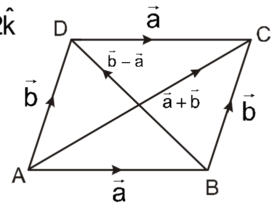
Sol. Let ABCD be a parallelogram such that $\overrightarrow{AB}=\vec{a}$ and $\overrightarrow{BC}=\vec{b}$.

Then, $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AC} = \vec{a} + \vec{b} = 3\hat{i} + 6\hat{j} - 2\hat{k}$

$|\overrightarrow{AC}| = \sqrt{9+36+4} = 7$

$\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$



$\Rightarrow \overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB} = \vec{b} - \vec{a} = \hat{i} + 2\hat{j} - 8\hat{k}$

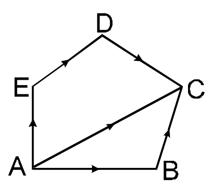
$|\overrightarrow{BD}| = \sqrt{1+4+64} = \sqrt{69}$

$\therefore$ Unit vector along $\overrightarrow{AC} = \dfrac{\overrightarrow{AC}}{|\overrightarrow{AC}|} = \dfrac{1}{7}\left(3\hat{i} + 6\hat{j} - 2\hat{k}\right)$

and Unit vector along $\overrightarrow{BD} = \dfrac{\overrightarrow{BD}}{|\overrightarrow{BD}|} = \dfrac{1}{\sqrt{69}}\left(\hat{i} + 2\hat{j} - 8\hat{k}\right)$

Ex.4 ABCDE is a pentagon. Prove that the resultant of the forces $\overrightarrow{AB}$, $\overrightarrow{AE}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, and $\overrightarrow{AC}$ is $3\overrightarrow{AC}$.

Sol. Let $\vec{R}$ be the resultant force



$\therefore \vec{R} = \overrightarrow{AB} + \overrightarrow{AE} + \overrightarrow{BC} + \overrightarrow{DC} + \overrightarrow{ED} + \overrightarrow{AC}$

$\Rightarrow \vec{R} = (\overrightarrow{AB} + \overrightarrow{BC}) + (\overrightarrow{AE} + \overrightarrow{ED} + \overrightarrow{DC}) + \overrightarrow{AC}$

$\Rightarrow \vec{R} = \overrightarrow{AC} + \overrightarrow{AC} + \overrightarrow{AC}$

$\Rightarrow \vec{R} = 3\overrightarrow{AC}$. Hence proved.

IMPORTANT PROPERTIES AND FORMULAE

1. **(a)** Triangle law of vector addition $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

   (b) Parallelogram law of vector addition : If ABCD is a parallelogram, then $\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$

   (c) If $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ then $\vec{r}_1 + \vec{r}_2 = (x_1+x_2)\hat{i} + (y_1+y_2)\hat{j} + (z_1+z_2)\hat{k}$ and $\vec{r}_1 = \vec{r}_2 \Leftrightarrow x_1 = x_2,\ y_1 = y_2,\ z_1 = z_2$

2. **(a)** $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} = m\vec{b}$ for same non-Zero scalar m.

   (b) $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ or $\vec{a} = |\vec{a}|\,\hat{a}$

   (c) Associative law : $m(n\vec{a}) = (mn)\vec{a} = n(m\vec{a})$

   (d) Distributive laws : $(m+n)\vec{a} = m\vec{a} + n\vec{a}$ and $n(\vec{a}+\vec{b}) = n\vec{a} + n\vec{b}$

   (e) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $m\vec{r} = mx\hat{i} + my\hat{j} + mz\hat{k}$.

   (f) $\vec{r}, \vec{a}, \vec{b}$ are coplaner if and only $\vec{r} = x\vec{a} + y\vec{b}$ for some scalars x and y.

   (g) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

   (h) The points A, B, C will be collinear if and only if $\overrightarrow{AB} = m\,\overrightarrow{AC}$, for some non-Zero scalar m.

   (i) Given vectors $x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$, $x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$, $x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$, where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, will be coplanar if and only if

   $$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

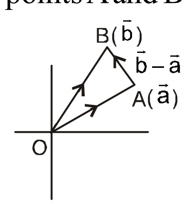
   (j) **Method to prove four points to be coplanar:** To prove that the four points A, B, C and D are coplanar. Find the vector $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ and then prove them to be coplanar by the method of coplanarity i.e. one of them is a linear combination of the other two.

   (k) $|\vec{a}| - |\vec{b}| \le |\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$

   $|\vec{a} - \vec{b}| \ge |\vec{a}| - |\vec{b}|$

POSITION VECTOR OF A POINT

Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B, then



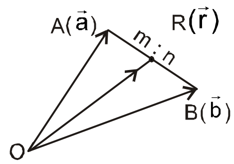
$\overrightarrow{AB} = \vec{b} - \vec{a}$ = position vector **(p.v.)** of B – position vector **(p.v.)** of A.

Distance formula :

Distance between the two points $A(\vec{a})$ and $B(\vec{b})$ is

$AB = \left|\vec{a}-\vec{b}\right|$

Section formula :



If $\vec{a}$ and $\vec{b}$ are the position vectors of two points A and B, then the p.v. of

a point which divides AB in the ratio m: n is given by

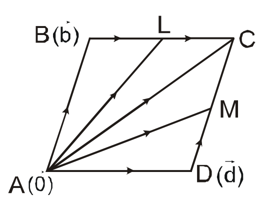
$\vec{r} = \dfrac{n\vec{a}+m\vec{b}}{m+n}$.

Note : Position vector of mid point of AB = $\dfrac{\vec{a}+\vec{b}}{2}$.

Solved Examples

Ex.5 ABCD is a parallelogram. If L, M be the middle point of BC and CD, express $\overrightarrow{AL}$ and $\overrightarrow{AM}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Also show that $\overrightarrow{AL}$ + $\overrightarrow{AM}$ = $\dfrac{3}{2}$ $\overrightarrow{AC}$.

Sol. Let the position vectors of points B and D be respectively $\vec{b}$ and $\vec{d}$ referred to A as origin of reference.



Then $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AD} + \overrightarrow{AB}$ $[\because \overrightarrow{DC} = \overrightarrow{AB}]$

$\Rightarrow \overrightarrow{AC} = \vec{d} + \vec{b}$ $\qquad \because \overrightarrow{AB} = \vec{b},\ \overrightarrow{AD} = \vec{d}$

i.e. position vector of C referred to A is $\vec{d} + \vec{b}$

$\therefore$ $\overrightarrow{AL}$ = p.v. of L, the mid point of $\overrightarrow{BC}$.

$= \dfrac{1}{2}$ [p.v. of B + p.v. of C] $= \dfrac{1}{2}\left(\vec{b}+\vec{d}+\vec{b}\right)$

$= \overrightarrow{AB} + \dfrac{1}{2}\overrightarrow{AD}$

Similarly $\overrightarrow{AM} = \dfrac{1}{2}\left(\vec{d}+\vec{d}+\vec{b}\right) = \overrightarrow{AD} + \dfrac{1}{2}\overrightarrow{AB}$

$\therefore$ $\overrightarrow{AL} + \overrightarrow{AM} = \vec{b} + \dfrac{1}{2}\vec{d} + \vec{d} + \dfrac{1}{2}\vec{b}$

$= \dfrac{3}{2}\vec{b} + \dfrac{3}{2}\vec{d} = \dfrac{3}{2}(\vec{b} + \vec{d}) = \dfrac{3}{2}\overrightarrow{AC}$.

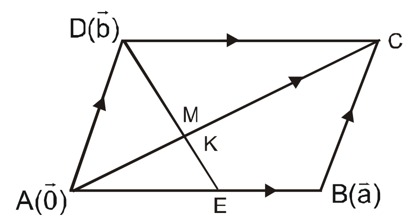
Ex.6 If ABCD is a parallelogram and E is the mid point of AB. Show by vector method that DE trisect AC and is trisected by AC.

Sol. Let $\overrightarrow{AB} = \vec{a}$ and AD = $\vec{b}$

Then $\overrightarrow{BC} = \overrightarrow{AD} = \vec{b}$ and $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} = \vec{a} + \vec{b}$

Also let K be a point on AC, such that AK : AC = 1 : 3

$\Rightarrow$ AK = $\dfrac{1}{3}$ AC



$\Rightarrow \overrightarrow{AK} = \dfrac{1}{3}(\vec{a} + \vec{b})$ ....(i)

Again E being the mid point of AB, we have

$\overrightarrow{AE} = \dfrac{1}{2}\vec{a}$

Let M be the point on DE such that DM : ME = 2 : 1

$\therefore$ $\overrightarrow{AM} = \dfrac{\overrightarrow{AD}+2\overrightarrow{AE}}{1+2} = \dfrac{\vec{b}+\vec{a}}{3}$ ....(ii)

From (i) and (ii) we find that

$\overrightarrow{AK} = \dfrac{1}{3}(\vec{a} + \vec{b}) = \overrightarrow{AM}$, and so we conclude that K and M coincide. i.e. DE trisect AC and is trisected by AC. Hence proved.

LINEAR COMBINATIONS

Given a finite set of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots\ldots$, then the vector $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + \ldots\ldots$ is called a linear combination of $\vec{a}, \vec{b}, \vec{c}, \ldots\ldots$ for any x, y, Z..... $\in$ R. We have the following results:

(a) If $\vec{a}, \vec{b}$ are non Zero, non–collinear vectors, then $x\vec{a} + y\vec{b} = x'\vec{a} + y'\vec{b} \Rightarrow x = x',\ \ y = y'$

(b) **Fundamental Theorem in plane :** Let $\vec{a}, \vec{b}$ be non Zero, non collinear vectors, then any vector $\vec{r}$ coplanar with $\vec{a}, \vec{b}$ can be expressed uniquely as a linear combination of $\vec{a}$ and $\vec{b}$

i.e. there exist some unique x, y $\in$ R such that $x\vec{a} + y\vec{b} = \vec{r}$.

(c) If $\vec{a}, \vec{b}, \vec{c}$ are non–Zero, non–coplanar vectors, then $x\vec{a} + y\vec{b} + z\vec{c} = x'\vec{a} + y'\vec{b} + z'\vec{c}$

$\Rightarrow x = x'\ ,\ y = y'\ ,\ z = z'$

(d) **Fundamental theorem in space:** Let $\vec{a}, \vec{b}, \vec{c}$ be non–Zero, non–coplanar vectors in space.

Then any vector $\vec{r}$ can be uniquely expressed as a linear combination of $\vec{a}, \vec{b}, \vec{c}$ i.e. there exist some unique x,y, Z $\in$ R such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{r}$.

(e) If $\vec{x}_1, \vec{x}_2, \ldots\ldots, \vec{x}_n$ are n non Zero vectors and $k_1, k_2, \ldots\ldots, k_n$ are n scalars and if the linear combination $k_1\vec{x}_1 + k_2\vec{x}_2 + \ldots\ldots + k_n\vec{x}_n = \vec{0}$

$\Rightarrow k_1 = 0, k_2 = 0, \ldots\ldots, k_n = 0$, then we say that vectors $\vec{x}_1, \vec{x}_2, \ldots\ldots, \vec{x}_n$ are **linearly independent vectors**.

(f) If $\vec{x}_1, \vec{x}_2, \ldots\ldots, \vec{x}_n$ are not **linearly independent** then they are said to be **linearly dependent vectors**. i.e. if $k_1\vec{x}_1 + k_2\vec{x}_2 + k_3\vec{x}_3 \ldots\ldots + k_r\vec{x}_r + \ldots\ldots + k_n\vec{x}_n = \vec{0}$ and if there exists at least one $k_r \neq 0$, then $\vec{x}_1, \vec{x}_2, \ldots\ldots, \vec{x}_n$ are said to be **linearly dependent vectors**.

Note 1:

If $k_r \neq 0$; $k_1\vec{x}_1 + k_2\vec{x}_2 + k_3\vec{x}_3 \ldots\ldots + k_r\vec{x}_r + \ldots\ldots + k_n\vec{x}_n = \vec{0}$

$$\Rightarrow -k_r\vec{x}_r = k_1\vec{x}_1 + k_2\vec{x}_2 + \ldots\ldots + k_{r-1}.\vec{x}_{r-1} + k_{r+1}.\vec{x}_{r+1} + \ldots\ldots + k_n\vec{x}_n$$

$$\Rightarrow -k_r\frac{1}{k_r}\vec{x}_r = k_1\frac{1}{k_r}\vec{x}_1 + k_2\frac{1}{k_r}\vec{x}_2 + \ldots\ldots + k_{r-1}.\frac{1}{k_r}\vec{x}_{r-1} + \ldots\ldots + k_n\frac{1}{k_r}\vec{x}_n$$

$$\Rightarrow \vec{x}_r = c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots\ldots + c_{r-1}\vec{x}_{r-1} + c_{r+1}\vec{x}_{r+1} + \ldots\ldots + c_n\vec{x}_n$$

i.e. $\vec{x}_r$ is expressed as a linear combination of vectors $\vec{x}_1, \vec{x}_2, \ldots\ldots, \vec{x}_{r-1}, \vec{x}_{r+1}, \ldots\ldots, \vec{x}_n$

Hence $\vec{x}_r$ with $\vec{x}_1, \vec{x}_2, \ldots\ldots, \vec{x}_{r-1}, \vec{x}_{r+1}, \ldots\ldots, \vec{x}_n$ forms a linearly dependent set of vectors.

Note 2:

* If $\vec{a} = 3\hat{i} + 2\hat{j} + 5\hat{k}$ then $\vec{a}$ is expressed as a **Linear Combination** of vectors $\hat{i}, \hat{j}, \hat{k}$. Also $\vec{a}, \hat{i}, \hat{j}, \hat{k}$ form a linearly dependent set of vectors. In general, in 3 dimensional space every set of four vectors is a linearly dependent system.

* $\hat{i}, \hat{j}, \hat{k}$ are **Linearly Independent** set of vectors. For

$K_1\hat{i} + K_2\hat{j} + K_3\hat{k} = \vec{0} \ \Rightarrow\ K_1 = K_2 = K_3 = 0$

* Two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent $\Rightarrow \vec{a}$ is parallel to $\vec{b}$ i.e. $\vec{a} \times \vec{b} = \vec{0} \Rightarrow$ linear dependence of $\vec{a}$ and $\vec{b}$. Conversely if $\vec{a} \times \vec{b} \neq \vec{0}$ then $\vec{a}$ and $\vec{b}$ are linearly independent.

* If three vectors $\vec{a}, \vec{b}, \vec{c}$ are linearly dependent, then they are coplanar i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = 0$. Conversely if $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$ then the vectors are linearly independent.

Solved Examples

Ex.7 Given that position vectors of points A, B, C are respectively

$\vec{a} - 2\vec{b} + 3\vec{c}$, $2\vec{a} + 3\vec{b} - 4\vec{c}$, $-7\vec{b} + 10\vec{c}$ then prove that vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent.

Sol. Let A, B, C be the given points and O be the point of reference then

$\overrightarrow{OA} = \vec{a} - 2\vec{b} + 3\vec{c}$, $\overrightarrow{OB} = 2\vec{a} + 3\vec{b} - 4\vec{c}$

and $\overrightarrow{OC} = -7\vec{b} + 10\vec{c}$

Now $\overrightarrow{AB}$ = p.v. of B – p.v. of A

$= \overrightarrow{OB} - \overrightarrow{OA} = (\vec{a} + 5\vec{b} - 7\vec{c})$ and

$\overrightarrow{AC}$ = p.v. of C – p.v of A

$= \overrightarrow{OC} - \overrightarrow{OA} = -(\vec{a} + 5\vec{b} - 7\vec{c}) = -\overrightarrow{AB}$

$\therefore$ $\overrightarrow{AC} = \lambda\overrightarrow{AB}$ where $\lambda = -1$. Hence $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are linearly dependent

TEST OF COLLINEARITY

Three points A,B,C with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ respectively are collinear, if & only if there exist scalars x, y, Z not all Zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where x + y + Z = 0.

