Differential Equation

INTRODUCTION

An equation involving independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**. There are two kinds of differential equation:

1. **Ordinary Differential Equation :** If the dependent variables depend on one independent variable x, then the differential equation is said to be ordinary.

for example $\frac{dy}{dx} + \frac{dz}{dx} = y + z$,

$\frac{dy}{dx} + xy = \sin x$, $\frac{d^3y}{dx^3} + 2\frac{dy}{dx} + y = e^x$,

$k\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}$, $y = x\frac{dy}{dx} + k\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$

2. **Partial differential equation :** If the dependent variables depend on two or more independent variables, then it is known as partial differential equation

for example $y^2\frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial y} = ax$, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

3. **Order of a Differential equation**

The order of a differential equation is the order of the highest derivative occurring in the differential equation

For example in the equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^x$, the order of highest order derivative is 2. So it is a differential equation of order 2. The equation $\frac{d^3y}{dx^3} - 6\left(\frac{dy}{dx}\right)^2 - 4y = 0$ is of order 3, because the order of highest order derivative in it is 3.

Note :- The order of a differential equation is a positive integer

4. **Degree of a differential Equation**

The degree of a differential equation is the power of the highest order derivative present in the equation when the involved derivatives are made free from radicals and fractions.

Thus the degree of a differential equation is the power of the highest order derivative present in the equation when it is written as a polynomial in involved derivatives.

Solved Examples

Ex.1 Find the order & degree of following differential equations.

(i) $\frac{d^2y}{dx^2} = \left[y + \left(\frac{dy}{dx}\right)^6\right]^{1/4}$ (ii) $y = e^{\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\right)}$

(iii) $\sin\left(\frac{dy}{dx} + \frac{d^2y}{dx^2}\right) = y$ (iv) $e^{y'''} - xy'' + y = 0$

Sol. (i) $\left(\frac{d^2y}{dx^2}\right)^4 = y + \left(\frac{dy}{dx}\right)^6$ $\therefore$ order = 2, degree = 4

(ii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \ell n y$ $\therefore$ order = 2, degree = 1

(iii) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \sin^{-1} y$ $\therefore$ order = 2, degree = 1

(iv) $e^{\frac{d^3y}{dx^3}} - x\frac{d^2y}{dx^2} + y = 0$

$\because$ equation can not be expressed as a polynomial in differential coefficients, so degree is not applicable but order is 3.

LINEAR & NON - LINEAR DIFFERENTIAL EQUATIONS

A differential equation in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together is called a linear differential equation.

The general and n^{th} order differential equation is given below

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + ... + P_{n-1}\frac{dy}{dx} + P_n y = Q$$

Where P_0, P_1, P_2, ... P_{n-1} and Q are either constants or functions of independent variable x.

Those equations which are not linear are called non-linear differential equations

For example

(i) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$ is a linear differential equation of order 2 and degree 1.

(ii) The differential equation $\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2 + 5y = x$ is a non-linear differential equation, because differential coefficient $\frac{dy}{dx}$ has exponent 2.

FORMATION OF DIFFERENTIAL EQUATION

Differential equation corresponding to a family of curve will have :

(a) Order exactly same as number of essential arbitrary constants in the equation of curve.

(b) No arbitrary constant present in it.

The differential equation corresponding to a family of curve can be obtained by using the following steps:

(i) Identify the number of essential arbitrary constants in equation of curve.

NOTE :

If arbitrary constants appear in addition, subtraction, multiplication or division, then we can club them to reduce into one new arbitrary constant.

(ii) Differentiate the equation of curve till the required order.

(iii) Eliminate the arbitrary constant from the equation of curve and additional equations obtained in step (ii) above.

Solved Examples

Ex.2 Form a differential equation of family of straight lines passing through origin.

Sol. Family of straight lines passing through origin is y = mx where 'm' is a parameter.

Differentiating w.r.t. x

$\frac{dy}{dx} = m$

Eliminating 'm' from both equations, we obtain

$\frac{dy}{dx} = \frac{y}{x}$ which is the required differential equation.

Ex.3 Form a differential equation of family of circles touching x-axis at the origin

Sol. Equation of family of circles touching x-axis at the origin is

$x^2 + y^2 + \lambda y = 0$ ....(i)

where λ is a parameter

$2x + 2y\dfrac{dy}{dx} + \lambda\dfrac{dy}{dx} = 0$ ....(ii)

Eliminating 'λ' from (i) and (ii)

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

which is required differential equation.

Ex.4 Form the differential equation corresponding to $y^2 = n(b^2 - x^2)$ by eliminating n and b.

Sol. We are given that $y^2 = n(b^2 - x^2)$ ..... (i)

Since the given equation contains two arbitrary constants, so we shall differentiate it two times and we shall get a differential equation of second order.

Differentiating both sides of (i) w.r.t. x, we get

$2y\dfrac{dy}{dx} = n(-2x) \Rightarrow y\dfrac{dy}{dx} = -nx$ ..... (ii)

Differentiating both sides of (ii) w.r.t. to x, we get

$y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = -n$ .... (iii)

From (ii) and (iii) we get

$$x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = y\frac{dy}{dx}$$

This is the required differential equation

Ex.5 The differential equation of all circles which passes through the origin and whose centre lies on y-axis, is

Sol. The system of circles pass through origin and centre lies on y-axis is $x^2 + y^2 - 2by = 0$

$$\Rightarrow 2x + 2y\frac{dy}{dx} - 2b\frac{dy}{dx} = 0 \Rightarrow 2b = 2y + 2x\frac{dx}{dy}$$

Therefore, the required differential equation is

$$x^2 + y^2 - 2y^2 - 2xy\frac{dx}{dy} = 0$$

$$\Rightarrow (x^2 - y^2)\frac{dy}{dx} - 2xy = 0$$

SOLUTION OF DIFFERENTIAL EQUATION

A solution of a differential equation is any function which when put into the equation changes it into an identity.

