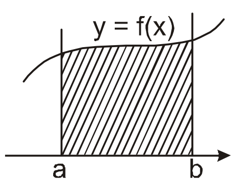
Area Under Curves

Area included between the curve y = f(x), x-axis and the ordinates x = a, x = b

(a) If $f(x) \geq 0$ for $x \in [a, b]$, then area bounded by curve $y = f(x)$, x-axis, $x = a$ and $x = b$ is $\int_a^b f(x)\,dx$

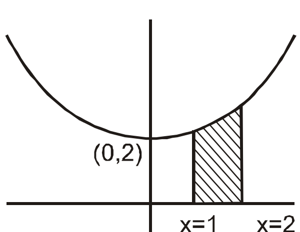


Graph of y = f(x)

Solved Examples

Ex.1 Find the area enclosed between the curve $y = x^2 + 2$, x-axis, $x = 1$ and $x = 2$.

Sol. Graph of $y = x^2 + 2$



$$\text{Area} = \int_1^2 (x^2 + 2)dx = \left[\frac{x^3}{3} + 2x\right]_1^2 = \frac{13}{3}$$

Ex.2 Find area bounded by the curve $y = \ell n\, x + \tan^{-1} x$ and x-axis between ordinates $x = 1$ and $x = 2$.

Sol. $y = \ell n\, x + \tan^{-1}x$

Domain $x > 0$, $\quad \frac{dy}{dx} = \frac{1}{x} + \frac{1}{1+x^2} > 0$

y is increasing and $x = 1, y = \frac{\pi}{4} \Rightarrow$ y is positive in $[1, 2]$

$\therefore$ Required area $= \int_1^2 (\ell n\, x + \tan^{-1} x)\,dx$

$$= \left[x\,\ell n\, x - x + x\tan^{-1} x - \frac{1}{2}\ell n\,(1+x^2)\right]_1^2$$

$$= 2\,\ell n\, 2 - 2 + 2\tan^{-1}2 - \frac{1}{2}\,\ell n\, 5 - 0 + 1 - \tan^{-1} 1 + \frac{1}{2}\,\ell n\, 2$$

$$= \frac{5}{2}\,\ell n\, 2 - \frac{1}{2}\,\ell n\, 5 + 2\tan^{-1}2 - \frac{\pi}{4} - 1$$

Note :

If a function is known to be positive valued then graph is not necessary.

Ex.3 The area cut off from a parabola by any double ordinate is k times the corresponding rectangle contained by the double ordinate and its distance from the vertex. Find the value of k ?

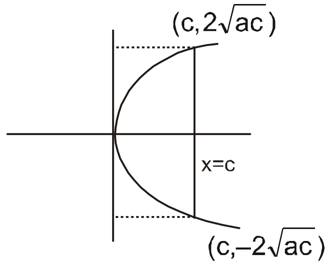
Sol. Consider $y^2 = 4ax$, $a > 0$ and $x = c$

Area by double ordinate $= 2\int_0^c 2\sqrt{a}\sqrt{x}\,dx = \frac{8}{3}\sqrt{a}\,c^{3/2}$

Area by double ordinate = k (Area of rectangle)

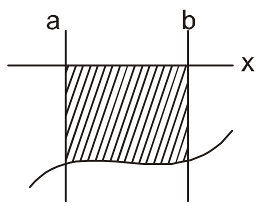
$\frac{8}{3}\sqrt{a}\,c^{3/2} = k\,4\sqrt{a}\,c^{3/2}$

$k = \frac{2}{3}$



Figure

(b) If $f(x) < 0$ for $x \in [a, b]$, then area bounded by curve $y = f(x)$, x-axis, $x = a$ and $x = b$ is $-\int_a^b f(x)\,dx$

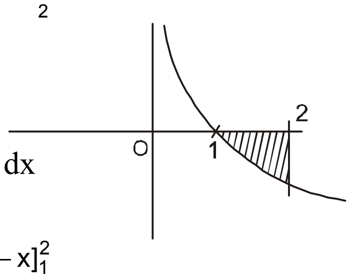


Graph of y = f(x)

Solved Examples

Ex.4 Find area bounded by $y = \log_{\frac{1}{2}} x$ and x-axis between $x = 1$ and $x = 2$