TEST OF COPLANARITY

Four points A, B, C, D with position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ respectively are coplanar if and only if there exist scalars x, y, Z, w not all Zero simultaneously such that $x\vec{a} + y\vec{b} + z\vec{c} + w\vec{d} = \vec{0}$, where x + y + Z + w = 0.

Solved Examples

Ex.8 Show that the vectors $2\vec{a} - \vec{b} + 3\vec{c}$, $\vec{a} + \vec{b} - 2\vec{c}$ and $\vec{a} + \vec{b} - 3\vec{c}$ are non-coplanar vectors.

Sol. Let, the given vectors be coplanar.

Then one of the given vectors is expressible in terms of the other two.

Let $2\vec{a} - \vec{b} + 3\vec{c} = x(\vec{a} + \vec{b} - 2\vec{c}) + y(\vec{a} + \vec{b} - 3\vec{c})$, for some scalars x and y.

$\Rightarrow 2\vec{a} - \vec{b} + 3\vec{c} = (x + y)\vec{a} + (x + y)\vec{b} + (-2x - 3y)\vec{c}$

$\Rightarrow 2 = x + y, -1 = x + y$ and $3 = -2x - 3y$.

Solving first and third of these equations, we get x = 9 and y = –7.

Clearly these values do not satisfy the second equation.

Hence the given vectors are not coplanar.

Ex.9 Prove that four points $2\vec{a} + 3\vec{b} - \vec{c}$, $\vec{a} - 2\vec{b} + 3\vec{c}$, $3\vec{a} + 4\vec{b} - 2\vec{c}$ and $\vec{a} - 6\vec{b} + 6\vec{c}$ are coplanar.

Sol. Let the given four points be P, Q, R and S respectively. These points are coplanar if the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are coplanar. These vectors are coplanar iff one of them can be expressed as a linear combination of other two. So let $\overrightarrow{PQ} = x\overrightarrow{PR} + y\overrightarrow{PS}$

$\Rightarrow -\vec{a} - 5\vec{b} + 4\vec{c} = x(\vec{a} + \vec{b} - \vec{c}) + y(-\vec{a} - 9\vec{b} + 7\vec{c})$

$\Rightarrow -\vec{a} - 5\vec{b} + 4\vec{c} = (x - y)\vec{a} + (x - 9y)\vec{b} + (-x + 7y)\vec{c}$

$\Rightarrow x - y = -1, x - 9y = -5, -x + 7y = 4$

[Equating coeff. of $\vec{a}, \vec{b}, \vec{c}$ on both sides]

Solving the first two equations of these three equations, we get $x = -\frac{1}{2}, y = \frac{1}{2}$.

These values also satisfy the third equation. Hence the given four points are coplanar.

ANGLE BETWEEN TWO VECTORS

It is the smaller angle formed when the initial points or the terminal points of the two vectors are brought together. Note that $0^\circ \le \theta \le 180^\circ$.

SCALAR PRODUCT OR DOT PRODUCT

(a) $\vec{a}\,.\,\vec{b} = ab\cos\theta$, where $0 \le \theta \le \pi$

(b) $\vec{a}\,.\,\vec{b} = 0 \Rightarrow \quad \vec{a} = 0$ or $\vec{b} = 0$

(c) Component of a vector $\vec{r}$ in the direction of $\vec{a}$ and perpendicular to $\vec{a}$ are $\left(\dfrac{\vec{r}.\vec{a}}{|\vec{a}|^2}\right)\vec{a}$ and $\vec{r} - \left\{\dfrac{(\vec{r}.\vec{a})}{|\vec{a}|^2}\right\}\vec{a}$ respectively.

(d) If $\vec{a}$ and $\vec{b}$ are the non-Zero vectors, then $\vec{a}\,.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$

(e) $\cos\theta = \hat{a}\,.\,\hat{b} = \dfrac{\vec{a}.\vec{b}}{|a|.|b|}$

(f) $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$

$\hat{i}.\hat{j} = \hat{j}.\hat{i} = \hat{j}.\hat{k} = \hat{k}.\hat{j} = \hat{k}.\hat{i} = \hat{i}.\hat{k} = 0$

(g) If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ i.e. if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

(i) $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

(ii) $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$

(iii) $\vec{a}$ and $\vec{b}$ will be perpendicular if and only if $a_1b_1 + a_2b_2 + a_3b_3 = 0$

(iv) $\vec{a}$ and $\vec{b}$ will be parallel if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

Note:

(a) If θ is acute, then $\vec{a}\,.\,\vec{b} > 0$ and if θ is obtuse, then $\vec{a}.\vec{b} < 0$.

(b) $\vec{a}\,.\,\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b} \quad (\vec{a} \neq 0, \vec{b} \neq 0)$

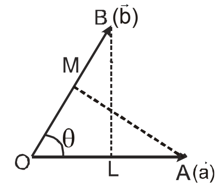
(c) Maximum value of $\vec{a}\,.\,\vec{b}$ is $|\vec{a}|\,|\vec{b}|$

(d) Minimum value of $\vec{a}\,.\,\vec{b}$ is $-|\vec{a}|\,|\vec{b}|$

Geometrical interpretation of scalar product :

Let $\vec{a}$ and $\vec{b}$ be vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Draw $BL \perp OA$ and $AM \perp OB$.

From ΔOBL and ΔOAM, we have $OL = OB\cos\theta$ and $OM = OA\cos\theta$.



Here OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$ respectively.

Now, $\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = |\vec{a}|(|\vec{b}|\cos\theta)$

$= |\vec{a}|(OB\cos\theta) = |\vec{a}|(OL)$

$=$ (Magnitude of $\vec{a}$) (Projection of $\vec{b}$ on $\vec{a}$) ....(i)

Again $\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = |\vec{b}|(|\vec{a}|\cos\theta)$

$= |\vec{b}|(OA\cos\theta) = |\vec{b}|(OM)$

$=$ (magnitude of $\vec{b}$) (Projection of $\vec{a}$ on $\vec{b}$) .....(ii)

Thus geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

(i) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

(ii) $\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{a}$ **(commutative)**

(iii) $\vec{a}\,.\,(\vec{b} + \vec{c}) = \vec{a}\,.\,\vec{b} + \vec{a}\,.\,\vec{c}$ **(distributive)**

(iv) $\vec{b} = \vec{a}\,.(m\vec{b}) = m(\vec{a}\,.\,\vec{b})$, where m is a scalar.

Solved Examples

Ex.10 Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are

(i) perpendicular (ii) parallel

Sol. (i) $\vec{a} \perp \vec{b}$

$\Rightarrow \quad \vec{a}\,.\,\vec{b} = 0$

$\Rightarrow \quad (3\hat{i} + 2\hat{j} + 9\hat{k})\,.\,(\hat{i} + p\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \quad 3 + 2p + 27 = 0 \Rightarrow \quad p = -15$

(ii) vectors $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b} = \hat{i}+p\hat{j}+3\hat{k}$ are parallel iff

$\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \Rightarrow 3 = \frac{2}{p} \Rightarrow \quad p = \frac{2}{3}$

Ex.11 If $\vec{a}+\vec{b}+\vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.

Sol. We have, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \vec{a}+\vec{b} = -\vec{c} \Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = (-\vec{c})\cdot(-\vec{c})$

$\Rightarrow |\vec{a}+\vec{b}|^2 = |\vec{c}|^2 \Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{c}|^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}|\,|\vec{b}|\cos\theta = |\vec{c}|^2$

$\Rightarrow 9 + 25 + 2(3)(5)\cos\theta = 49$

$\Rightarrow \cos\theta = \frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{\pi}{3}$.

Ex.12 Find the values of x for which the angle between the vectors $\vec{a} = 2x^2\,\hat{i} + 4x\,\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + x\,\hat{k}$ is obtuse.

Sol. The angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by

$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

Now, θ is obtuse $\Rightarrow \cos\theta < 0 \Rightarrow \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} < 0$

$\Rightarrow \vec{a}.\vec{b} < 0 \qquad [\because |\vec{a}|, |\vec{b}| > 0]$

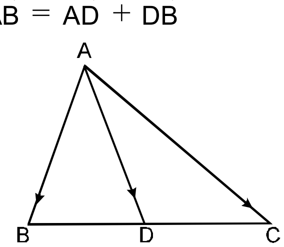
$\Rightarrow 14x^2 - 8x + x < 0 \qquad \Rightarrow 7x(2x-1) < 0$

$\Rightarrow x(2x-1) < 0 \qquad \Rightarrow 0 < x < \frac{1}{2}$

Hence, the angle between the given vectors is obtuse if $x \in (0, 1/2)$

Ex.13 D is the mid point of the side BC of a ΔABC, show that $AB^2 + AC^2 = 2(AD^2 + BD^2)$

Sol. We have $\vec{AB} = \vec{AD} + \vec{DB}$



$\Rightarrow AB^2 = (\vec{AD}+\vec{DB})^2$

$\Rightarrow AB^2 = AD^2 + DB^2 + 2\vec{AD}\cdot\vec{DB}$ ....(i)

Also we have $\vec{AC} = \vec{AD} + \vec{DC}$

$\Rightarrow AC^2 = (\vec{AD}+\vec{DC})^2$

$\Rightarrow AC^2 = AD^2 + DC^2 + 2\vec{AD}\cdot\vec{DC}$ ....(ii)

Adding (i) and (ii), we get $AB^2 + AC^2 = 2AD^2 + 2BD^2 + 2\vec{AD}\cdot(\vec{DB}+\vec{DC})$

$\Rightarrow AB^2 + AC^2 = 2(AD^2 + BD^2) \quad \because \vec{DB} + \vec{DC} = \vec{0}$

Ex.14 If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$, then find

(i) Component of $\vec{b}$ along $\vec{a}$.

(ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to $\vec{a}$.

Sol. (i) Component of $\vec{b}$ along $\vec{a}$ is $\left(\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\vec{a}$

Here $\vec{a}\,.\,\vec{b} = 2 - 1 + 3 = 4$

$|\vec{a}|^2 = 3$

Hence $\left(\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \frac{4}{3}\vec{a} = \frac{4}{3}(\hat{i} + \hat{j} + \hat{k})$

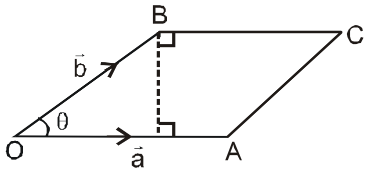
(ii) Component of $\vec{b}$ in plane of $\vec{a}$ & $\vec{b}$ but $\perp$ to $\vec{a}$ is

$\vec{b} - \left(\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\vec{a}. = \frac{1}{3}(2\hat{i} - 7\hat{j} + 5\hat{k})$

VECTOR PRODUCT (CROSS PRODUCT) OF TWO VECTORS

(i) If $\vec{a}, \vec{b}$ are two vectors and θ is the angle between them, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}$ and $\hat{n}$ forms a right handed screw system.

(ii) Geometrically $|\vec{a} \times \vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ and $\vec{b}$.



(iii) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ **(not commutative)**

(iv) $(m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b}) = m(\vec{a} \times \vec{b})$, where m is a scalar.

(v) $\vec{a} \times (\vec{b}+\vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ **(distributive)**

(vi) $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel (collinear) $(\vec{a} \neq 0\ ,\ \vec{b} \neq 0)$ i.e. $\vec{a} = K\vec{b}$, where K is a scalar.

(vii) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$; $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

(viii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(ix) Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\hat{n} = \pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

(x) A vector of magnitude 'r' and perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is $\pm \dfrac{r(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$

(xi) If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$

(xii) If two vectors $\vec{a}$ and $\vec{b}$ are parallel, then $\theta = 0$ or π i.e. $\sin\theta = 0$ in both cases.

$\because\ (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 = 0$

$\Rightarrow a_1b_2 - a_2b_1 = 0,\ a_2b_3 - a_3b_2 = 0,\ a_3b_1 - a_1b_3 = 0$

$\Rightarrow \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2},\ \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3},\ \dfrac{a_3}{b_3} = \dfrac{a_1}{b_1} \Rightarrow \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

(xiii) If $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of 3 points A, B and C respectively, then the vector area of $\Delta ABC = \dfrac{1}{2}\left[\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right]$. The points A, B and C are collinear if $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$

(xiv) Area of any quadrilateral whose diagonal vectors are $\vec{d}_1$ and $\vec{d}_2$ is given by $\dfrac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$

(xv) **Lagrange's Identity :** For any two vectors $\vec{a}$ and $\vec{b}$; $(\vec{a} \times \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$

Solved Examples

Ex.15 Find a vector of magnitude 9, which is perpendicular to both the vectors $4\hat{i} - \hat{j} + 3\hat{k}$ and $-2\hat{i} + \hat{j} - 2\hat{k}$.

Sol. Let $\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$. Then

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = (2-3)\,\hat{i} - (-8+6)\,\hat{j} + (4-2)\,\hat{k} = -\hat{i} + 2\hat{j} + 2\hat{k}$

$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3$

$\therefore$ Required vector $= \pm 9\left(\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}\right)$

$= \pm \dfrac{9}{3}(-\hat{i} + 2\hat{j} + 2\hat{k}) = \pm(-3\hat{i} + 6\hat{j} + 6\hat{k})$

Ex.16 For any three vectors $\vec{a}, \vec{b}, \vec{c}$, show that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$.

Sol. We have, $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b}$

[Using distributive law]

$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} - \vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{b} \times \vec{c} = \vec{0}$

$[\because \vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$ etc$]$

Ex.17 For any vector $\vec{a}$, prove that

$$|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = 2\,|\vec{a}|^2$$

Sol. Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$. Then

$\vec{a}\times\hat{i} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{i} = a_1(\hat{i}\times\hat{i}) + a_2(\hat{j}\times\hat{i}) + a_3(\hat{k}\times\hat{i}) = -a_2\hat{k}+a_3\hat{j}$

$\Rightarrow |\vec{a}\times\hat{i}|^2 = a_2^2 + a_3^2$

$\vec{a}\times\hat{j} = (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times\hat{j} = a_1\hat{k}-a_3\hat{i}$

$\Rightarrow |\vec{a}\times\hat{j}|^2 = a^2_1 + a_3^2$

$\vec{a}\times\hat{k} = (a_i\hat{i}+a_2\hat{j}+a_3\hat{k})\times\vec{k} = -a_i\hat{j}+a_2\hat{i}$

$\Rightarrow |\vec{a}\times\hat{k}|^2 = a_1^2 + a_2^2$

$\therefore |\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = a_2^2 + a_3^3 + a_1^2 + a_3^2 + a_1^2 + a_2^2 \quad = 2\,(a_1^2 + a_2^2 + a_3^2) = 2\,|\vec{a}|^2$

Ex.18 Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = 10\vec{a} + 2\vec{b}$ and $\overrightarrow{OC} = \vec{b}$ where O is origin. Let p denote the area of the quadrilateral OABC and q denote the area of the parallelogram with OA and OC as adjacent sides. Prove that p = 6q.