1. **General Solution :** The solution which contains a number of arbitrary constants equal to the order of the equation is called general solution or complete integral or complete primitive of differential equation.

eg. $y = ce^x$ is the general solution of $\dfrac{dy}{dx} = y$

Solved Examples

Ex.6 Find the general solution of $x^2\dfrac{dy}{dx} = 6$

Sol. $\dfrac{dy}{dx} = \dfrac{6}{x^2} \Rightarrow dy = \dfrac{6}{x^2}dx$

Now integrate it. We get $y = -\dfrac{6}{x} + c$.

2. **Particular solution :** Solution obtained from the general solution by giving particular values to the constants are called particular solutions.

eg. $y = e^x$ is a particular solution of $\dfrac{dy}{dx} = y$

Solved Examples

Ex.7 Verify that $y = cx + \dfrac{a}{c}$ is solution of the differential equation :

$$y = x\frac{dy}{dx} + a\frac{dx}{dy}$$

Sol. We have, $y = cx + \dfrac{a}{c}$ ....(1)

Diff. with respect to x

$\dfrac{dy}{dx} = c \quad \Rightarrow \quad \dfrac{dx}{dy} = \dfrac{1}{c}$

$\therefore\ x\dfrac{dy}{dx} + a\dfrac{dx}{dy} = x(c) + a\dfrac{1}{c} = y$

which is the required verification

Differential Equation of First Order and First Degree :

A differential equation of first order and first degree is of the type $\frac{dy}{dx} + f(x, y) = 0$, which can also be written as : $Mdx + Ndy = 0$, where M and N are functions of x and y.

SOLUTION METHODS OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS :

Variables separable :

If the differential equation can be put in the form, $f(x)\,dx = \phi(y)\,dy$ we say that variables are separable and solution can be obtained by integrating each side separately.

A general solution of this will be

$\int f(x)dx = \int \phi(y)\,dy + c$, where c is an arbitrary constant

Solved Examples

Ex.8 Solve the differential equation $(1 + x)\, y\, dx = (y - 1)\, x\, dy$

Sol. The equation can be written as -

$$\left(\frac{1+x}{x}\right)dx = \left(\frac{y-1}{y}\right)dy$$

$$\Rightarrow \int\left(\frac{1}{x}+1\right)dx = \int\left(1-\frac{1}{y}\right)dy$$

$$\ell n\, x + x = y - \ell n y + c$$

$$\Rightarrow \ell n y + \ell n x = y - x + c \qquad xy = ce^{y-x}$$

Ex.9 Solve : $\frac{dy}{dx} = (e^x + 1)(1 + y^2)$

Sol. The equation can be written as

$$\frac{dy}{1+y^2} = (e^x + 1)dx$$

Integrating both sides,

$\tan^{-1} y = e^x + x + c.$

Ex.10 Solve : $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

Sol. The equation can be written as

$$y - ay^2 = (x + a)\frac{dy}{dx}$$

$$\frac{dx}{x+a} = \frac{dy}{y-ay^2}$$

$$\frac{dx}{x+a} = \frac{1}{y(1-ay)}\,dy$$

$$\frac{dx}{x+a} = \left(\frac{1}{y}+\frac{a}{1-ay}\right)dy$$

Integrating both sides,

$$\ell n\,(x + a) = \ell n\, y - \ell n\,(1 - ay) + \ell n\, c$$

$$\ell n\,(x + a) = \ell n\left(\frac{cy}{1-ay}\right)$$

$$cy = (x + a)(1 - ay)$$

where 'c' is an arbitrary constant.

Ex.11 Find the general solution of the differential equations $\frac{dy}{dx} = x^4 + x^3 - x^2 + \frac{2}{x}$

Sol. We have $\frac{dy}{dx} = x^4 + x^3 - x^2 + \frac{2}{x}$

Integrating, $y = \int\left(x^4 + x^3 - x^2 + \frac{2}{x}\right)dx + c$

$$= \int x^4\,dx + \int x^3dx - \int x^2dx + \int\frac{2}{x} + c$$

$$\Rightarrow y = \frac{x^5}{5} + \frac{x^4}{4} - \frac{x^3}{3} + 2\log|x| + c$$

Which is the reqd. general solution

Ex.12 Find the solution of the differential equation

$$\frac{dy}{dx} = \operatorname{cosec}x[\cot x - \operatorname{cosec}x]$$

Sol. $\frac{dy}{dx} = \operatorname{cosec}x[\cot x - \operatorname{cosec}x]$

$$\Rightarrow \frac{dy}{dx} = \operatorname{cosec}x\,\cot x - \operatorname{cosec}^2 x$$

Now integrating both sides, we get ;

$y = -\operatorname{cosec}x + \cot x + c$

Ex.13 Solve the differential equation

$$(x^3 - y^2x^3)\frac{dy}{dx} + y^3 + x^2y^3 = 0$$

Sol. The given equation $(x^3 - y^2x^3)\frac{dy}{dx} + y^3 + x^2y^3 = 0$

$$\Rightarrow \frac{1-y^2}{y^3}dy + \frac{1+x^2}{x^3}dx = 0$$

$$\Rightarrow \int\left(\frac{1}{y^3} - \frac{1}{y}\right)dy + \int\left(\frac{1}{x^3} + \frac{1}{x}\right)dx = 0$$

$$\log\left(\frac{x}{y}\right) = \frac{1}{2}\left(\frac{1}{y^2} + \frac{1}{x^2}\right) + c$$

Ex.14 Solve the diff. equation $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