Sol. We have,

p = Area of the quadrilateral OABC

$\Rightarrow p = \frac{1}{2}\,|\overrightarrow{OB}\times\overrightarrow{AC}| = \frac{1}{2}\,|\overrightarrow{OB}\times(\overrightarrow{OC}-\overrightarrow{OA})|$

$\Rightarrow p = \frac{1}{2}\,|(10\vec{a}+2\vec{b})\times(\vec{b}-\vec{a})|$

$= \frac{1}{2}\,|10(\vec{a}\times\vec{b}) - 10(\vec{a}\times\vec{a}) + 2(\vec{b}\times\vec{b}) - 2(\vec{b}\times\vec{a})|$

$\Rightarrow p = \frac{1}{2}\,|10(\vec{a}\times\vec{b}) - 0 + 0 + 2(\vec{a}\times\vec{b})| = 6\,|\vec{a}\times\vec{b}|$ .....(i)

and q = Area of the parallelogram with OA and OC as adjacent sides

$\Rightarrow \quad q = |\overrightarrow{OA}\times\overrightarrow{OC}| = |\vec{a}\times\vec{b}|$ ....(ii)

From (i) and (ii), we get p = 6q

PROJECTION OF A LINE SEGMENT ON A LINE

(i) If the coordinates of P and Q are (x_1, y_1, Z_1) and (x_2, y_2, Z_2), then the projection of the line segments PQ on a line having direction cosines ℓ, m, n is

$$\left|\ell(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)\right|$$

(ii) **Vector form :** projection of a vector $\vec{a}$ on another vector $\vec{b}$ is $\vec{a}\,.\,\hat{b} = \frac{\vec{a}\,.\,\vec{b}}{|\vec{b}|}$

In the above case we can consider $\overrightarrow{PQ}$ as $(x_2-x_1)\hat{i} + (y_2-y_1)\hat{j} + (Z_2-Z_1)\hat{k}$ in place of $\vec{a}$ and $\ell\hat{i} + m\hat{j} + n\hat{k}$ in place of $\vec{b}$.

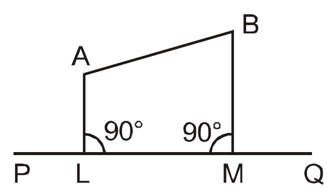
(iii) $\ell|\vec{r}|$, $m|\vec{r}|$ & $n|\vec{r}|$ are the projection of $\vec{r}$ in OX, OY & OZ axes.

(iv) $\vec{r} = |\vec{r}|(\ell\hat{i} + m\hat{j} + n\hat{k})$

Solved Examples

Ex.19 Find the projection of the line joining (1, 2, 3) and (–1, 4, 2) on the line having direction ratios 2, 3, – 6.

Sol. Let A ≡ (1, 2, 3), B ≡ (–1, 4, 2)



Direction ratios of the given line PQ are 2, 3, – 6

$\sqrt{2^2+3^2+(-6)^2} = 7$

$\therefore$ direction cosines of PQ are $\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$

Projection of AB on PQ

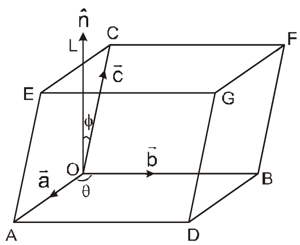
$= |\ell\,(x_2-x_1) + m(y_2-y_1) + n(Z_2-Z_1)|$

$= \left|\frac{2}{7}(-1-1) + \frac{3}{7}(4-2) - \frac{6}{7}(2-3)\right| = \left|\frac{-4+6+6}{7}\right|$

$= \frac{8}{7}$

Scalar triple product (Box Product) (S.T.P.) :

(i) The scalar triple product of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined as: $\vec{a} \times \vec{b} \cdot \vec{c} = |\vec{a}|\,|\vec{b}|\,|\vec{c}| \sin\theta \cdot \cos\phi$, where θ is the angle between $\vec{a}$, $\vec{b}$ **(i.e.** $\vec{a} \wedge \vec{b} = \theta$**)** and ϕ is the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ **(i.e.** $(\vec{a} \times \vec{b}) \wedge \vec{c} = \phi$**)**. It is (i.e. $\vec{a} \times \vec{b} \cdot \vec{c}$) also written as $[\vec{a}\ \vec{b}\ \vec{c}]$ and spelled as box product.



(ii) Scalar triple product geometrically represents the volume of the parallelopiped whose three coterminous edges are represented by $\vec{a}, \vec{b}$ and $\vec{c}$ i.e. $V = |[\vec{a}\ \vec{b}\ \vec{c}]|$

(iii) In a scalar triple product the position of dot and cross can be interchanged i.e.
$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} \quad \Rightarrow \quad [\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$

(iv) $\vec{a} \cdot (\vec{b} \times \vec{c}) = -\vec{a} \cdot (\vec{c} \times \vec{b})$ i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{a}\,\vec{c}\,\vec{b}]$

(a) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(d) If any two of the vectors $\vec{a}, \vec{b}, \vec{c}$ are equal, then $[a\ b\ c] = 0$

(v) If $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.

(vi) The value of a scalar triple product is Zero, if two of its vectors are parallel.

(vii) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-coplanar, then $[\vec{a}\ \vec{b}\ \vec{c}] > 0$ for right handed system and $[\vec{a}\ \vec{b}\ \vec{c}] < 0$ for left handed system.

(viii) $[\hat{i}\ \hat{j}\ \hat{k}] = 1$

(ix) $[K\vec{a}\ \vec{b}\ \vec{c}] = K[\vec{a}\ \vec{b}\ \vec{c}]$

(x) $[(\vec{a} + \vec{b})\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

(xi) $[\vec{a} - \vec{b}\quad \vec{b} - \vec{c}\quad \vec{c} - \vec{a}] = 0$ and $[\vec{a} + \vec{b}\quad \vec{b} + \vec{c}\quad \vec{c} + \vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$.

(xii) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

In general, if $\vec{a} = a_1\vec{\ell} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{\ell} + b_2\vec{m} + b_3\vec{n}$ and $\vec{c} = c_1\vec{\ell} + c_2\vec{m} + c_3\vec{n}$

then $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{\ell}\,\vec{m}\,\vec{n}]$, where $\vec{\ell}$, $\vec{m}$ and $\vec{n}$ are non-coplanar vectors.

(xiii) $$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$$

(xiv) Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are coplanar if and only if $[\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}] = 0$ i.e. if and only if $[\vec{b} - \vec{a}\quad \vec{c} - \vec{a}\quad \vec{d} - \vec{a}] = 0$

(xv) Volume of a tetrahedron with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{6}|[\vec{a}\ \vec{b}\ \vec{c}]|$

(xvi) Volume of prism on a triangular base with three coterminous edges $\vec{a}, \vec{b}, \vec{c} = \frac{1}{2}|[\vec{a}\ \vec{b}\ \vec{c}]|$

Tetrahedron and its properties :

(a) The volume of the tetrahedron OABC with O as origin and the position vectors of A, B and C being $\vec{a}, \vec{b}$ and $\vec{c}$ respectively is given by $V = \frac{1}{6}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$

(b) If the position vectors of the vertices of tetrahedron are $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$, then the position vector of its centroid is given by $\frac{1}{4}(\vec{a} + \vec{b} + \vec{c} + \vec{d})$.

NOTE :

This is also the point of concurrency of the lines joining the vertices to the centroids of the opposite faces and is also called the centre of the tetrahedron. In case the tetrahedron is regular it is equidistant from the vertices and the four faces of the tetrahedron.

Solved Examples

Ex.20 Find the volume of a parallelopiped whose sides are given by $-3\hat{i}+7\hat{j}+5\hat{k}$, $-5\hat{i}+7\hat{j}-3\hat{k}$ and $7\hat{i}-5\hat{j}-3\hat{k}$

Sol. Let $\vec{a} = -3\hat{i}+7\hat{j}+5\hat{k}$, $\vec{b} = -5\hat{i}+7\hat{j}-3\hat{k}$ and $\vec{c} = 7\hat{i}-5\hat{j}-3\hat{k}$.

We know that the volume of a parallelopiped whose three adjacent edges are $\vec{a}, \vec{b}, \vec{c}$ is $\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$.

Now $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$

$= -3\,(-21 - 15) - 7\,(15 + 21) + 5\,(25 - 49)$

$= 108 - 252 - 120 = -264$

So required volume of the parallelopiped

$= \left|[\vec{a}\ \vec{b}\ \vec{c}]\right| = |-264| = 264$ cubic units

Ex.21 Simplify $[\vec{a}-\vec{b}\quad \vec{b}-\vec{c}\quad \vec{c}-\vec{a}]$

Sol. We have :

$[\vec{a}-\vec{b}\quad \vec{b}-\vec{c}\quad \vec{c}-\vec{a}] = \{(\vec{a}-\vec{b})\times(\vec{b}-\vec{c})\}\cdot(\vec{c}-\vec{a})$ [By definition]

$= (\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{b}+\vec{b}\times\vec{c})\cdot(\vec{c}-\vec{a})$ [By distribution law]

$= (\vec{a}\times\vec{b}+\vec{c}\times\vec{a}+\vec{b}\times\vec{c})\cdot(\vec{c}-\vec{a})$ $[\because \vec{b}\times\vec{b}=\vec{0}]$

$= (\vec{a}\times\vec{b})\cdot\vec{c} - (\vec{a}\times\vec{b})\cdot\vec{a} + (\vec{c}\times\vec{a})\cdot\vec{c} - (\vec{c}\times\vec{a})\cdot\vec{a} + (\vec{b}\times\vec{c})\cdot\vec{c} - (\vec{b}\times\vec{c})\cdot\vec{a}$ [By distribution law]

$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{a}] + [\vec{c}\ \vec{a}\ \vec{c}] - [\vec{c}\ \vec{a}\ \vec{a}] + [\vec{b}\ \vec{c}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}]$ [$\because$ When any two vectors are equal, scalar triple product is Zero]

$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$[\because [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]]$

Ex.22 Find the volume of the tetrahedron whose four vertices have position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$.

Sol. Let four vertices be A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively.

$\therefore\ \overrightarrow{DA} = (\vec{a} - \vec{d})$

$\overrightarrow{DB} = (\vec{b} - \vec{d})$

$\overrightarrow{DC} = (\vec{c} - \vec{d})$

Hence volume $V = \frac{1}{6}[\vec{a}-\vec{d}\quad \vec{b}-\vec{d}\quad \vec{c}-\vec{d}]$

$= \frac{1}{6}(\vec{a}-\vec{d})\cdot[(\vec{b}-\vec{d})\times(\vec{c}-\vec{d})]$

$= \frac{1}{6}(\vec{a}-\vec{d})\cdot[\vec{b}\times\vec{c}-\vec{b}\times\vec{d}+\vec{c}\times\vec{d}]$

$= \frac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{b}\ \vec{d}]+[\vec{a}\ \vec{c}\ \vec{d}]-[\vec{d}\ \vec{b}\ \vec{c}]\}$

$= \frac{1}{6}\{[\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{b}\ \vec{d}]+[\vec{a}\ \vec{c}\ \vec{d}]-[\vec{b}\ \vec{c}\ \vec{d}]\}$

Ex.23 Show that the vectors $\vec{a} = -2\hat{i} + 4\hat{j} - 2\hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} - 2\hat{k}$ and $\vec{c} = -2\hat{i} - 2\hat{j} + 4\hat{k}$ are coplanar.

Sol. $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} -2 & 4 & -2 \\ 4 & -2 & -2 \\ -2 & -2 & 4 \end{vmatrix}$

$= -2(-8-4) - 4(16-4) - 2(-8-4)$

$= 24 - 48 + 24 = 0$

So vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar

VECTOR TRIPLE PRODUCT

If $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then $(\vec{a} \times \vec{b}) \times \vec{c}$ and $\vec{a} \times (\vec{b} \times \vec{c})$ are known as vector triple product.

(a) $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b}).\vec{c}$

$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}.\vec{c})\,\vec{b} - (\vec{b}.\vec{c}).\vec{a}$

(b) $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector in the plane of vectors $\vec{b}$ and $\vec{c}$

(c) The vector triple product is not commutative i.e

$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

(d) Lagrange's identity :

$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$

$= (\vec{a}.\vec{c})\,(\vec{b}.\vec{d}) - (\vec{a}.\vec{d})\,(\vec{b}.\vec{c})$

Solved Examples

Ex.24 For any vector $\vec{a}$, prove that

$\hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k}) = 2\vec{a}$

Sol. Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

Then $\hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\hat{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k})$

$= \{(\hat{i}.\hat{i})\vec{a} - (\hat{i}.\vec{a})\hat{i}\} + \{(\hat{j}.\hat{j})\vec{a} - (\hat{j}.\vec{a})\hat{j}\} + \{(\hat{k}.\hat{k})\vec{a} - (\hat{k}.\vec{a})\hat{k}\}$

$= \{(\vec{a} - (\hat{i}.\vec{a})\,\hat{i}\} + \{\vec{a} - (\hat{j}.\vec{a})\,\hat{j}\} + \{\vec{a} - (\hat{k}.\vec{a})\hat{k}\}$

$= 3\vec{a} - \{(\hat{i}.\vec{a})\hat{i} + (\hat{j}.\vec{a})\hat{j} + (\hat{k}.\vec{a})\hat{k}$

$= 3\vec{a} - (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) = 3\vec{a} - \vec{a} = 2\vec{a}$

Ex.25 Prove that $\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{d})\} = (\vec{b}.\vec{d})(\vec{a} \times \vec{c}) - (\vec{b}.\vec{c})(\vec{a} \times \vec{d})$

Sol. We have, $\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{d})\} = \vec{a} \times \{(\vec{b}.\vec{d})\,\vec{c} - (\vec{b}.\vec{c})\,\vec{d}\}$

$= \vec{a} \times \{(\vec{b}.\vec{d})\,\vec{c}\} - \vec{a} \times \{(\vec{b}.\vec{c})\,\vec{d}\}$ [by dist. law]

$= (\vec{b}.\vec{d})\,(\vec{a} \times \vec{c}) - (\vec{b}.\vec{c})\,(\vec{a} \times \vec{d})$

Ex.26 Let $\vec{a} = \alpha\hat{i} + 2\hat{j} - 3\hat{k}$, $\vec{b} = \hat{i} + 2\alpha\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \alpha\hat{j} + \hat{k}$. Find the value(s) of α, if any, such that $\left\{\left(\vec{a} \times \vec{b}\right) \times \left(\vec{b} \times \vec{c}\right)\right\} \times \left(\vec{c} \times \vec{a}\right) = \vec{0}$.

Sol. $\left\{\left(\vec{a} \times \vec{b}\right) \times \left(\vec{b} \times \vec{c}\right)\right\} \times \left(\vec{c} \times \vec{a}\right)$

$= \left[\vec{a}\ \vec{b}\ \vec{c}\right] \vec{b} \times \left(\vec{c} \times \vec{a}\right)$

$= \left[\vec{a}\ \vec{b}\ \vec{c}\right] \left\{\left(\vec{a}.\vec{b}\right)\vec{c} - \left(\vec{b}.\vec{c}\right)\vec{a}\right\}$

which vanishes if

(i) $\left(\vec{a}.\vec{b}\right)\vec{c} = \left(\vec{b}.\vec{c}\right)\vec{a}$ (ii) $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

(i) $\left(\vec{a}.\vec{b}\right)\vec{c} = \left(\vec{b}.\vec{c}\right)\vec{a}$ leads to the equation $2\alpha^3 + 10\alpha + 12 = 0$, $\alpha^2 + 6\alpha = 0$ and $6\alpha - 6 = 0$, which do not have a common solution.

(ii) $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0 \Rightarrow \begin{vmatrix} \alpha & 2 & -3 \\ 1 & 2\alpha & -2 \\ 2 & -\alpha & 1 \end{vmatrix} = 0$

$\Rightarrow 3\alpha = 2 \Rightarrow \alpha = \frac{2}{3}$

Ex.27 If $\vec{A} + \vec{B} = \vec{a}$, $\vec{A}.\vec{a} = 1$ and $\vec{A} \times \vec{B} = \vec{b}$, then prove that $\vec{A} = \frac{\vec{a} \times \vec{b} + \vec{a}}{|\vec{a}|^2}$ and $\vec{B} = \frac{\vec{b} \times \vec{a} + \vec{a}\left(|\vec{a}|^2 - 1\right)}{|\vec{a}|^2}$.

Sol. Given $\vec{A} + \vec{B} = \vec{a}$ .....(i)

$\Rightarrow \vec{a}.\left(\vec{A} + \vec{B}\right) = \vec{a}.\vec{a}$

$\Rightarrow \vec{a}.\vec{A} + \vec{a}.\vec{B} = \vec{a}.\vec{a} \Rightarrow 1 + \vec{a}.\vec{B} = |\vec{a}|^2$

$\Rightarrow \vec{a}.\vec{B} = |\vec{a}|^2 - 1$ .....(ii)

Given $\vec{A} \times \vec{B} = \vec{b}$

$\Rightarrow \vec{a} \times \left(\vec{A} \times \vec{B}\right) = \vec{a} \times \vec{b}$

$\Rightarrow \left(\vec{a}.\vec{B}\right)\vec{A} - \left(\vec{a}.\vec{A}\right)\vec{B} = \vec{a} \times \vec{b}$

$\Rightarrow \left(|\vec{a}|^2 - 1\right)\vec{A} - \vec{B} = \vec{a} \times \vec{b}$ .....(iii)

[Using equation (ii)]

solving equation (i) and (iii) simultaneously, we get

$\vec{A} = \frac{\vec{a} \times \vec{b} + \vec{a}}{|\vec{a}|^2}$ and $\vec{B} = \frac{\vec{b} \times \vec{a} + \vec{a}\left(|\vec{a}|^2 - 1\right)}{|\vec{a}|^2}$

Ex.28 Solve for $\vec{r}$ satisfying the simultaneous equations $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$, $\vec{r}.\vec{a}=0$ provided $\vec{a}$ is not perpendicular to $\vec{b}$.

Sol. $(\vec{r}-\vec{c})\times\vec{b}=\vec{0}$ $\Rightarrow$ $\vec{r}-\vec{c}$ and $\vec{b}$ are collinear

$\therefore\ \vec{r}-\vec{c}=k\vec{b}$

$\Rightarrow\ \vec{r}=\vec{c}+k\vec{b}$ ....(i)

$\because\ \vec{r}.\vec{a}=0 \Rightarrow (\vec{c}+k\vec{b}).\vec{a}=0$

$\Rightarrow k=-\dfrac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}$ putting in (i) we get

$\vec{r}=\vec{c}-\dfrac{\vec{a}.\vec{c}}{\vec{a}.\vec{b}}\vec{b}$

Ex.29 If $\vec{x}\times\vec{a}+k\vec{x}=\vec{b}$, where k is a scalar and $\vec{a},\vec{b}$ are any two vectors, then determine $\vec{x}$ in terms of $\vec{a},\vec{b}$ and k.

Sol. $\vec{x}\times\vec{a}+k\vec{x}=\vec{b}$ ....(i)

Premultiply the given equation vectorially by $\vec{a}$

$\vec{a}\times(\vec{x}\times\vec{a})+k(\vec{a}\times\vec{x})=\vec{a}\times\vec{b}$

$\Rightarrow (\vec{a}.\vec{a})\vec{x}-(\vec{a}.\vec{x})\vec{a}+k(\vec{a}\times\vec{x})=\vec{a}\times\vec{b}$ ....(ii)

Premultiply (i) scalarly by $\vec{a}$

$[\vec{a}\ \vec{x}\ \vec{a}]+k(\vec{a}.\vec{x})=\vec{a}.\vec{b}$

$k(\vec{a}.\vec{x})=\vec{a}.\vec{b}$ ....(iii)

Substituting $\vec{x}\times\vec{a}$ from (i) and $\vec{a}.\vec{x}$ from (iii) in (ii) we get

$$\vec{x}=\frac{1}{a^2+k^2}\left[k\vec{b}+(\vec{a}\times\vec{b})+\frac{(\vec{a}.\vec{b})}{k}\vec{a}\right]$$

RECIPROCAL SYSTEM OF VECTORS

(a) If $\vec{a},\vec{b},\vec{c}$ be any three non-coplanar vectors so that $[\vec{a}\ \vec{b}\ \vec{c}]\neq 0$, then the three vectors $\vec{a}',\vec{b}',\vec{c}'$ defined by the equations $\vec{a}'=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{b}'=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $\vec{c}'=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$ are called the reciprocal system of vectors to the given vectors $\vec{a},\vec{b},\vec{c}$.

(b) Properties :

(i) $\vec{a}.\vec{a}'=\vec{b}.\vec{b}'=\vec{c}.\vec{c}'=1$

(ii) The scalar product of any vector of one system with a vector of the other system which does not correspond to it, is Zero, i.e. $\vec{a}.\vec{b}'=\vec{a}.\vec{c}'=\vec{b}.\vec{a}'=\vec{b}.\vec{c}'=\vec{c}.\vec{a}'=\vec{c}.\vec{b}'=0$

(iii) $[\vec{a}\ \vec{b}\ \vec{c}]\,[\vec{a}'\ \vec{b}'\ \vec{c}']=1$

(iv) $\vec{i}'=\vec{i},\ \vec{j}'=\vec{j},\ \vec{k}'=\vec{k}$

(v) If $\{\vec{a}',\vec{b}',\vec{c}'\}$ is reciprocal system of $\{\vec{a},\vec{b},\vec{c}\}$ and $\vec{r}$ is any vector, then

$\vec{r}=(\vec{r}.\vec{a})\,\vec{a}'+(\vec{r}.\vec{b})\,\vec{b}'+(\vec{r}.\vec{c})\,\vec{c}'$

$\vec{r}=(\vec{r}.\vec{a}')\,\vec{a}+(\vec{r}.\vec{b}')\,\vec{b}+(\vec{r}.\vec{c}')\,\vec{c}$

APPLICATION OF VECTOR IN GEOMETRY

(a) Vector equation of a straight line passing through a point $\vec{a}$ and parallel to $\vec{b}$ is $\vec{r}=\vec{a}+t\vec{b}$ where t is an arbitrary constant.

(b) Vector equation of a straight line passing through two points $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+t(\vec{b}-\vec{a})$.

(c) Vector equation of a plane passing through a point $\vec{a}$ and parallel to two given vectors $\vec{b}$ and $\vec{c}$ is $\vec{r}=\vec{a}+s\vec{b}+t\vec{c}$, where t and s are arbitrary constants. or $[\vec{r}\ \vec{b}\ \vec{c}]=[\vec{a}\ \vec{b}\ \vec{c}]$

(d) Vector equation of a plane passing through the points $\vec{a},\vec{b},\vec{c}$ is $\vec{r}=(1-s-t)\,\vec{a}+s\,\vec{b}+t\,\vec{c}$.

or $\vec{r}.(\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b})=[\vec{a}\ \vec{b}\ \vec{c}]$

(e) Vector equation of a plane passing through the point $\vec{a}$ and perpendicular to $\vec{n}$ is $\vec{r}.\vec{n}=\vec{a}.\vec{n}$.

Note : Perpendicular distance of the plane from the origin $=\dfrac{\vec{a}.\vec{n}}{|\vec{n}|}$

(f) Perpendicular distance of a point P(r) from a line passing through $\vec{a}$ and parallel to $\vec{b}$ is given by

$$PM=\frac{|(\vec{r}-\vec{a})\times\vec{b}|}{|\vec{b}|}=\left[(\vec{r}-\vec{a})^2-\left\{\frac{(\vec{r}-\vec{a}).\vec{b}}{|\vec{b}|}\right\}^2\right]^{1/2}$$

(g) Perpendicular distance of a point P(r) from a plane passing through $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is given by $PM=\dfrac{(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}$

(h) Perpendicular distance of a point P(r) from a plane passing through the points $\vec{a}$, $\vec{b}$ and $\vec{c}$ is given by

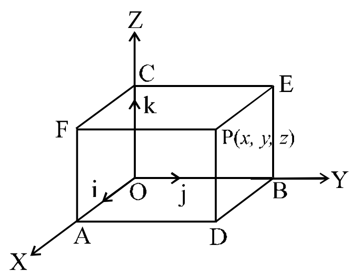
$$PM=\frac{(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b})}{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|}$$

THREE DIMENSIONAL GEOMETRY

COORDINATES OF A POINT IN SPACE

Let O be a fixed point, known as origin and let OX, OY and OZ be three mutually perpendicular lines, taken as x-axis, y-axis and z-axis respectively, in such a way that they form a right handed system.

The planes XOY, YOZ and ZOX are known as xy-plane, yz-plane and zx-plane respectively.



Let P be a point in space and distances of P from y-z, z-x and x-y planes be x, y, z respectively (with proper signs) then we say that coordinates of P are (x, y, z).

Also OA = x, OB = y, OC = z

DISTANCE FORMULA

The distance between two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is given by

$$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

(a) Distance from Origin :

Let O be the origin and P (x, y, z) be any point, then

$$OP = \sqrt{(x^2 + y^2 + z^2)}$$

(b) Distance of a point from coordinate axes :

Let P(x, y, z) be any point in the space. Let PA, PB and PC be the perpendiculars drawn from P to the axes OX, OY and OZ respectively. Then

$$PA = \sqrt{(y^2 + z^2)}\ ;\quad PB = \sqrt{(z^2 + x^2)}\ ;$$

$$PC = \sqrt{(x^2 + y^2)}$$

Solved Examples

Ex.30 Show that the points (0, 7, 10), (– 1, 6, 6) and (– 4, 9, 6) form a right angled isosceles triangle.

Sol. Let A ≡ (0, 7, 10), B ≡ (–1, 6, 6), C ≡ (– 4, 9, 6)

$AB^2 = (0 + 1)^2 + (7 – 6)^2 + (10 – 6)^2 = 18$

$\therefore$ AB = $3\sqrt{2}$

Similarly

$\therefore$ BC = $3\sqrt{2}$, & AC = 6

Clearly $AB^2 + BC^2 = AC^2$

$\therefore$ $\angle ABC = 90°$

Also AB = BC

Hence ΔABC is right angled isosceles.

Ex.31 Show by using distance formula that the points (4, 5, –5), (0, –11, 3) and (2, –3, –1) are collinear.

Sol. Let A ≡ (4, 5, –5), B ≡ (0, –11, 3), C ≡ (2, –3, –1).

$$AB = \sqrt{(4-0)^2 + (5+11)^2 + (-5-3)^2}$$

$$= \sqrt{336} = \sqrt{4 \times 84} = 2\sqrt{84}$$

$$BC = \sqrt{(0-2)^2 + (-11+3)^2 + (3+1)^2} = \sqrt{84}$$

$$AC = \sqrt{(4-2)^2 + (5+3)^2 + (-5+1)^2} = \sqrt{84}$$

BC + AC = AB

Hence points A, B, C are collinear and C lies between A and B.

Ex.32 Find the locus of a point which moves such that the sum of its distances from points $A(0, 0, -\alpha)$ and $B(0, 0, \alpha)$ is constant.

Sol. Let the variable point whose locus is required be P(x, y, z)

Given PA + PB = constant = 2a (say)

$$\therefore \sqrt{(x-0)^2 + (y-0)^2 + (z+\alpha)^2} + \sqrt{(x-0)^2 + (y-0)^2 + (z-\alpha)^2} = 2a$$

$$\Rightarrow \sqrt{x^2+y^2+(z+\alpha)^2} = 2a - \sqrt{x^2+y^2+(z-\alpha)^2}$$

$$\Rightarrow x^2 + y^2 + z^2 + \alpha^2 + 2z\alpha = 4a^2 + x^2 + y^2 + z^2 + \alpha^2$$

$$- 2z\alpha - 4a\sqrt{x^2+y^2+(z-\alpha)^2}$$

$$\Rightarrow 4z\alpha - 4a^2 = -4a\sqrt{x^2+y^2+(z-\alpha)^2}$$

$$\Rightarrow \frac{z^2\alpha^2}{a^2} + a^2 - 2z\alpha = x^2 + y^2 + z^2 + \alpha^2 - 2z\alpha$$

$$\text{or, } x^2 + y^2 + z^2\left(1-\frac{\alpha^2}{a^2}\right) = a^2 - \alpha^2$$

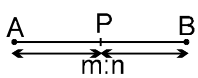
$$\Rightarrow \frac{x^2}{a^2-\alpha^2} + \frac{y^2}{a^2-\alpha^2} + \frac{z^2}{a^2} = 1$$

This is the required locus.

Section formula

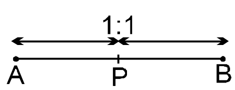
If point P divides the distance between the points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) in the ratio of m : n, internally then coordinates of P are given as

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$$



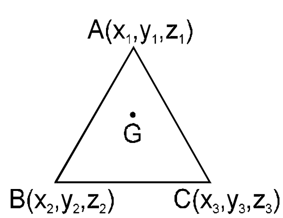
Note :-Mid point

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$



Centroid of a triangle

$$G \equiv \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$



Incentre of triangle ABC

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}, \frac{az_1+bz_2+cz_3}{a+b+c}\right)$$

Where AB = c, BC = a, CA = b

Centroid of a tetrahedron

A (x_1, y_1, z_1) B (x_2, y_2, z_2) C (x_3, y_3, z_3) and D (x_4, y_4, z_4) are the vertices of a tetrahedron, then coordinate of its centroid (G) is given as

$$\left(\frac{\sum_{i=1}^{4} x_i}{4}, \frac{\sum_{i=1}^{4} y_i}{4}, \frac{\sum_{i=1}^{4} z_i}{4}\right)$$

Solved Examples

Ex.33 Show that the points A(2, 3, 4), B(–1, 2, –3) and C(–4, 1, –10) are collinear. Also find the ratio in which C divides AB.

Sol. Given A ≡ (2, 3, 4), B ≡ (–1, 2, –3), C ≡ (– 4, 1, –10).

Let C divide AB internally in the ratio k : 1, then

$$C \equiv \left(\frac{-k+2}{k+1}, \frac{2k+3}{k+1}, \frac{-3k+4}{k+1}\right)$$

$$\therefore \frac{-k+2}{k+1} = -4 \quad \Rightarrow \quad 3k = -6 \Rightarrow \quad k = -2$$

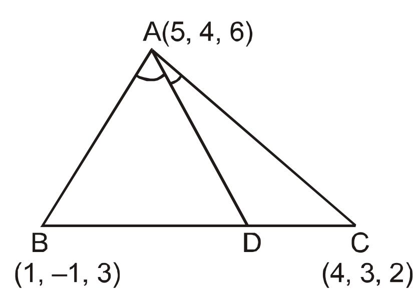
For this value of k, $\frac{2k+3}{k+1} = 1$, and $\frac{-3k+4}{k+1} = -10$

Since k < 0, therefore C divides AB externally in the ratio 2 : 1 and points A, B, C are collinear.