Sol. The given equation is $\frac{dy}{dx} = e^{x+y} + x^2e^y$

$$\Rightarrow \frac{dy}{dx} = e^x . e^y + x^2e^y \Rightarrow e^{-y}\, dy = (e^x + x^2)\, dx$$

[variables separable]

Integrating, $\int e^{-y}\, dy = \int (e^x + x^2)dx + c$

$$\Rightarrow \frac{e^{-y}}{-1} = e^x + \frac{x^3}{3} + c \quad \Rightarrow -\frac{1}{e^y} = e^x + \frac{1}{3}x^3 + c$$

Ex.15 Solve the differential equation $\frac{dy}{dx} = y^2(e^{2x} + 1)$

Sol. $\frac{dy}{dx} = y^2(e^{2x} + 1) \Rightarrow \frac{dy}{y^2} = (e^{2x} + 1)dx$

integrating both sides, we get $\frac{e^{2x}}{2} + \frac{1}{y} + x + c$

Ex.16 Solve the differential equation $\frac{dy}{dx} + \frac{1+\cos 2y}{1-\cos 2x} = 0$

Sol. $\frac{dy}{dx} = -\frac{1+\cos 2y}{1-\cos 2x} \Rightarrow \frac{dy}{dx} = -\frac{2\cos^2 y}{2\sin^2 x} \Rightarrow \sec^2 y \; dy$
$= -\operatorname{cosec}^2 x\, dx$

On integrating both sides , we get

$\tan y = \cot x + c \Rightarrow \tan y - \cot x = c$

Polar coordinates transformations :

Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials:

(a) If $x = r \cos\theta$; $y = r \sin\theta$ then,

(i) $x\, dx + y\, dy = r\, dr$

(ii) $dx^2 + dy^2 = dr^2 + r^2d\theta^2$

(iii) $x\, dy - y\, dx = r^2d\theta$

(b) If $x = r \sec\theta$ & $y = r \tan\theta$ then

(i) $x\, dx - y\, dy = r\, dr$

(ii) $x\, dy - y\, dx = r^2 \sec\theta\, d\theta$.

Solved Examples

Ex.17 Solve the differential equation

$xdx + ydy = x\,(xdy - ydx)$

Sol. Taking $x = r \cos\theta$, $y = r \sin\theta$

$x^2 + y^2 = r^2$

$2x\, dx + 2ydy = 2rdr$

$xdx + ydy = rdr$ ....(i)

$\frac{y}{x} = \tan\theta$

$$\frac{x\frac{dy}{dx} - y}{x^2} = \sec^2\theta . \frac{d\theta}{dx}$$

$xdy - y\, dx = x^2 \sec^2\theta . d\theta$

$xdy - ydx = r^2\, d\theta$ ....(ii)

Using (i) & (ii) in the given differential equation then it becomes

$r\, dr = r \cos\theta.\, r^2\, d\theta$

$\frac{dr}{r^2} = \cos\theta\, d\theta$

$-\frac{1}{r} = \sin\theta + \lambda$

$$-\frac{1}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} + \lambda$$

$\frac{y+1}{\sqrt{x^2+y^2}} = c$ where $-\lambda' = c$

$(y + 1)^2 = c(x^2 + y^2)$

Equations Reducible to the Variables Separable form :

If a differential equation can be reduced into a variables separable form by a proper substitution, then it is said to be "Reducible to the variables separable type". Its general form is

$\frac{dy}{dx} = f(ax + by + c)$ a, b ≠ 0. To solve this, put ax + by + c = t.

Solved Examples

Ex.18 Solve $\frac{dy}{dx} = (4x + y + 1)^2$

Sol. Putting $4x + y + 1 = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 4$$

Given equation becomes

$$\frac{dt}{dx} - 4 = t^2 \Rightarrow \frac{dt}{t^2+4} = dx$$

(Variables are separated)

Integrating both sides, $\int \frac{dt}{4+t^2} = \int dx$

$$\Rightarrow \frac{1}{2}\tan^{-1}\frac{t}{2} = x + c \Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$$

Ex.19 Solve $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Sol. $\frac{dy}{dx} = \sin(x + y)$

putting $x + y = t$

$$\frac{dy}{dx} = \frac{dt}{dx} - 1 \quad \therefore \quad \frac{dt}{dx} - 1 = \sin t$$

$$\Rightarrow \frac{dt}{dx} = 1 + \sin t \Rightarrow \frac{dt}{1+\sin t} = dx$$

Integrating both sides,

$$\int \frac{dt}{1+\sin t} = \int dx \Rightarrow \int \frac{1-\sin t}{\cos^2 t}\, dt = x + c$$

$$\Rightarrow \int (\sec^2 t - \sec t \tan t)\, dt = x + c$$

$$\tan t - \sec t = x + c$$

$$-\frac{1-\sin t}{\cos t} = x + c$$

$$\Rightarrow \sin t - 1 = x \cos t + c \cos t$$

substituting the value of t

$$\sin(x + y) = x \cos(x + y) + c \cos(x + y) + 1$$

Differential equation of the form of

$$\frac{dy}{dx} = f(ax + by + c)$$

Solution of this differential equation by putting

$$ax + by + c = v \text{ and } \frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$$

$$\therefore \frac{dv}{a + bf(v)} = dx$$

So solution is by integrate $\int \frac{dv}{a + bf(v)} = \int dx$

Solved Examples

Ex.20 Find the solution of differential equation $\frac{dy}{dx} = \cos(x + y)$

Sol. We are given that $\frac{dy}{dx} = \cos(x + y)$

Put $x + y = v$, so that $1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

So, the given equation becomes

$$\frac{dv}{dx} - 1 = \cos v \Rightarrow \frac{dv}{dx} = 1 + \cos v \Rightarrow \frac{1}{1+\cos v}dv = dx$$

$$\Rightarrow \frac{1}{2}\sec^2\frac{v}{2}dv = dx$$

Integrating both sides, we get

$$\int \frac{1}{2}\sec^2\frac{v}{2}dv = \int 1.dx$$

$$\Rightarrow \tan\frac{v}{2} = x + C \Rightarrow \tan\left(\frac{x+y}{2}\right) = x + C$$