Ex.34 The vertices of a triangle are A(5, 4, 6), B(1, –1, 3) and C(4, 3, 2). The internal bisector of ∠BAC meets BC in D. Find AD.

Sol. AB = $\sqrt{4^2+5^2+3^2} = 5\sqrt{2}$

AC = $\sqrt{1^2+1^2+4^2} = 3\sqrt{2}$



Since AD is the internal bisector of $\angle BAC$

$\therefore \dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{5}{3}$

$\therefore$ D divides BC internally in the ratio 5 : 3

$\therefore D \equiv \left(\dfrac{5\times4+3\times1}{5+3}, \dfrac{5\times3+3(-1)}{5+3}, \dfrac{5\times2+3\times3}{5+3}\right)$

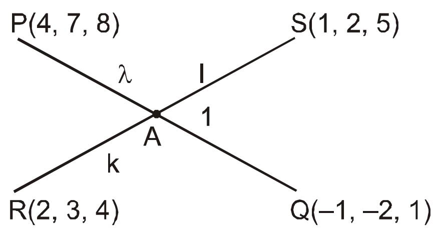
or, $D = \left(\dfrac{23}{8}, \dfrac{12}{8}, \dfrac{19}{8}\right)$

$\therefore AD = \sqrt{\left(5-\dfrac{23}{8}\right)^2 + \left(4-\dfrac{12}{8}\right)^2 + \left(6-\dfrac{19}{8}\right)^2}$

$= \dfrac{\sqrt{1530}}{8}$ unit

Ex.35 If the points P, Q, R, S are (4, 7, 8), (– 1, – 2, 1), (2, 3, 4) and (1,2,5) respectively, show that PQ and RS intersect. Also find the point of intersection.

Sol. Let the lines PQ and RS intersect at point A.



Let A divide PQ in the ratio $\lambda : 1$, then

$A \equiv \left(\dfrac{-\lambda+4}{\lambda+1}, \dfrac{-2\lambda+7}{\lambda+1}, \dfrac{\lambda+8}{\lambda+1}\right)$. .... (1)

Let A divide RS in the ratio k : 1, then

$A \equiv \left(\dfrac{k+2}{k+1}, \dfrac{2k+3}{k+1}, \dfrac{5k+4}{k+1}\right)$ ..... (2)

From (1) and (2), we have,

$\dfrac{-\lambda+4}{\lambda+1} = \dfrac{k+2}{k+1}$ ..... (3)

$\dfrac{-2\lambda+7}{\lambda+1} = \dfrac{2k+3}{k+1}$ .... (4)

$\dfrac{\lambda+8}{\lambda+1} = \dfrac{5k+4}{k+1}$ ..... (5)

From (3), $-\lambda k - \lambda + 4k + 4 = \lambda k + 2\lambda + k + 2$

or $2\lambda k + 3\lambda - 3k - 2 = 0$ ..... (6)

From (4), $-2\lambda k - 2\lambda + 7k + 7 = 2\lambda k + 3\lambda + 2k + 3$

or $4\lambda k + 5\lambda - 5k - 4 = 0$ ..... (7)

Multiplying equation (6) by 2, and subtracting from equation (7), we get $-\lambda + k = 0$ or , $\lambda = k$

Putting $\lambda = k$ in equation (6),

we get $2\lambda^2 + 3\lambda - 3\lambda - 2 = 0$ or, $\lambda = \pm 1$.

But $\lambda \neq -1$, as the co-ordinates of P would be undefined and in this case

PQ || RS, which is not true.

$\therefore \lambda = 1 = k$.

Clearly $\lambda = k = 1$ satisfies eqn. (5).

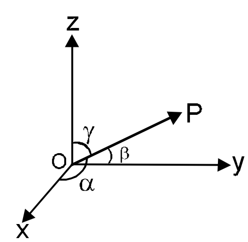
Hence our assumption is correct

$\therefore A \equiv \left(\dfrac{-1+4}{2}, \dfrac{-2+7}{2}, \dfrac{1+8}{2}\right)$ or, $A \equiv \left(\dfrac{3}{2}, \dfrac{5}{2}, \dfrac{9}{2}\right)$.

DIRECTION COSINES AND DIRECTION RATIOS

(i) Direction cosines : Let α, β, γ be the angles which a directed line makes with the positive directions of the axes of x, y and z respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the direction cosines of the line. The direction cosines are usually denoted by ℓ, m, n.

Thus $\ell = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.



(ii) If ℓ, m, n be the direction cosines of a line, then $\ell^2 + m^2 + n^2 = 1$

(iii) Direction ratios : Let a, b, c be proportional to the direction cosines ℓ, m, n then a, b, c are called the direction ratios.

If a, b, c, are the direction ratios of any line L, then $a\hat{i} + b\hat{j} + c\hat{k}$ will be a vector parallel to the line L.

If ℓ, m, n are direction cosines of line L, then $\ell\hat{i} + m\hat{j} + n\hat{k}$ is a unit vector parallel to the line L.

(iv) If ℓ, m, n be the direction cosines and a, b, c be the direction ratios of a vector, then

$$\left(\ell=\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\frac{b}{\sqrt{a^2+b^2+c^2}}, n=\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$$

or $\ell = \frac{-a}{\sqrt{a^2+b^2+c^2}}$, $m = \frac{-b}{\sqrt{a^2+b^2+c^2}}$,

$n = \frac{-c}{\sqrt{a^2+b^2+c^2}}$

(v) If OP = r, when O is the origin and the direction cosines of OP are ℓ, m, n then the coordinates of P are (ℓr, mr, nr).

If direction cosines of the line AB are ℓ, m,n, |AB| = r, and the coordinates of A is (x_1, y_1, z_1) then the coordinates of B is given as $(x_1 + r\ell, y_1 + rm, z_1 + rn)$

(vi) If the coordinates P and Q are (x_1, y_1, z_1) and (x_2, y_2, z_2), then the direction ratios of line PQ are, $a = x_2 - x_1$, $b = y_2 - y_1$ & $c = z_2 - z_1$ and the direction cosines of line PQ are $\ell=\frac{x_2-x_1}{|PQ|}$,

$m=\frac{y_2-y_1}{|PQ|}$ and $n=\frac{z_2-z_1}{|PQ|}$.

(vii) Direction cosines of axes : Since the positive x–axis makes angles 0º, 90º, 90º with axes of x, y and z respectively. Therefore

Direction cosines of x–axis are (1, 0, 0)

Direction cosines of y–axis are (0, 1, 0)

Direction cosines of z–axis are (0, 0, 1)

Solved Examples

Ex.36 If a line makes angles α, β, γ with the co-ordinate axes, prove that $\sin^2\alpha + \sin^2\beta + \sin\gamma^2 = 2$.

Sol. Since a line makes angles α, β, γ with the co-ordinate axes,

hence cosα, cosβ, cosγ are its direction cosines

$\therefore\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$

Ex.37 Find the direction cosines ℓ, m, n of a line which are connected by the relations

$\ell + m + n = 0,\ 2mn + 2m\ell - n\ell = 0$

Sol. Given, $\ell + m + n = 0$ ..... (1)

$2mn + 2m\ell - n\ell = 0$ ..... (2)

From (1), $n = -(\ell + m)$.

Putting $n = -(\ell + m)$ in equation (2), we get,

$-2m(\ell + m) + 2m\ell + (\ell + m)\ell = 0$

or, $-2m\ell - 2m^2 + 2m\ell + \ell^2 + m\ell = 0$

or, $\ell^2 + m\ell - 2m^2 = 0$

or, $\left(\frac{\ell}{m}\right)^2 + \left(\frac{\ell}{m}\right) - 2 = 0$ [dividing by m^2]

or $\frac{\ell}{m} = \frac{-1\pm\sqrt{1+8}}{2} = \frac{-1\pm 3}{2} = 1, -2$

Case I. when $\frac{\ell}{m} = 1$: In this case $m = \ell$

From (1), $2\ell + n = 0 \Rightarrow n = -2\ell$

$\therefore\ \ell : m : n = 1 : 1 : -2$

$\therefore$ Direction ratios of the line are 1, 1, – 2

$\therefore$ Direction cosines are

$\pm\frac{1}{\sqrt{1^2+1^2+(-2)^2}}, \pm\frac{1}{\sqrt{1^2+1^2+(-2)^2}}, \pm\frac{-2}{\sqrt{1^2+1^2+(-2)^2}}$

$\Rightarrow \frac{1}{\sqrt6}, \frac{1}{\sqrt6}, \frac{-2}{\sqrt6}$ or $-\frac{1}{\sqrt6}, -\frac{1}{\sqrt6}, \frac{2}{\sqrt6}$

Case II. When $\frac{\ell}{m} = -2$: In this case $\ell = -2m$

From (1), $-2m + m + n = 0 \Rightarrow n = m$

$\therefore\ \ell : m : n = -2m : m : m = -2 : 1 : 1$

$\therefore$ Direction ratios of the line are – 2, 1, 1.

$\therefore$ Direction cosines are

$\pm\frac{-2}{\sqrt{(-2)^2+1^2+1^2}}, \pm\frac{1}{\sqrt{(-2)^2+1^2+1^2}}, \pm\frac{1}{\sqrt{(-2)^2+1^2+1^2}}$

$\frac{-2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}$ or $\frac{2}{\sqrt6}, \frac{-1}{\sqrt6}, \frac{-1}{\sqrt6}$

(viii) Projection of a line on another line :

Let PQ be a line segment with $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and let L be a straight line whose d.c.'s are l, m, n. Note that l, m, n are d.c.'s of line L, not d.r.'s. Then the length of projection of PQ on the line L is Projection $= |l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$

(ix) Angle between two lines :

Let θ be the angle between the lines with d.c.'s l_1, m_1, n_1 and l_2, m_2, n_2 then $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$
Also $\sin^2\theta = \Sigma(l_1 m_2 - l_2 m_1)^2$

(x) Perpendicularity and parallelism :

Let the two lines have their d.c.'s given by l_1, m_1, n_1 and l_2, m_2, n_2 respectively then they are perpendicular if $\theta = 90^\circ$ i.e. $\cos\theta = 0$, *i.e.* $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.

Also the two lines are parallel if $\theta = 0$ i.e. $\sin\theta = 0$, i.e. $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$

* If instead of d.c.'s, d.r.'s a_1, b_1, c_1 and a_2, b_2, c_2 are given, then the lines are perpendicular if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$ and parallel if $a_1/a_2 = b_1/b_2 = c_1/c_2$.

* If θ is the angle between the lines then

$$\cos\theta = \frac{(a_1 a_2 + b_1 b_2 + c_1 c_2)}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

A LINE IN SPACE

Equation of a line :

(i) A straight line in space is characterised by the intersection of two planes which are not parallel and therefore, the equation of a straight line is a solution of the system constituted by the equations of the two planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$. This form is also known as non–symmetrical form.

(ii) The equation of a line passing through the point (x_1, y_1, z_1) and having direction ratios a, b, c is $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} = r$. This form is called symmetric form. A general point on the line is given by $(x_1 + ar, y_1 + br, z_1 + cr)$.

(iii) Vector equation: Vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$ where λ is a scalar. **H**

(iv) Equation of the line through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Reduction of non-symmetrical form to symmetrical form :

(v) Vector equation of a straight line passing through two points with position vectors $\vec{a}$ & $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

(vi) Reduction of cartesion form of equation of a line to vector form & vice versa

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \Leftrightarrow$$

$$\vec{r} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}).$$

Some particular straight lines :

	Straight lines	**Equation**
(i)	Through the origin	$y = mx, z = nx$
(ii)	x-axis	$y = 0, z = 0$
(iii)	y-axis	$x = 0, z = 0$
(iv)	z-axis	$x = 0, y = 0$
(v)	|| to x-axis	$y = p, z = q$
(vi)	|| to y-axis	$x = h, z = q$
(vii)	|| to z-axis	$x = h, y = p$

Solved Examples

Ex.38 Find the equation of the line through the points (3, 4, –7) and (1, – 1, 6) in vector form as well as in cartesian form.

Sol. Let $A \equiv (3, 4, -7), B \equiv (1, -1, 6)$

Now $\vec{a} = \overrightarrow{OA} = 3\hat{i} + 4\hat{j} - 7\hat{k}$,

$\vec{b} = \overrightarrow{OB} = \hat{i} - \hat{j} + 6\hat{k}$

Equation of the line through A($\vec{a}$) and B($\vec{b}$) is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

or $\vec{r} = 3\hat{i} + 4\hat{j} - 7\hat{k} + t(-2\hat{i} - 5\hat{j} + 13\hat{k})$ ... (1)

Equation in cartesian form :

Equation of AB is $\frac{x-3}{3-1} = \frac{y-4}{4+1} = \frac{z+7}{-7-6}$ or,

$\frac{x-3}{2} = \frac{y-4}{5} = \frac{z+7}{-13}$

Ex.39 Find the co-ordinates of those points on the line $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ which is at a distance of 3 units from point (1, – 2, 3).

Sol. Given line is $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{6}$ ..... (1)

Let $P \equiv (1, -2, 3)$

Direction ratios of line (1) are 2, 3, 6

$\therefore$ Direction cosines of line (1) are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

Equation of line (1) may be written as

$$\frac{x-1}{\frac{2}{7}} = \frac{y+2}{\frac{3}{7}} = \frac{z-3}{\frac{6}{7}} \quad ..... (2)$$

Co-ordinates of any point on line (2) may be taken as $\left(\frac{2}{7}r + 1, \frac{3}{7}r - 2, \frac{6}{7}r + 3\right)$

Let $Q \equiv \left(\frac{2}{7}r + 1, \frac{3}{7}r - 2, \frac{6}{7}r + 3\right)$

Distance of Q from P = | r |

According to question | r | = 3 $\quad \therefore \quad r = \pm 3$

Putting the value of r, we have

$Q \equiv \left(\frac{13}{7}, -\frac{5}{7}, \frac{39}{7}\right)$ or $Q \equiv \left(\frac{1}{7}, -\frac{23}{7}, \frac{3}{7}\right)$

Ex.40 Find the equation of the line drawn through point (1, 0, 2) to meet at right angles the line

$$\frac{x+1}{3} = \frac{y-2}{-2} = \frac{z+1}{-1}$$

Sol. Given line is $\frac{x+1}{3} = \frac{y-2}{-2} = \frac{z+1}{-1}$ ..... (1)

Let $P \equiv (1, 0, 2)$

Co-ordinates of any point on line (1) may be taken as

$Q \equiv (3r - 1, -2r + 2, -r - 1)$

Direction ratios of PQ are $3r - 2, -2r + 2, -r - 3$

Direction ratios of line AB are 3, – 2, – 1

Since PQ $\perp$ AB

$\therefore$ $3(3r - 2) - 2(-2r + 2) - 1(-r - 3) = 0$

$\Rightarrow 9r - 6 + 4r - 4 + r + 3 = 0 \quad \Rightarrow 14r = 7$

$\Rightarrow r = \frac{1}{2}$

Therefore, direction ratios of PQ are $-\frac{1}{2}, 1, -\frac{7}{2}$

or, – 1, 2, – 7

Equation of line PQ is $\frac{x-1}{-1} = \frac{y-0}{2} = \frac{z-2}{-7}$ or,

$\frac{x-1}{1} = \frac{y}{-2} = \frac{z-2}{7}$

Ex.41 Show that the two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect. Find also the point of intersection of these lines.