Which is the required solution

Ex.21 Find the solution of differential equation $(x+y)(dx - dy) = dx + dy$

Sol. $(x + y - 1)\, dx = (x + y + 1)\, dy$

Put $x + y = v$ $\qquad \therefore\ 1+\frac{dy}{dx}=\frac{dv}{dx}$

$\Rightarrow (v-1)=(v+1)\left(\frac{dv}{dx}-1\right), \quad \therefore\ \frac{dv}{dx}=\frac{v-1}{v+1}+1$

$\Rightarrow \frac{dv}{dx}=\frac{2v}{v+1}, \quad \frac{(v+1)dv}{2v}=dx, \quad \frac{1}{2}\left(1+\frac{1}{v}\right)dv = dx$

$\Rightarrow \frac{1}{2}[v + \ell n v] = x + c$ or $\ell n\, v = 2(x + c) - v$

$\Rightarrow \ell n(x + y) = 2(x + c) - (x + y)$ or

$\ell n(x + y) = x - y + c$

Homogeneous Differential Equations :

A differential equation of the form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$ where f and g are homogeneous function of x and y, and of the same degree, is called homogeneous differential equaiton and can be solved easily by putting $y = vx$.

Solved Examples

Ex.22 Solve $2\frac{y}{x}+\left(\left(\frac{y}{x}\right)^2-1\right)\frac{dy}{dx}$

Sol. Putting $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$2v + (v^2 - 1)\left(v + x\frac{dv}{dx}\right) = 0$

$v + x\frac{dv}{dx} = -\frac{2v}{v^2-1} \qquad x\frac{dv}{dx}=\frac{-v(1+v^2)}{v^2-1}$

$\int \frac{v^2-1}{v(1+v^2)}\,dv = -\int\frac{dx}{x}$

$\int\left(\frac{2v}{1+v^2}-\frac{1}{v}\right)dv = -\ell n\, x + c$

$\ell n(1 + v^2) - \ell n\, v = -\ell n\, x + c$

$\ell n\left|\frac{1+v^2}{v}.x\right| = c \qquad \ell n\left|\frac{x^2+y^2}{y}\right| = c$

$x^2 + y^2 = yc'$ $\qquad$ where $c' = e^c$

Ex.23 Solve : $(x^2 - y^2)\, dx + 2xydy = 0$ given that $y = 1$ when $x = 1$

Sol. $\frac{dy}{dx} = -\frac{x^2-y^2}{2xy}$

$y = vx$

$\frac{dy}{dx} = v + \frac{dv}{dx}$

$\therefore\ v + x\frac{dv}{dx} = -\frac{1-v^2}{2v}$

$\int\frac{2v}{1+v^2}\,dv = -\int\frac{dx}{x}$

$\ell n(1 + v^2) = -\ell nx + c$

at $x = 1, y = 1$ $\qquad \therefore \qquad v = 1$

$\ell n\, 2 = c$

$\therefore\ \ell n\left\{\left(1+\frac{y^2}{x^2}\right).x\right\} = \ell n 2$

$x^2 + y^2 = 2x$

Ex.24 Solve the different equation $dy/dx = y / x + \tan y /x$

Sol. Put $y/x = v$, or $y = xv$ or $dy/ dx = v + x\, dv / dx$

$x\frac{dv}{dx}+v = v+\tan v \qquad \Rightarrow \qquad x\frac{dv}{dx}=\tan v$

$\Rightarrow \frac{dv}{\tan v}=\frac{dx}{x}$, $\cot v\, dv = \frac{dx}{x}$

or $\ell n \sin v = \ell n\, x + \ell n C$

$\Rightarrow \frac{\sin v}{x}=C$ or $\frac{\sin(y/x)}{x}=C$ or $x = C' \sin(y/x)$

Ex.25 Solve the differential equation $(1 + 2e^{x/y})\, dx + 2e^{x/y}(1 - x/y)\, dy = 0$

Sol. The equation is homogeneous of degree 0. The appearance of x/y throughout the equation suggests the use of the transformation

$x = vy, dx = v\, dy + y\, dv,$

Then

$(1 + 2e^v)(v\, dy + y\, dv) + 2e^v(1 - v)\, dy = 0$

$\Rightarrow (v + 2e^v)\, dy + y(1 + 2e^v)\, dv = 0$

$\frac{dy}{y}+\frac{1+2e^v}{v+2e^v}\,dv = 0$

Integrating and replacing v by x/y

$\ell n\, y + \ell n(v + 2e^v) = \ell n C$ and $x + 2ye^{x/y} = c$

Differential Equation reducible to Homogeneous forms

A differential equation of the form $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, where $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$, can be reduced to homogeneous form by adopting the following procedure

Put $x = X + h$, $y = Y + k$, so that $\frac{dY}{dX}=\frac{dy}{dx}$

The equation then transformed to

$$\frac{dY}{dX}=\frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)}$$

Now choose h and k such that $a_1h + b_1k + c_1 = 0$ and $a_2h + b_2k + c_2 = 0$. Then for these values of h and k, the equation becomes

$$\frac{dY}{dX}=\frac{a_1X+b_1Y}{a_2X+b_2Y}$$

This is a homogeneous equation which can be solved by putting Y = vX and then Y and X should be replaced by $y - k$ and $x - h$

Solved Examples

Ex.26 Solve the differential equation

$((x+y-1)\,dx+(2x+2y-3)\,dy=0$

Sol. The given differential equation is

$$\frac{dy}{dx}=-\left(\frac{x+y-1}{2x+2y-3}\right)$$

Put $x + y = t \Rightarrow \frac{dy}{dx}=\frac{dt}{dx}-1$

$$\frac{dy}{dx}=\frac{1-t}{2t-3}\Rightarrow\frac{dt}{dx}=\frac{t-2}{2t-3}\Rightarrow\frac{2t-3}{t-2}dt=dx$$

Integrating both sides, we get

$$\int\frac{2t-4}{t-2}dt-\int\frac{3-4}{t-2}dt=\int 1.dx$$

$2t + \log(t-2) = x + c$

Equations Reducible to the Homogeneous form

Equations of the form

$$\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C} \qquad ....(1)$$

can be made homogeneous (in new variables X and Y) by substituting $x = X + h$ and $y = Y + k$, where h and k are constants to obtain, $\frac{dY}{dX}$

$$=\frac{aX+bY+(ah+bk+c)}{AX+BY+(Ah+Bk+C)}. \qquad ....(2)$$

These constants are chosen such that $ah + bk + c = 0$, and $Ah + Bk + C = 0$. Thus we obtain the following differential equation $\frac{dY}{dX}=\frac{aX+bY}{AX+BY}$.

The differential equation can now be solved by substituting $Y = vX$.

Solved Examples

Ex.27 Solve the differential equation $\frac{dy}{dx}=\frac{x+2y-5}{2x+y-4}$

Sol. Let $x = X + h$, $\quad y = Y + k$

$$\frac{dy}{dX}=\frac{d}{dX}(Y+k)$$

$$\frac{dy}{dX}=\frac{dY}{dX} \qquad ....(i)$$

$$\frac{dx}{dX}=1+0 \qquad ....(ii)$$

on dividing (i) by (ii) $\frac{dy}{dx}=\frac{dY}{dX}$

$$\therefore\ \frac{dY}{dX}=\frac{X+h+2(Y+k)-5}{2X+2h+Y+k-4}$$

$$=\frac{X+2Y+(h+2k-5)}{2X+Y+(2h+k-4)}$$

h & k are such that $h + 2k - 5 = 0$ & $2h + k - 4 = 0$

$h = 1, k = 2$

$$\therefore\ \frac{dY}{dX}=\frac{X+2Y}{2X+Y}$$

which is homogeneous differential equation.

Now, substituting Y = vX

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$\therefore\ X\frac{dv}{dX} = \frac{1+2v}{2+v} - v$$

$$\int \frac{2+v}{1-v^2}\,dv = \int \frac{dX}{X}$$

$$\int\left(\frac{1}{2(v+1)} + \frac{3}{2(1-v)}\right)dv = \ell n\,X + c$$

$$\frac{1}{2}\,\ell n\,(v+1) - \frac{3}{2}\,\ell n\,(1-v) = \ell n\,X + c$$

$$\ell n\left|\frac{v+1}{(1-v)^3}\right| = \ell n X^2 + 2c$$

$$\frac{(Y+X)}{(X-Y)^3}\,\frac{X^2}{X^2} = e^{2c}$$

$X + Y = c'(X - Y)^3$ where $e^{2c} = c^1$

$x - 1 + y - 2 = c'\,(x - 1 - y + 2)^3$

$x + y - 3 = c'\,(x - y + 1)^3$

Special case :

Case - 1

In equation (1) if $\frac{a}{A} = \frac{b}{B}$, then the substitution $ax + by = v$ will reduce it to the form in which variables are separable.

Solved Examples

Ex.28 Solve $\frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$

Sol. Putting $u = 2x + 3y$

$$\frac{du}{dx} = 2 + 3\,.\,\frac{dy}{dx}$$

$$\frac{1}{3}\left(\frac{du}{dx} - 2\right) = \frac{u-1}{2u-5}$$

$$\frac{du}{dx} = \frac{3u-3+4u-10}{2u-5}$$

$$\int \frac{2u-5}{7u-13}\,dx = \int dx$$

$$\Rightarrow \frac{2}{7}\int 1.du - \frac{9}{7}\int \frac{1}{7u-13}\,.\,du = x + c$$

$$\Rightarrow \frac{2}{7}u - \frac{9}{7}\,.\,\frac{1}{7}\,\ell n\,(7u - 13) = x + c$$

$$\Rightarrow 4x + 6y - \frac{9}{7}\,\ell n\,(14x + 21y - 13) = 7x + 7c$$

$$\Rightarrow -3x + 6y - \frac{9}{7}\,\ell n\,(14x + 21y - 13) = c'$$

Case - 2

In equation (1), if $b + A = 0$, then by a simple cross multiplication equation (1) becomes an **exact differential equation**.

Solved Examples

Ex.29 Solve $\frac{dy}{dx} = \frac{x-2y+5}{2x+y-1}$

Sol. Cross multiplying,

$2xdy + y\,dy - dy = xdx - 2ydx + 5dx$

$2\,(xdy + y\,dx) + ydy - dy = xdx + 5\,dx$

$2\,d(xy) + y\,dy - dy = xdx + 5dx$

On integrating,

$$2xy + \frac{y^2}{2} - y = \frac{x^2}{2} + 5x + c$$

$\Rightarrow x^2 - 4xy - y^2 + 10x + 2y = c'$ where $c' = -2c$

(C) If the homogeneous equation is of the form :

$yf(xy)\,dx + xg(xy)dy = 0$, the variables can be separated by the substitution $xy = v$.