Sol. Given lines are $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ ..... (1)

and $\frac{x-4}{5} = \frac{y-1}{2} = \frac{z-0}{1}$ ..... (2)

Any point on line (1) is P $(2r + 1, 3r + 2, 4r + 3)$

and any point on line (2) is Q $(5\lambda + 4, 2\lambda + 1, \lambda)$

Lines (1) and (2) will intersect if P and Q coincide for some value of λ and r.

$\therefore\ 2r + 1 = 5\lambda + 4 \Rightarrow 2r - 5\lambda = 3$ ..... (3)

$3r + 2 = 2\lambda + 1 \Rightarrow 3r - 2\lambda = -1$ ..... (4)

$4r + 3 = \lambda \Rightarrow 4r - \lambda = -3$ ..... (5)

Solving (3) and (4), we get $r = -1, \lambda = -1$

Clearly these values of r and λ satisfy eqn. (5)

Now $P \equiv (-1, -1, -1)$

Hence lines (1) and (2) intersect at $(-1, -1, -1)$.

ANGLE BETWEEN TWO LINE SEGMENTS

If two lines have direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively, then we can consider two vectors parallel to the lines as $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and angle between them can be given as.

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

(i) The lines will be perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) The lines will be parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) Two parallel lines have same direction cosines i.e. $\ell_1 = \ell_2, m_1 = m_2, n_1 = n_2$

Solved Examples

Ex.42 What is the angle between the lines whose direction cosines are

$$-\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2} \text{ and } -\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}.$$

Sol Let θ be the required angle, then

$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

$$= \left(-\frac{\sqrt{3}}{4}\right)\left(-\frac{\sqrt{3}}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \left(-\frac{\sqrt{3}}{2}\right).\left(\frac{\sqrt{3}}{2}\right)$$

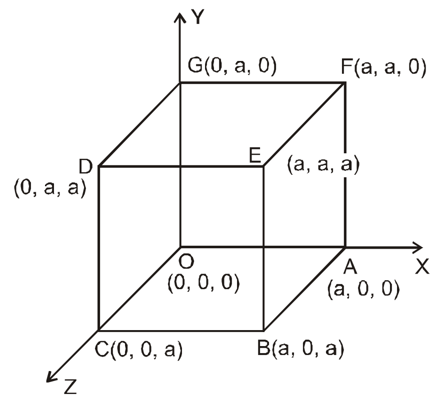
$$= \frac{3}{16} + \frac{1}{16} - \frac{3}{4} = -\frac{1}{2}$$

$\Rightarrow \theta = 120°.$

Ex.43 Find the angle between any two diagonals of a cube.

Sol. The cube has four diagonals

OE, AD, CF and GB



The direction ratios of OE are a, a, a or 1, 1, 1

$\therefore$ its direction cosines are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Direction ratios of AD are $-a, a, a$ or $-1, 1, 1$.

$\therefore$ its direction cosines are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Similarly, direction cosines of CF and GB respectively are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

We take any two diagonals, say OE and AD

Let θ be the acute angle between them, then

$$\cos\theta = \left|\left(\frac{1}{\sqrt{3}}\right)\left(\frac{-1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right).\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right).\left(\frac{1}{\sqrt{3}}\right)\right| = \frac{1}{3}$$

or, $\theta = \cos^{-1}\left(\frac{1}{3}\right)$.

Ex.44 Find the angle between the lines $x - 3y - 4 = 0$, $4y - z + 5 = 0$ and $x + 3y - 11 = 0$, $2y - z + 6 = 0$.

Sol. Given lines are $\left.\begin{matrix} x - 3y - 4 = 0 \\ 4y - z + 5 = 0 \end{matrix}\right\}$ ..... (1)

and $\left.\begin{matrix} x + 3y - 11 = 0 \\ 2y - z + 6 = 0 \end{matrix}\right\}$ ..... (2)

Let ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 be the direction cosines of lines (1) and (2) respectively

$\because$ line (1) is perpendicular to the normals of each of the planes $x - 3y - 4 = 0$ and $4y - z + 5 = 0$

$\therefore\ \ell_1 - 3m_1 + 0.n_1 = 0$ ..... (3)

and $0\ell_1 + 4m_1 - n_1 = 0$ ..... (4)

Solving equations (3) and (4), we get $\frac{\ell_1}{3-0} = \frac{m_1}{0-(-1)} = \frac{n_1}{4-0}$ or, $\frac{\ell_1}{3} = \frac{m_1}{1} = \frac{n_1}{4} = k$ (let).

Since line (2) is perpendicular to the normals of each of the planes

$x + 3y - 11 = 0$ and $2y - z + 6 = 0$,

$\therefore\ \ell_2 + 3m_2 = 0$ ..... (5)

and $2m_2 - n_2 = 0$ ..... (6)

$\therefore\ \ell_2 = -3m_2$ or, $\frac{\ell_2}{-3} = m_2$ and $n_2 = 2m_2$ or, $\frac{n_2}{2} = m_2$.

$\therefore\ \frac{\ell_2}{-3} = \frac{m_2}{1} = \frac{n_2}{2} = t$ (let).

If θ be the angle between lines (1) and (2), then

$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$

$= (3k)(-3t) + (k)(t) + (4k)(2t)$

$= -9kt + kt + 8kt = 0$

$\therefore\ \theta = 90°$.

To find image of a point w. r. t a line :

Let $L \equiv \frac{x - x_2}{a} = \frac{y - y_2}{b} = \frac{z - z_2}{c}$ is a given line

Let (x', y', z') is the image of the point $P(x_1, y_1, z_1)$ with respect to the line L. Then

(i) $a(x_1 - x') + b(y_1 - y') + c(z_1 - z') = 0$

(ii) $$\frac{\frac{x_1 + x'}{2} - x_2}{a} = \frac{\frac{y_1 + y'}{2} - y_2}{b} = \frac{\frac{z_1 + z'}{2} - z_2}{c} = \lambda$$

from (ii) get the value of x', y', z' in terms of λ as $x' = 2a\lambda + 2x_2 - x_1$,

$y' = 2b\lambda + 2y_2 - y_1$, $z' = 2c\lambda + 2z_2 - z_1$

now put the values of x', y', z' in (i) get λ and resubtitute the value of λ to to get $(x'\ y'\ z')$.

Solved Examples

Ex.45 Find the length of the perpendicular from P $(2, -3, 1)$ to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$.

Sol. Given line is $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$ ..... (1)

$P \equiv (2, -3, 1)$

Co-ordinates of any point on line (1) may be taken as $Q \equiv (2r - 1, 3r + 3, -r - 2)$

Direction ratios of PQ are $2r - 3, 3r + 6, -r - 3$

Direction ratios of AB are $2, 3, -1$

Since $PQ \perp AB$

$\therefore\ 2(2r - 3) + 3(3r + 6) - 1(-r - 3) = 0$

or, $14r + 15 = 0$ $\therefore$ $r = \frac{-15}{14}$

$\therefore\ Q \equiv \left(\frac{-22}{7}, \frac{-3}{14}, \frac{-13}{14}\right)$

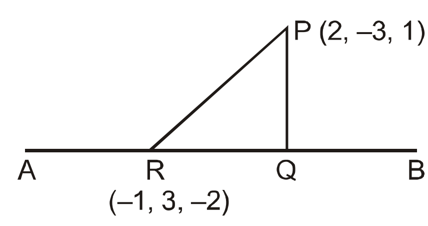
$\therefore\ PQ = \sqrt{\frac{531}{14}}$ units.

Second method : Given line is

$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$

$P \equiv (2, -3, 1)$

Direction ratios of line (1) are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}$



RQ = length of projection of RP on AB

$= \left|\frac{2}{\sqrt{14}}(2+1) + \frac{3}{\sqrt{14}}(-3-3) - \frac{1}{\sqrt{14}}(1+2)\right| = \frac{15}{\sqrt{14}}$

$PR^2 = 3^2 + 6^2 + 3^2 = 54$

$\therefore\ PQ = \sqrt{PR^2 - RQ^2} = \sqrt{54 - \frac{225}{14}} = \sqrt{\frac{531}{14}}$ units.

Condition that two given lines should intersect i.e. be coplanar i.e. testing of skewness or coplanarity of two given lines. Equation of plane containing two intersecting lines :

Let the two lines be

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \quad .... (1)$$

and $$\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2} \quad .... (2)$$

These lines will coplanar if

$$\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

the plane containing the two lines is

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Condition of coplanarity if both lines are in general from :

Let the lines be

$ax+by+cz+d=0=a'x+b'y+c'z+d'$

and $\alpha x+\beta y+\gamma z+\delta=0=\alpha' x+\beta' y+\gamma' z+\delta'$

These are coplanar if

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix}=0$$

Skew lines :

The straight lines which are not parallel and non-coplanar i.e. non-intersecting are called skew lines.

Shortest distance between two skew straight lines :

Shortest distance between two skew lines is perpendicular to both.

* **If the equations are in cartesian form :**

Suppose the equation of the lines are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad .... (1)$$

and $$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'} \quad .... (2)$$

Then shortest distance between them is given by

$$\text{S.D.}=\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \sqrt{\{\Sigma(mn'-m'n)^2\}}$$

* If the lines intersect, the S.D.between them is zero. Therefore

$$\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix}=0$$

* **If the equations are in vector form :**

Suppose the equation of the lines are

$\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$

Then shortest distance between them is given by

$$\text{S.D.}=\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$$

Solved Examples

Ex.46 Find the shortest distance and the vector equation of the line of shortest distance between the lines given by

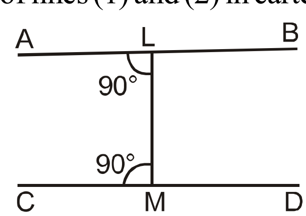
$\vec{r}=3\hat{i}+8\hat{j}+3\hat{k}+\lambda(3\hat{i}-\hat{j}+\hat{k})$ and

$\vec{r}=-3\hat{i}-7\hat{j}+6\hat{k}+\mu(-3\hat{i}+2\hat{j}+4\hat{k})$

Sol. Given lines are $\vec{r}=3\hat{i}+8\hat{j}+3\hat{k}+\lambda(3\hat{i}-\hat{j}+\hat{k})$ .... (1)

and $\vec{r}=-3\hat{i}-7\hat{j}+6\hat{k}+\mu(-3\hat{i}+2\hat{j}+4\hat{k})$ ..... (2)

Equation of lines (1) and (2) in cartesian form is



AB : $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}=\lambda$

and CD : $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}=\mu$

Let $L\equiv(3\lambda+3,-\lambda+8,\lambda+3)$

and $M\equiv(-3\mu-3,2\mu-7,4\mu+6)$

Direction ratios of LM are

$3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3.$

Since $\quad LM \perp AB$

$\therefore\ 3(3\lambda + 3\mu + 6) - 1(-\lambda - 2\mu + 15) + 1(\lambda - 4\mu - 3) = 0$

or, $11\lambda + 7\mu = 0 \qquad \text{..... (5)}$

Again $LM \perp CD$

$\therefore\ -3(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$

or, $-7\lambda - 29\mu = 0 \qquad \text{..... (6)}$

Solving (5) and (6), we get $\lambda = 0, \mu = 0$

$\therefore\ L \equiv (3, 8, 3), M \equiv (-3, -7, 6)$

Hence shortest distance LM

$= \sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2} = \sqrt{270} = 3\sqrt{30}$ units

Vector equation of LM is

$\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + t\left(6\hat{i} + 15\hat{j} - 3\hat{k}\right)$

Note : Cartesian equation of LM is $\dfrac{x-3}{6} = \dfrac{y-8}{15} = \dfrac{z-3}{-3}$.

Ex.47 Prove that the shortest distance between any two opposite edges of a tetrahedron formed by the planes $y + z = 0, x + z = 0, x + y = 0, x + y + z = \sqrt{3}\,a$ is $\sqrt{2}\,a$.

Sol. Given planes are $y + z = 0 \qquad \text{..... (i)}$

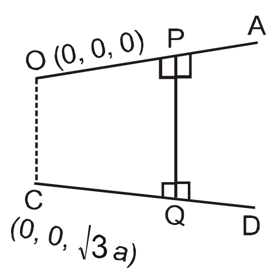
$x + z = 0 \qquad \text{..... (ii)}$

$x + y = 0 \qquad \text{..... (iii)}$

$x + y + z = \sqrt{3}\,a \qquad \text{..... (iv)}$

Clearly planes (i), (ii) and (iii) meet at O(0, 0, 0)

Let the tetrahedron be OABC



Let the equation to one of the pair of opposite edges OA and BC be

$y + z = 0, x + z = 0 \qquad \text{..... (1)}$

$x + y = 0, x + y + z = \sqrt{3}\,a \qquad \text{..... (2)}$

equation (1) and (2) can be expressed in symmetrical form as

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{-1} \qquad \text{..... (3)}$$

and, $$\frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-\sqrt{3}\,a}{0} \qquad \text{..... (4)}$$

d. r. of OA and BC are respectively (1, 1, – 1) and (1, – 1, 0).

Let PQ be the shortest distance between OA and BC having direction cosines (ℓ, m, n)

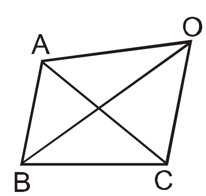
$\therefore$ PQ is perpendicular to both OA and BC.

$\therefore\ \ell + m - n = 0 \quad$ and $\quad \ell - m = 0$

Solving (5) and (6), we get, $\dfrac{\ell}{1} = \dfrac{m}{1} = \dfrac{n}{2} = k$ (say)

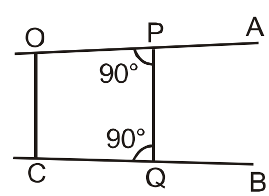
also, $\ell^2 + m^2 + n^2 = 1$

$\therefore\ k^2 + k^2 + 4k^2 = 1 \Rightarrow k = \pm\dfrac{1}{\sqrt{6}}$



$\therefore\ \ell = \pm\dfrac{1}{\sqrt{6}}, m = \pm\dfrac{1}{\sqrt{6}}, n = \pm\dfrac{2}{\sqrt{6}}$

Shortest distance between OA and BC



i.e. PQ = The length of projection of OC on PQ

$= |(x_2 - x_1)\ell + (y_2 - y_1)m + (z_2 - z_1)n|$

$= \left|0 \cdot \dfrac{1}{\sqrt{6}} + 0 \cdot \dfrac{1}{\sqrt{6}} + \sqrt{3}\,a \cdot \dfrac{2}{\sqrt{6}}\right| = \sqrt{2}\,a.$

A PLANE

If line joining any two points on a surface lies completely on it then the surface is a plane.

OR

If line joining any two points on a surface is perpendicular to some fixed straight line. Then this surface is called a plane. This fixed line is called the normal to the plane.

Equations of a Plane :

The equation of every plane is of the first degree i.e. of the form $ax + by + cz + d = 0$, in which a, b, c are constants, where $a^2 + b^2 + c^2 \neq 0$ (i.e. $a, b, c \neq 0$ simultaneously).

Plane Parallel to the Coordinate Planes :

(i) Equation of y-z plane is $x = 0$.

(ii) Equation of z-x plane is $y = 0$.

(iii) Equation of x-y plane is $z = 0$.

(iv) Equation of the plane parallel to x-y plane at a distance c (if $c > 0$, towards positive z-axis) is $z = c$. Similarly, planes parallel to y-z plane and z-x plane are respectively $x = c$ and $y = c$.

(v) The equation $(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$ represents a plane containing the point with position vector $\vec{r}_0$, where $\vec{n}$ is a vector normal to the plane.