Linear differential equations

A differential equation is said to be linear differential equation of first order if the dependent variable and its derivative occur only in first degree and are not multiplied together. The general form of such an equation is

$$\frac{dy}{dx} + Py = Q \quad .... (1)$$

where P and Q are constants or functions of x. Solution of this equation is given by

$$y\,e^{\int pdx} = \int Q e^{\int pdx} + c$$

Solved Examples

Ex.30 Find the solution of differential equation

$$\cos x\frac{dy}{dx}+y\sin x=1$$

Sol. $\cos x\frac{dy}{dx}+y\sin x=1$

Given equation can be written as

$$\frac{dy}{dx}+y\tan x=\sec x$$

I.F. $= e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$

Hence solution is $y\sec x = \int \sec^2 x + c = \tan x + c$

Ex.31 Solve the differential equation $(1-x^2)\frac{dy}{dx}+2xy=x\sqrt{1-x^2}$

Sol. $\frac{dy}{dx}+\frac{2x}{(1-x^2)}y=\frac{x\sqrt{1-x^2}}{(1-x^2)}$

$$\int P dx = \int \frac{2x}{1-x^2}dx = -\ell n(1-x^2)$$

$$\text{I.F.} = e^{\int P dx} = e^{-\ell n(1-x^2)} = \frac{1}{1-x^2}$$

$$y.\left(\frac{1}{(1-x^2)}\right)=\int\frac{x}{\sqrt{(1-x^2)}}.\frac{1}{1-x^2}dx+c=\int\frac{x}{(1-x^2)^{3/2}}dx+c$$

$$=\frac{-1}{2}\int\frac{-2x}{(1-x^2)^{3/2}}dx+c \text{ , Put } 1-x^2=t$$

$$\Rightarrow -2x\,dx = dt$$

$$=\frac{-1}{2}\int t^{-3/2}dt+c=2t^{-1/2}/2+c$$

$$y\frac{1}{(1-x^2)}=\frac{1}{\sqrt{1-x^2}}+c$$

Ex.32 Solve $\frac{dy}{dx}+\frac{3x^2}{1+x^3}y=\frac{\sin^2 x}{1+x^3}$

Sol. $\frac{dy}{dx}+Py=Q$

$$P=\frac{3x^2}{1+x^3}$$

$$\text{IF} = e^{\int P.dx} = e^{\int\frac{3x^2}{1+x^3}dx} = e^{\ell n(1+x^3)} = 1+x^3$$

$\therefore$ General solution is

$$y(\text{IF}) = \int Q(\text{IF}).dx + c$$

$$y(1+x^3) = \int \frac{\sin^2 x}{1+x^3}(1+x^3)\,dx + c$$

$$y(1+x^3) = \int \frac{1-\cos 2x}{2}\,dx + c$$

$$y(1+x^3) = \frac{1}{2}x - \frac{\sin 2x}{4} + c$$

Ex.33 Solve : $x\,\ell nx\frac{dy}{dx}+y=2\,\ell n\,x$

Sol. $\frac{dy}{dx}+\frac{1}{x\ell nx}y=\frac{2}{x}$

$$P=\frac{1}{x\ell nx},\ Q=\frac{2}{x}$$

$$\text{IF} = e^{\int P.dx} = e^{\int\frac{1}{x\ell nx}dx} = e^{\ell n(\ell nx)} = \ell n\,x$$

$\therefore$ General solution is

$$y.(\ell n\,x) = \int \frac{2}{x}.\ell nx.dx + c$$

$$y(\ell n\,x) = (\ell n\,x)^2 + c$$

Ex.34 Solve the differential equation

$t(1+t^2)\,dx = (x + xt^2 - t^2)\,dt$ and it given that $x = -\pi/4$ at $t = 1$

Sol. $t(1+t^2)\,dx = [x(1+t^2) - t^2]\,dt$

$$\frac{dx}{dt}=\frac{x}{t}-\frac{t}{(1+t^2)}$$

$$\frac{dx}{dt}-\frac{x}{t}=-\frac{t}{1+t^2} \quad \text{which is linear in } \frac{dx}{dt}$$

Here, $P = -\frac{1}{t}$, $Q = -\frac{t}{1+t^2}$ IF = $e^{-\int \frac{1}{t}dt} = e^{-\ell nt} = \frac{1}{t}$

$\therefore$ General solution is -

$$x \cdot \frac{1}{t} = \int \frac{1}{t} \cdot \left(-\frac{t}{1+t^2}\right) dt + c$$

$$\frac{x}{t} = -\tan^{-1} t + c$$

putting $x = -\pi/4, t = 1$

$-\pi/4 = -\pi/4 + c \quad \Rightarrow \quad c = 0$

$\therefore$ $x = -t\tan^{-1} t$

Equation reducible to linear form :-

(i) Bernoulli's equation :- A differential equation of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x alone is called Bernoulli's equation. This form can be reduced to linear form by dividing y^n and then putting $y^{1-n} = v$

Dividing by y^n, we get $y^{-n}\frac{dy}{dx} + y^{-(n-1)} . P = Q$

Putting $y^{-(n-1)} = Y$, so that $\frac{(1-n)}{y^n}\frac{dy}{dx} = \frac{dY}{dx}$,

we get $\frac{dY}{dx} + (1-n)P.Y = (1-n)Q$

which is a linear differential equation.

(ii) If the given equation is of the form $\frac{dy}{dx} + P.f(y) = Q.g(y)$, where P and Q are functions of x alone, we divide the equation by g(y) and get

$$\frac{1}{g(y)}\frac{dy}{dx} + P.\frac{f(y)}{g(y)} = Q$$

Now substitute $\frac{f(y)}{g(y)}$ = v and solve

Solved Examples

Ex.35 Solve the differential equation $x\frac{dy}{dx} + y = x^3y^6$ -

Sol. The given differential equation can be written as

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{xy^5} = x^2$$

Putting $y^{-5} = v$ so that

$$-5y^{-6}\frac{dy}{dx} = \frac{dv}{dx} \text{ or } y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dv}{dx} \quad \text{we get}$$

$$-\frac{1}{5}\frac{dv}{dx} + \frac{1}{x}v = x^2 \Rightarrow \frac{dv}{dx} - \frac{5}{x}v = -5x^2 \qquad ......(1)$$

This is the standard form of the linear deferential equation having integrating factor

$$\text{I.F} = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = \frac{1}{x^5}$$

Multiplying both sides of (1) by I.F. and integrating w.r.t. r

We get $v . \frac{1}{x^5} = \int -5x^2 . \frac{1}{x^5}dx$

$$\Rightarrow \frac{v}{x^5} = \frac{5}{2}x^{-2} + c$$

$\Rightarrow y^{-5}x^{-5} = \frac{5}{2}x^{-2} + c$ which is the required solution.

Ex.36 Find the solution of differential equation

$$\frac{dy}{dx} - y\tan x = -y^2 \sec x$$

Sol. $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$

$$\frac{1}{y} = v \; ; \; \frac{-1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{-dv}{dx} - v\tan x = -\sec x$$

$\frac{dv}{dx} + v\tan x = \sec x$, Here P = tan x, Q = sec x

I.F. = $e^{\int \tan x dx} = \sec x \qquad v\sec x = \int \sec^2 x\, dx + c$

Hence the solution is $\qquad y^{-1}\sec x = \tan x + c$

Ex.37 Solve : $y \sin x \frac{dy}{dx} = \cos x (\sin x - y^2)$

Sol. The given differential equation can be reduced to linear form by change of variable by a suitable subtitution.

Substituting $y^2 = z$

$2y \frac{dy}{dx} = \frac{dz}{dx}$ differential equation becomes

$$\frac{\sin x}{2}\frac{dz}{dx} + \cos x . z = \sin x \cos x$$

$\frac{dz}{dx} + 2 \cot x . z = 2 \cos x$ which is linear in $\frac{dz}{dx}$

$$\text{IF} = e^{\int 2\cot x\, dx} = e^{2\ell n \sin x} = \sin^2 x$$

$\therefore$ General solution is -

$$z. \sin^2 x = \int 2\cos x.\sin^2 x.dx + c$$

$$y^2 \sin^2 x = \frac{2}{3} \sin^3 x + c$$

Ex.38 Solve : $\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$ (Bernoulli's equation)

Sol. Dividing both sides by y^2

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{xy} = \frac{1}{x^2} \quad ..... (1)$$

Putting $\frac{1}{y} = t$

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$\therefore$ differential equation (1) becomes,

$$-\frac{dt}{dx} - \frac{t}{x} = \frac{1}{x^2}$$

$$\frac{dt}{dx} + \frac{t}{x} = -\frac{1}{x^2}$$

which is linear differential equation in $\frac{dt}{dx}$

$$\text{IF} = e^{\int \frac{1}{x}dx} = e^{\ell nx} = x$$

$\therefore$ General solution is -

$$t.\ x = \int -\frac{1}{x^2} \cdot x\, dx + c$$

$$tx = -\ \ell nx + c$$

$$\frac{x}{y} = -\ \ell nx + c$$

If differential equation is of the form of $\frac{d^2y}{dx^2} = f(x)$

Solution of the differential equation $\frac{d^2y}{dx^2} = f(x)$ is obtained by integrating it with respect to x twice.

Solved Examples

Ex.39 Find the solution of the differential equation

$$\cos^2 x \frac{d^2y}{dx^2} = 1$$

Sol. $\cos^2 x \frac{d^2y}{dx^2} = 1 \Rightarrow \frac{d^2y}{dx^2} = \sec^2 x$

On integrating, we get $\frac{dy}{dx} = \tan x \pm c_1$

Again integrating, we get $y = \log \sec x \pm c_1 x \pm c_2$

ORTHOGONAL TRAJECTORY

An orthogonal trajectory of a given system of curves is defined to be a curve which cuts every member of a given family of curve at right angle.

Steps to find orthogonal trajectory :

(i) Let $f(x, y, c) = 0$ be the equation of the given family of curves, where 'c' is an arbitrary constant.

(ii) Differentiate the given equation w.r.t. x and then eliminate c.

(iii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the equation obtained in (ii).

(iv) Solve the differential equation obtained in (iii).

Hence solution obtained in (iv) is the required orthogonal trajectory.

Solved Examples

Ex.40 Find the orthogonal trajectory of family of straight lines passing through the origin.

Sol. Family of straight lines passing through the origin is -

$y = mx$ ..... (i)

where 'm' is an arbitrary constant.

Differentiating wrt x

$\frac{dy}{dx} = m$ ..... (ii)

Eliminate 'm' from (i) & (ii)

$y = \frac{dy}{dx} x$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get $y = -\frac{dx}{dy} x$

$x\,dx + y\,dy = 0$ Integrating each term,

$\frac{x^2}{2} + \frac{y^2}{2} = c \Rightarrow x^2 + y^2 = 2c$

which is the required orthogonal trajectory.

Ex.41 Find the orthogonal trajectory of $y^2 = 4ax$ (a being the parameter).

Sol. $y^2 = 4ax$ ..... (i) $2y\frac{dy}{dx} = 4a$ ..... (ii)

Eliminating 'a' from (i) & (ii)

$y^2 = 2y\frac{dy}{dx} x$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$y = 2\left(-\frac{dx}{dy}\right) x \Rightarrow 2\,x\,dx + y\,dy = 0$

Integrating each term,

$x^2 + \frac{y^2}{2} = c \Rightarrow 2x^2 + y^2 = 2c$

which is the required orthogonal trajectories.