The above equation can also be written as $\vec{r} \cdot \vec{n} = d$, where $d = \vec{r}_0 \cdot \vec{n}$

Equations of Planes Parallel to the Axes :

If $a = 0$, the plane is parallel to x-axis i.e. equation of the plane parallel to x-axis is $by + cz + d = 0$

Similarly, equations of planes $\parallel$ to y-axis and $\parallel$ to z-axis are $ax + cz + d = 0$ and $ax + by + d = 0$, respectively.

Equation of a Plane through Origin :

Equation of plane passing through origin is $ax + by + cz = 0$

Equation of a Plane through a given Point :

Let the plane $ax + by + cz + d = 0$ passes through the point (x_1, y_1, z_1) then $ax_1 + by_1 + cz_1 + d = 0$.

$\therefore$ Subtracting, we get $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ which is the required equation.

* Vector form: The equation of a plane passing through a point having position vector $\vec{a}$ & normal to vector $\vec{n}$ is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ or $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

Equation of a Plane in Intercept Form :

Equation of the plane which cuts off intercepts a, b, c from the axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Equation of a Plane in Normal Form :

If the length of the perpendicular distance of the plane from the origin is p and direction cosines of this perpendicular are (l, m, n), then the equation of the plane is $lx + my + nz = p$.

* In solving problems of plane, first consider its normal. In the equation $ax + by + cz + d = 0$, a, b, c are the direction ratios of the normal of the plane.

* Vector equation of a plane normal to unit vector $\hat{n}$ and at a distance d from the origin is $\vec{r} \cdot \vec{n} = d$

Equation of a Plane through three points :

The equation of the plane through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Angle between two planes :

Consider two planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$. Angle between these planes is the angle between their normals.

$$\cos\theta = \frac{aa' + bb' + cc'}{\sqrt{a^2 + b^2 + c^2}\sqrt{a'^2 + b'^2 + c'^2}}$$

$\therefore$ Planes are perpendicular if $aa' + bb' + cc' = 0$ and they are parallel if $a/a' = b/b' = c/c'$.

* **Transformation of the equation of a plane to the normal form:** To reduce any equation $ax + by + cz - d = 0$ to the normal form, first write the constant term on the right hand side and make it positive, then divide each term by $\sqrt{a^2+b^2+c^2}$, where a, b, c are coefficients of x, y and z respectively e.g.

$$\frac{ax}{\pm\sqrt{a^2+b^2+c^2}}+\frac{by}{\pm\sqrt{a^2+b^2+c^2}}+\frac{cz}{\pm\sqrt{a^2+b^2+c^2}}=\frac{d}{\pm\sqrt{a^2+b^2+c^2}}$$

Where (+) sign is to be taken if $d > 0$ and (–) sign is to be taken if $d < 0$.

Planes parallel to a given Plane :

Equation of a plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + d' = 0$. d' is to be found by other given function.

* Any plane parallel to the given plane $ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$.
Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is $\frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

* **Equation of a plane passing through a given point & parallel to the given vectors:**
The equation of a plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ & $\vec{c}$ is $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ (parametric form) where λ & μ are scalars.
or $\vec{r}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot(\vec{b}\times\vec{c})$ (non parametric form)

* A plane $ax+by+cz+d=0$ divides the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2). in the ratio $\left(-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$

* The xy–plane divides the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $-\frac{z_1}{z_2}$.
Similarly yz–plane in $-\frac{x_1}{x_2}$ and zx–plane in $-\frac{y_1}{y_2}$

* Coplanarity of four points
The points $A(x_1\, y_1\, z_1)$, $B(x_2\, y_2\, z_2)$ $C(x_3\, y_3\, z_3)$ and $D(x_4\, y_4\, z_4)$ are coplaner then

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix} = 0$$

very similar in vector method the points $A(\vec{r}_1)$, $B(\vec{r}_2)$, $C(\vec{r}_3)$ and $D(\vec{r}_4)$ are coplanar if

$[\vec{r}_4 - \vec{r}_1, \vec{r}_4 - \vec{r}_2, \vec{r}_4 - \vec{r}_3] = 0$

Solved Examples

Ex.48 Find the equation of the plane upon which the length of normal from origin is 10 and direction ratios of this normal are 3, 2, 6.

Sol. If p be the length of perpendicular from origin to the plane and ℓ, m, n be the direction cosines of this normal, then its equation is

$\ell x + my + nz = p$ ..... (1)

Here $p = 10$

Direction ratios of normal to the plane are 3, 2, 6

$\sqrt{3^2+2^2+6^2} = 7$

$\therefore$ Direction cosines of normal to the required plane are

$\ell = \frac{3}{7}$, $m = \frac{2}{7}$, $n = \frac{6}{7}$

Putting the values of ℓ, m, n, p in (1), equation of required plane is $\frac{3}{7}x + \frac{2}{7}y + \frac{6}{7}z = 10$

or, $3x + 2y + 6z = 70$

Ex.49 Show that the points (0, – 1, 0), (2, 1, – 1), (1, 1, 1), (3, 3, 0) are coplanar.

Sol. Let A ≡ (0, – 1, 0), B ≡ (2, 1, – 1), C ≡ (1, 1, 1) and D ≡ (3, 3, 0)

Equation of a plane through A (0, – 1, 0) is

a (x – 0) + b (y + 1) + c (z – 0) = 0

or, ax + by + cz + b = 0 ..... (1)

If plane (1) passes through B (2, 1, – 1) and C (1, 1, 1)

Then 2a + 2b – c = 0 ..... (2)

and a + 2b + c = 0 ..... (3)

From (2) and (3), we have $\frac{a}{2+2} = \frac{b}{-1-2} = \frac{c}{4-2}$

or, $\frac{a}{4} = \frac{b}{-3} = \frac{c}{2}$ = k (say)

Putting the value of a, b, c, in (1), equation of required plane is 4kx – 3k(y + 1) + 2kz = 0

or, 4x – 3y + 2z – 3 = 0 ..... (2)

Clearly point D (3, 3, 0) lies on plane (2)

Thus point D lies on the plane passing through A, B, C and hence points A, B, C and D are coplanar.

Ex.50 If P be any point on the plane $\ell x + my + nz = p$ and Q be a point on the line OP such that OP . OQ = p^2, show that the locus of the point Q is $p(\ell x + my + nz) = x^2 + y^2 + z^2$.

Sol. Let P ≡ (α, β, γ), Q ≡ (x_1, y_1, z_1)

Direction ratios of OP are α, β, γ and direction ratios of OQ are x_1, y_1, z_1.

Since O, Q, P are collinear, we have

$\frac{\alpha}{x_1} = \frac{\beta}{y_1} = \frac{\gamma}{z_1}$ = k (say) ..... (1)

As P (α, β, γ) lies on the plane $\ell x + my + nz = p$,

$\ell\alpha + m\beta + n\gamma = p$ or $k(\ell x_1 + my_1 + nz_1) = p$ .... (2)

Given OP . OQ = p^2

$\therefore \sqrt{\alpha^2+\beta^2+\gamma^2}\,\sqrt{x_1^2+y_1^2+z_1^2} = p^2$

or, $\sqrt{k^2(x_1^2+y_1^2+z_1^2)}\,\sqrt{x_1^2+y_1^2+z_1^2} = p^2$

or, $k(x_1^2+y_1^2+z_1^2) = p^2$ ..... (3)

On dividing (2) by (3), we get $\frac{\ell x_1 + my_1 + nz_1}{x_1^2+y_1^2+z_1^2} = \frac{1}{p}$

or, $p(\ell x_1 + my_1 + nz_1) = x_1^2+y_1^2+z_1^2$

Hence the locus of point Q is $p(\ell x + my + nz) = x^2 + y^2 + z^2$.

Ex.51 A point P moves on a plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. A plane through P and perpendicular to OP meets the co-ordinate axes in A, B and C. If the planes through A, B and C parallel to the planes x = 0, y = 0, z = 0 intersect in Q, find the locus of Q.

Sol. Given plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ..... (1)

Let P ≡ (h, k, ℓ)

Then $\frac{h}{a} + \frac{k}{b} + \frac{\ell}{c} = 1$ ..... (2)

OP = $\sqrt{h^2+k^2+\ell^2}$

Direction cosines of OP are

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}, \frac{k}{\sqrt{h^2+k^2+\ell^2}}, \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}$$

∴ Equation of the plane through P and normal to OP is

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}x + \frac{k}{\sqrt{h^2+k^2+\ell^2}}y + \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}z = \sqrt{h^2+k^2+\ell^2}$$

or, $hx + ky + \ell z = (h^2 + k^2 + \ell^2)$

$\therefore A \equiv \left(\frac{h^2+k^2+\ell^2}{h}, 0, 0\right), B \equiv \left(0, \frac{h^2+k^2+\ell^2}{k}, 0\right)$,

$C \equiv \left(0, 0, \frac{h^2+k^2+\ell^2}{\ell}\right)$

Let Q ≡ (α, β, γ), then $\alpha = \frac{h^2+k^2+\ell^2}{h}$,

$\beta = \frac{h^2+k^2+\ell^2}{k}, \gamma = \frac{h^2+k^2+\ell^2}{\ell}$ ..... (3)

Now

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{h^2+k^2+\ell^2}{(h^2+k^2+\ell^2)^2} = \frac{1}{(h^2+k^2+\ell^2)}$ .... (4)

From (3), $h = \dfrac{h^2 + k^2 + \ell^2}{\alpha}$

$\therefore \dfrac{h}{a} = \dfrac{h^2 + k^2 + \ell^2}{a\alpha}$

Similarly $\dfrac{k}{b} = \dfrac{h^2 + k^2 + \ell^2}{b\beta}$ and $\dfrac{\ell}{c} = \dfrac{h^2 + k^2 + \ell^2}{c\gamma}$

$\therefore \dfrac{h^2+k^2+\ell^2}{a\alpha} + \dfrac{h^2+k^2+\ell^2}{b\beta} + \dfrac{h^2+k^2+\ell^2}{c\gamma} = \dfrac{h}{a} + \dfrac{k}{b} + \dfrac{\ell}{c}$

= 1 [from (2)]

or, $\dfrac{1}{a\alpha} + \dfrac{1}{b\beta} + \dfrac{1}{c\gamma} = \dfrac{1}{h^2 + k^2 + \ell^2} = \dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$

[from (4)]

$\therefore$ Required locus of Q (α, β, γ) is

$\dfrac{1}{ax} + \dfrac{1}{by} + \dfrac{1}{cz} = \dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2}$.

Position of two points w.r.t. a plane:

A plane divides the three dimensional space in two equal parts. Two points A $(x_1\ y_1\ z_1)$ and B $(x_2\ y_2\ z_2)$ are on the same side of the plane $ax + by + cz + d = 0$ if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are both positive or both negative and are opposite side of plane if both of these values are in opposite sign.

Solved Examples

Ex.52 Show that the points (1, 2, 3) and (2, – 1, 4) lie on opposite sides of the plane $x + 4y + z - 3 = 0$.

Sol. Since the numbers $1 + 4 \times 2 + 3 - 3 = 9$ and $2 - 4 + 4 - 3 = -1$ are of opposite sign, then points are on opposite sides of the plane.

A plane & a point

(i) Distance of the point (x', y', z') from the plane $ax + by + cz + d = 0$ is given by $\left|\dfrac{ax'+by'+cz'+d}{\sqrt{a^2+b^2+c^2}}\right|$.

(ii) The length of the perpendicular from a point having position vector $\vec{a}$ to plane $\vec{r}\cdot\vec{n} = d$ is $p = \dfrac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}$.

(iii) The coordinates of the foot of perpendicular from the point (x_1, y_1, z_1) to the plane

$ax + by + cz + d = 0$ are $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$

$= -\dfrac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

(iv) To find image of a point w.r.t. a plane.

Let P (x_1, y_1, z_1) is a given point and

$ax + by + cz + d = 0$ is given plane.

Let (x', y', z') is the image of the point, then

(v) $x' - x_1 = \lambda a, \quad y' - y_1 = \lambda b, \quad z' - z_1 = \lambda c$

$\Rightarrow x' = \lambda a + x_1, \quad y' = \lambda b + y_1, \quad z' = \lambda c + z_1$ ...... (i)

(vi) $a\left(\dfrac{x' + x_1}{2}\right) + b\left(\dfrac{y' + y_1}{2}\right) + c\left(\dfrac{z' + z_1}{2}\right) = 0$ ..... (ii)

from (i) put the values of x', y', z' in (ii) and get the values of λ and subtitute in (i) to get $(x'\ y'\ z')$.

The coordinate of the image of point (x_1, y_1, z_1) w.r.t the plane $ax + by + cz + d = 0$ are given

by $\dfrac{x' - x_1}{a} = \dfrac{y' - y_1}{b} = \dfrac{z' - z_1}{c} = -2\,\dfrac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

(vii) The distance between two parallel planes $ax + by + cx + d = 0$ and $ax + by + cx + d' = 0$

is $\dfrac{|d - d'|}{\sqrt{a^2 + b^2 + c^2}}$

Solved Examples

Ex.53 Find the image of the point P (3, 5, 7) in the plane $2x + y + z = 0$.

Sol. Given plane is $2x + y + z = 0$ ..... (1)

$P \equiv (3, 5, 7)$

Direction ratios of normal to plane (1) are 2, 1, 1

Let Q be the image of point P in plane (1). Let PQ meet plane (1) in R

then PQ $\perp$ plane (1)

Let $R \equiv (2r + 3, r + 5, r + 7)$

Since R lies on plane (1)

$\therefore\ 2(2r + 3) + r + 5 + r + 7 = 0$ or, $6r + 18 = 0$

$\therefore\ r = -3$

$\therefore\ R \equiv (-3, 2, 4)$

Let $Q \equiv (\alpha, \beta, \gamma)$

Since R is the middle point of PQ

$\therefore\ -3 = \dfrac{\alpha + 3}{2} \Rightarrow \alpha = -9$

$2 = \dfrac{\beta + 5}{2} \Rightarrow \beta = -1$

$4 = \dfrac{\gamma + 7}{2} \Rightarrow \gamma = 1$

$\therefore\ Q = (-9, -1, 1)$.

Ex.54 Find the distance between the planes

$2x - y + 2z = 4$ and $6x - 3y + 6z = 2$.

Sol. Given planes are $2x - y + 2z - 4 = 0$ .....(1)

and $6x - 3y + 6z - 2 = 0$ .....(2)

We find that $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

Hence planes (1) and (2) are parallel.

Plane (2) may be written as $2x - y + 2z - \dfrac{2}{3} = 0$ ..... (3)

$\therefore$ Required distance between the planes $= \dfrac{\left|4 - \frac{2}{3}\right|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \dfrac{10}{3.3} = \dfrac{10}{9}$

Bisectors of angles between two planes :

Let the equations of the two planes be $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$.

Then equations of bisectors of angles between them are given by

$$\frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}} = \pm \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}} \quad (*)$$

(i) Equation of bisector of the angle containing origin : First make both constant terms positive. Then +ve sign in (*) give the bisector of the angle which contains the origin.

(ii) Bisector of acute/obtuse angle : First making both constant terms positive, $aa_1 + bb_1 + cc_1 > 0$

$\Rightarrow$ origin lies in obtuse angle < 0

$\Rightarrow$ origin lies in acute angle

Family of planes

(i) Any plane passing through the line of intersection of non–parallel planes or equation of the plane through the given line in non symmetric form.