GEOMETRICAL APPLICATION OF DIFFERENTIAL EQUATION

Form a differential equation from a given geometrical problem. Often following formulae are useful to remember

(i) Length of tangent $(L_T) = \left|\frac{y\sqrt{1+m^2}}{m}\right|$

(ii) Length of normal $(L_N) = \left|y\sqrt{1+m^2}\right|$

(iii) Length of sub-tangent $(L_{ST}) = \left|\frac{y}{m}\right|$

(iv) Length of subnormal $(L_{SN}) = |my|$

where y is the ordinate of the point, m is the slope of the tangent $\left(\frac{dy}{dx}\right)$

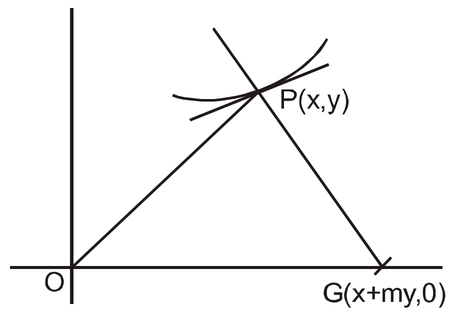
Solved Examples

Ex.42 Find the nature of the curve for which the length of the normal at a point 'P' is equal to the radius vector of the point 'P'.

Sol. Let the equation of the curve be y = f(x). P(x, y) be any point on the curve.

Slope of the tangent at P(x, y) is $\frac{dy}{dx} = m$

$\therefore$ Slope of the normal at P is $m' = -\frac{1}{m}$



Equation of the normal at 'P'

$Y - y = -\frac{1}{m}(X - x)$

Co-ordinates of G (x + my, 0)

Now, $OP^2 = PG^2$

$x^2 + y^2 = m^2y^2 + y^2$

$m = \pm\frac{x}{y} \Rightarrow \frac{dy}{dx} = \pm\frac{x}{y}$

Taking as the sign

$\frac{dy}{dx} = \frac{x}{y} \Rightarrow y\,.\,dy = x\,.\,dx$

$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + \lambda \Rightarrow x^2 - y^2 = -2\lambda$

$x^2 - y^2 = c$ (Rectangular hyperbola)

Again taking as –ve sign

$\frac{dy}{dx} = -\frac{x}{y}$

$\Rightarrow y\,dy = -x\,dx \Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + \lambda'$

$\Rightarrow x^2 + y^2 = 2\lambda'$ $x^2 + y^2 = c'$ (circle)

Ex.43 Find the curves for which the portion of the tangent included between the co-ordinate axes is bisected at the point of contact.

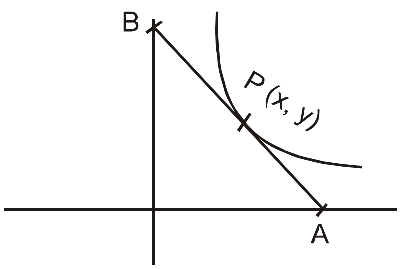
Sol. Let P (x, y) be any point on the curve.

Equation of tangent at P (x, y) is -

$Y - y = m(X - x)$ where $m = \frac{dy}{dx}$

is slope of the tangent at P (x, y).

Co-ordinates of $A\left(\frac{mx - y}{m}, 0\right)$ & B (0, y–mx)



P is the middle point of A & B

$\therefore \frac{mx - y}{m} = 2x$

$\Rightarrow mx - y = 2mx \qquad \Rightarrow \qquad mx = -y$

$\Rightarrow \frac{dy}{dx} x = -y \qquad \Rightarrow \qquad \frac{dx}{x} + \frac{dy}{y} = 0$

$\Rightarrow \ell nx + \ell ny = \ell nc \qquad \therefore \qquad xy = c$

Ex.44 Show that (4x + 3y + 1) dx + (3x + 2y + 1) dy = 0 represents a hyperbola having the lines x + y = 0 and 2x + y + 1 = 0 as asymptotes

Sol. $(4x + 3y + 1)\, dx + (3x + 2y + 1)\, dy = 0$

$4x dx + 3(y\, dx + x\, dy) + dx + 2y\, dy + dy = 0$

Integrating each term,

$2x^2 + 3xy + x + y^2 + y + c = 0$

$2x^2 + 3xy + y^2 + x + y + c = 0$

which is the equation of hyperbola when $h^2 > ab$ & $\Delta \neq 0$.

Now, combined equation of its asymptotes is -

$2x^2 + 3xy + y^2 + x + y + \lambda = 0$

which is pair of straight lines

$\therefore \Delta = 0$

$\Rightarrow 2.1\,\lambda + 2 \cdot \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2} - 2 \cdot \frac{1}{4} - 1 \cdot \frac{1}{4} - \lambda\, \frac{9}{4} = 0$

$\Rightarrow \lambda = 0$

$\therefore 2x^2 + 3xy + y^2 + x + y = 0$

$\Rightarrow (x + y)(2x + y) + (x + y) = 0$

$\Rightarrow (x + y)(2x + y + 1) = 0$

$\Rightarrow x + y = 0 \qquad$ or $\qquad 2x + y + 1 = 0$

Ex.45 The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1, 1)

Sol. Let P (x, y) be any point on the curve

Equation of tangent at 'P' is -

$Y - y = m(X - x)$

$mX - Y + y - mx = 0$

Now $\left(\frac{y - mx}{\sqrt{1 + m^2}}\right) = x$

$y^2 + m^2x^2 - 2mxy = x^2(1 + m^2)$

$\frac{y^2 - x^2}{2xy} = \frac{dy}{dx}$ which is homogeneous equation

Putting y = vx

$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \therefore \qquad v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$

$x\frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v} \qquad \Rightarrow \qquad \int \frac{2v}{v^2 + 1} dv = -\int \frac{dx}{x}$

$\ell n(v^2 + 1) = -\ell n\, x + \ell n\, c$

$x\left(\frac{y^2}{x^2} + 1\right) = c$

Curve is passing through (1, 1)

$\therefore c = 2$

$x^2 + y^2 - 2x = 0$