$a_1x + b_1y + c_1z + d_1 = 0$ &

$a_2x + b_2y + c_2z + d_2 = 0$ is

$a_1x + b_1y + c_1z + d_1 + \lambda (a_2x + b_2y + c_2z + d_2) = 0$, where $\lambda \in R$

(ii) The equation of plane passing through the intersection of the planes $\vec{r}.\vec{n}_1 = d_1$ & $\vec{r}.\vec{n}_2 = d_2$ is $\vec{r}.(n_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$ where λ is arbitrary scalar

Solved Examples

Ex.55 The plane $x - y - z = 4$ is rotated through 90° about its line of intersection with the plane $x + y + 2z = 4$. Find its equation in the new position.

Sol. Given planes are $x - y - z = 4$ ..... (1)

and $x + y + 2z = 4$ ..... (2)

Since the required plane passes through the line of intersection of planes (1) and (2)

$\therefore$ its equation may be taken as

$x + y + 2z - 4 + k (x - y - z - 4) = 0$

or $(1 + k)x + (1 - k)y + (2 - k)z - 4 - 4k = 0$ ... (3)

Since planes (1) and (3) are mutually perpendicular,

$\therefore\ (1 + k) - (1 - k) - (2 - k) = 0$

or, $1 + k - 1 + k - 2 + k = 0$ or $k = \dfrac{2}{3}$

Putting $k = \dfrac{2}{3}$ in equation (3), we get $5x + y + 4z = 20$

This is the equation of the required plane.

Ex.56 Find the equation of the plane through the point (1, 1, 1) which passes through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$.

Sol. Given planes are $x + y + z - 6 = 0$ .... (1)

and $2x + 3y + 4z + 5 = 0$ ..... (2)

Given point is P (1, 1, 1).

Equation of any plane through the line of intersection of planes (1) and (2) is

$x + y + z - 6 + k(2x + 3y + 4z + 5) = 0$ ..... (3)

If plane (3) passes through point P, then

$1 + 1 + 1 - 6 + k(2 + 3 + 4 + 5) = 0$ or, $k = \frac{3}{14}$

From (3) required plane is $20x + 23y + 26z - 69 = 0$

Ex.57 Find the planes bisecting the angles between planes $2x + y + 2z = 9$ and $3x - 4y + 12z + 13 = 0$. Which of these bisector planes bisects the acute angle between the given planes. Does origin lie in the acute angle or obtuse angle between the given planes ?

Sol. Given planes are $-2x - y - 2z + 9 = 0$ ..... (1)

and $3x - 4y + 12z + 13 = 0$ ..... (2)

Equations of bisecting planes are

$$\frac{-2x - y - 2z + 9}{\sqrt{(-2)^2 + (-1)^2 + (-2)^2}} = \pm \frac{3x - 4y + 12z + 13}{\sqrt{3^2 + (-4)^2 + (12)^2}}$$

or, $13[-2x - y - 2z + 9] = \pm 3(3x - 4y + 12z + 13)$

or, $35x + y + 62z = 78$, ... (3)
[Taking +ve sign]

and $17x + 25y - 10z = 156$ ..... (4)
[Taking – ve sign]

Now

$a_1a_2 + b_1b_2 + c_1c_2 = (-2)(3) + (-1)(-4) + (-2)(12)$
$= -6 + 4 - 24 = -26 < 0$

$\therefore$ Bisector of acute angle is given by

$35x + y + 62z = 78$

$\because$ $a_1a_2 + b_1b_2 + c_1c_2 < 0$, origin lies in the acute angle between the planes.

Reduction of non-symmetrical form to symmetrical form

Let equation of the line in non-symmetrical form be

$a_1 x + b_1 y + c_1 z + d_1 = 0;$

$a_2 x + b_2 y + c_2 z + d_2 = 0$

To find the equation of the line is symmetrical form, we must know (i) its direction ratios (ii) coordinates of any point on it.

(i) Direction Ratios : Let *l, m, n* be the direction ratios of the line. Since the line lies in both the planes, it must be perpendicular to normals of both planes. So

$a_1 l + b_1 m + c_1 n = 0; \quad a_2 l + b_2 m + c_2 n = 0$

From these equations, proportional values of *l, m, n* can be found by cross-multiplication as

$$\frac{l}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$$

(ii) Point on the line : Note that as *l, m, n* can not be zero simultaneously, so at least one must be non-zero. Let $a_1b_2 - a_2b_1 \neq 0$. Then the line cannot be parallel to *x-y* plane so it intersect it. Let it intersect *x-y* plane in $(x_1, y_1, 0)$. Then

$a_1 x_1 + b_1 y_1 + d_1 = 0$ and $a_2 x_1 + b_2 y_1 + d_2 = 0$

Solving these, we get a point on the line.

Then its equations becomes

$$\frac{x - x_1}{b_1c_2 - b_2c_1} = \frac{y - y_1}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

* If $l \neq 0$, take a point on *y-z* plane as $(0, y_1, z_1)$ and if $m \neq 0$, take a point on *x-z* plane as $(x_1, 0, z_1)$.

Solved Examples

Ex.58 Find the equation of the line of intersection of planes $4x + 4y - 5z = 12$, $8x + 12y - 13z = 32$ in the symmetric form.

Sol. Given planes are $4x + 4y - 5z - 12 = 0$ ..... (1)

and $8x + 12y - 13z - 32 = 0$ ..... (2)

Let ℓ, m, n be the direction ratios of the line of intersection :

then $4\ell + 4m - 5n = 0$ ..... (3)

and $8\ell + 12m - 13n = 0$

$\therefore \dfrac{\ell}{-52+60} = \dfrac{m}{-40+52} = \dfrac{n}{48-32}$

or, $\dfrac{\ell}{8} = \dfrac{m}{12} = \dfrac{n}{16}$ or, $\dfrac{\ell}{2} = \dfrac{m}{3} = \dfrac{n}{4}$

Hence direction ratios of line of intersection are 2, 3, 4.

Here $4 \neq 0$, therefore line of intersection is not parallel to xy-plane.

Let the line of intersection meet the xy-plane at P $(\alpha, \beta, 0)$.

Then P lies on planes (1) and (2)

$\therefore\ 4\alpha + 4\beta - 12 = 0$ or, $\alpha + \beta - 3 = 0$ ..... (5)

and $8\alpha + 12\beta - 32 = 0$ or, $2\alpha + 3\beta - 8 = 0$ ... (6)

Solving (5) and (6), we get

$\dfrac{\alpha}{-8+9} = \dfrac{\beta}{-6+8} = \dfrac{1}{3-2}$ or, $\dfrac{\alpha}{1} = \dfrac{\beta}{2} = \dfrac{1}{1}$

$\therefore\ \alpha = 1, \beta = 2$

Hence equation of line of intersection in symmetrical form is $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-0}{4}$.

Area of a triangle :

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of a triangle ABC. Form two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then area is given by

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\begin{vmatrix} i & j & k \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

Solved Examples

Ex.59 Through a point P (h, k, ℓ) a plane is drawn at right angles to OP to meet the co-ordinate axes in A, B and C. If OP = p, show that the area of ΔABC is $\left|\dfrac{p^5}{2hk\ell}\right|$, where O is the origin.

Sol. $OP = \sqrt{h^2 + k^2 + \ell^2} = p$

Direction cosines of OP are

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}, \frac{k}{\sqrt{h^2+k^2+\ell^2}}, \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}$$

Since OP is normal to the plane, therefore, equation of the plane will be,

$$\frac{h}{\sqrt{h^2+k^2+\ell^2}}x + \frac{k}{\sqrt{h^2+k^2+\ell^2}}y + \frac{\ell}{\sqrt{h^2+k^2+\ell^2}}z$$

$$= \sqrt{h^2+k^2+\ell^2}$$

or, $hx + ky + \ell z = h^2 + k^2 + \ell^2 = p^2$ ..... (1)

$\therefore\ A \equiv \left(\dfrac{p^2}{h}, 0, 0\right)$, $B \equiv \left(0, \dfrac{p^2}{k}, 0\right)$, $C \equiv \left(0, 0, \dfrac{p^2}{\ell}\right)$

Now area of ΔABC, $\Delta = \sqrt{A^2_{xy} + A^2_{yz} + A^2_{zx}}$

Now A_{xy} = area of projection of ΔABC on xy-plane = area of ΔAOB

$$= \text{Mod of } \frac{1}{2}\begin{vmatrix} \frac{p^2}{h} & 0 & 1 \\ 0 & \frac{p^2}{k} & 1 \\ 0 & 0 & 1 \end{vmatrix} = \frac{1}{2}\frac{p^4}{|hk|}$$

Similarly, $A_{yz} = \dfrac{1}{2}\dfrac{p^4}{|k\ell|}$ and $A_{zx} = \dfrac{1}{2}\dfrac{p^4}{|\ell h|}$

$$\therefore\ \Delta^2 = \frac{1}{4}\frac{p^8}{h^2k^2} + \frac{1}{4}\frac{p^8}{k^2\ell^2} + \frac{1}{4}\frac{p^8}{\ell^2h^2}$$

$$= \frac{p^8}{4h^2k^2\ell^2}(\ell^2 + k^2 + h^2) = \frac{p^{10}}{4h^2k^2\ell^2} \quad \text{or}$$

$$\Delta = \left|\frac{p^5}{2hk\ell}\right|.$$

Volume of a Tetrahedron :

Volume of the tetrahedron with vertices $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$ is given by

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Angle between a plane and a line:

(i) If θ is the angle between line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$, then $\sin\theta = \left[\frac{a\ell+bm+cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{\ell^2+m^2+n^2}}\right]$.

(ii) Vector form: If θ is the angle between a line $\vec{r} = (\vec{a}+\lambda\vec{b})$ and $\vec{r}.\vec{n} = d$ then $\sin\theta = \left[\frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}\right]$.

(iii) Condition for perpendicularity

$\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$ $\qquad$ $\vec{b} \times \vec{n} = 0$

(iv) Condition for parallel

$a\ell + bm + cn = 0$ $\qquad$ $\vec{b}.\vec{n} = 0$

Condition for a line to lie in a plane

(i) **Cartesian form:** Line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ would lie in a plane $ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d = 0$ & $a\ell + bm + cn = 0$.

(ii) **Vector form:** Line $\vec{r} = \vec{a} + \lambda\vec{b}$ would lie in the plane $\vec{r}.\vec{n} = d$ if $\vec{b}.\vec{n} = 0$ & $\vec{a}.\vec{n} = d$

Coplanar lines :

(i) If the given lines are $\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and $\frac{x-\alpha'}{\ell'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$, then condition for intersection/coplanarity is $\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} = 0$ & equation of plane containing the above two lines is $\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} = 0$ or

$$\begin{vmatrix} x-\alpha' & y-\beta' & z-\gamma' \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix} = 0$$

(ii) Condition of coplanarity if both the lines are in general form Let the lines be
$ax + by + cz + d = 0 = a'x + b'y + c'z + d'$
& $\alpha x + \beta y + \gamma z + \delta = 0 = \alpha' x + \beta' y + \gamma' z + \delta'$

They are coplanar if $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ \alpha & \beta & \gamma & \delta \\ \alpha' & \beta' & \gamma' & \delta' \end{vmatrix} = 0$

Alternative method

get vector along the line of shortest distance as

$$\vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \ell & m & n \\ \ell' & m' & n' \end{vmatrix}$$

Now get unit vector along $\vec{u}$, let it be $\hat{u}$

Let $\qquad \vec{v} = (\alpha-\alpha')\hat{i} + (\beta-\beta')\hat{j} + (\gamma-\gamma')\hat{k}$

S. D. $\qquad = \hat{u}.\vec{v}$

Solved Examples

Ex.60 Find the distance of the point $(1, 0, -3)$ from the plane $x - y - z = 9$ measured parallel to the line $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$.

Sol. Given plane is $x - y - z = 9$ ..... (1)

Given line AB is $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$ ..... (2)

Equation of a line passing through the point $Q(1, 0, -3)$ and parallel to line (2) is

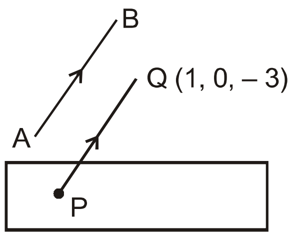
$\frac{x-1}{2} = \frac{y}{3} = \frac{z+3}{-6} = r.$ ..... (3)

Co-ordinates of any point on line (3) may be taken as

P (2r + 1, 3r, – 6r – 3)

If P is the point of intersection of line (3) and plane (1), then P lies on plane (1),

$\therefore$ (2r + 1) – (3r) – (– 6r – 3) = 9



r = 1 or, P ≡ (3, 3, – 9)

Distance between points Q (1, 0, – 3) and P (3, 3, – 9)

$PQ = \sqrt{(3-1)^2 + (3-0)^2 + (-9-(-3))^2}$

$= \sqrt{4+9+36} = 7.$

Ex.61 Find the equation of the plane passing through (1, 2, 0) which contains the line

$\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}.$

Sol. Equation of any plane passing through (1, 2, 0) may be taken as

a (x – 1) + b (y – 2) + c (z – 0) = 0 ..... (1)

where a, b, c are the direction ratios of the normal to the plane. Given line is

$\frac{x+3}{3} = \frac{y-1}{4} = \frac{z-2}{-2}$ . .... (2)

If plane (1) contains the given line, then

3a + 4b – 2c = 0 ..... (3)

Also point (– 3, 1, 2) on line (2) lies in plane (1)

$\therefore$ a (– 3 – 1) + b (1 – 2) + c (2 – 0) = 0

or, – 4a – b + 2c = 0 ..... (4)

Solving equations (3) and (4),

we get $\frac{a}{8-2} = \frac{b}{8-6} = \frac{c}{-3+16}$

or, $\frac{a}{6} = \frac{b}{2} = \frac{c}{13}$ = k (say). ..... (5)

Substituting the values of a, b and c in equation (1), we get

6 (x – 1) + 2 (y – 2) + 13 (z – 0) = 0.

or, 6x + 2y + 13z – 10 = 0. This is the required equation.

Ex.62 Show that the lines $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1}$ and $\frac{x-7}{-3} = \frac{y}{1} = \frac{z+7}{2}$ are coplanar. Also find the equation of the plane containing them.

Sol. Given lines are $\frac{x-3}{2} = \frac{y+1}{-3} = \frac{z+2}{1}$ = r (say) ... (1)

and $\frac{x-7}{-3} = \frac{y}{1} = \frac{z+7}{2}$ = R (say) ..... (2)

If possible, let lines (1) and (2) intersect at P.

Any point on line (1) may be taken as (2r + 3, – 3r – 1, r – 2) = P (let).

Any point on line (2) may be taken as

(– 3R + 7, R, 2R – 7) = P (let).

$\therefore$ 2r + 3 = – 3R + 7 or, 2r + 3R = 4 ..... (3)

Also – 3r – 1 = R or, – 3r – R = 1 ..... (4)

and r – 2 = 2R – 7 or, r – 2R = – 5. ..... (5)

Solving equations (3) and (4), we get, r = – 1, R = 2

Clearly r = – 1, R = 2 satisfies equation (5).

Hence lines (1) and (2) intersect.

$\therefore$ lines (1) and (2) are coplanar.

Equation of the plane containing lines (1) and (2) is

$$\begin{vmatrix} x-3 & y+1 & z+2 \\ 2 & -3 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 0$$

or, (x – 3) (– 6 – 1) – (y + 1) (4 + 3) + (z + 2) (2 – 9) = 0

or, – 7 (x – 3) – 7 (y + 1) – 7 (z + 2) = 0

or, x – 3 + y + 1 + z + 2 = 0

or, x + y + z = 0.