Sol. A rough graph of $y = \log_{\frac{1}{2}} x$ is as follows



Area $= -\int_1^2 \log_{\frac{1}{2}} x\,dx$

$= -\int_1^2 \log_e x \,.\, \log_{\frac{1}{2}} e\,dx$

$= -\log_{\frac{1}{2}} e \,.\, [x\log_e x - x]_1^2$

$= -\log_{\frac{1}{2}} e \,.\, (2\log_e 2 - 2 - 0 + 1)$

$= -\log_{\frac{1}{2}} e \,.\, (2\log_e 2 - 1)$

Note :

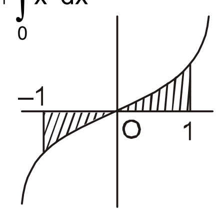
If $y = f(x)$ does not change sign in $[a, b]$, then area bounded by $y = f(x)$, x-axis between ordinates $x = a$, $x = b$ is $\left|\int_a^b f(x)\,dx\right|$

(c) If $f(x) \geq 0$ for $x \in [a,c]$ and $f(x) \leq 0$ for $x \in [c,b]$ $(a < c < b)$ then area bounded by curve $y = f(x)$ and x-axis between $x = a$ and $x = b$ is $\int_a^c f(x)dx - \int_c^b f(x)dx$

Solved Examples

Ex.5 Find the area bounded by $y = x^3$ and x- axis between ordinates $x = -1$ and $x = 1$

Sol. Required area $= \int_{-1}^{0} -x^3 dx + \int_0^1 x^3 dx$



Graph of y = x^3

$= \left[-\frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^3}{4}\right]_0^1$

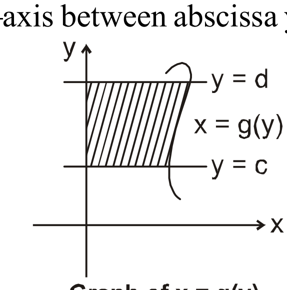
$= 0 - \left(-\frac{1}{4}\right) + \frac{1}{4} - 0 = \frac{1}{2}$

Note :

Most general formula for area bounded by curve $y = f(x)$ and x- axis between ordinates $x = a$ and $x = b$ is $\int_a^b |f(x)|\,dx$

Area included between the curve x = g(y), y-axis and the abscissas y = c, y = d

(a) If $g(y) \geq 0$ for $y \in [c,d]$ then area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and $y = d$ is $\int_{y=c}^{d} g(y)dy$



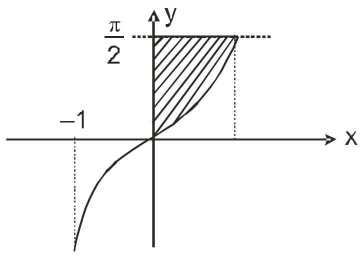
Graph of x = g(y)

Ex.6 Find area bounded between $y = \sin^{-1}x$ and y–axis between $y = 0$ and $y = \frac{\pi}{2}$.

Sol. $y = \sin^{-1} x$

$\Rightarrow x = \sin y$

Required area



Graph of $y = \sin^{-1}x$

$$= \int_0^{\pi/2} \sin y \, dy$$

$$= -\cos y\Big]_0^{\pi/2} = -(0 - 1) = 1$$

Note :

The area in above example can also evaluated by integration with respect to x.

Area = (area of rectangle formed by $x = 0$, $y = 0$, $x = 1$, $y = \frac{\pi}{2}$) – (area bounded by $y = \sin^{-1}x$, x–axis between $x = 0$ and $x = 1$)

$$= \frac{\pi}{2} \times 1 - \int_0^1 \sin^{-1} x \, dx = \frac{\pi}{2} - \left(x\sin^{-1}x + \sqrt{1-x^2}\right)_0^1$$

$$= \frac{\pi}{2} - \left(\frac{\pi}{2} + 0 - 0 - 1\right) = 1$$

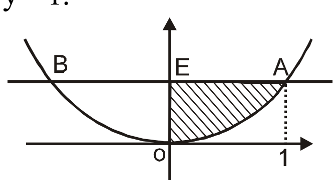
Solved Examples

Ex.7 Find the area bounded by the parabola $x^2 = y$, y-axis and the line $y = 1$.

Sol. Graph of $y = x^2$

Area OEBO

= Area OAEO



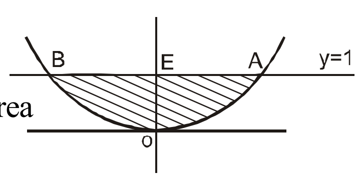
$$= \int_0^1 |x|\, dy = \int_0^1 \sqrt{y}\, dy = \frac{2}{3}$$

Ex.8 Find the area bounded by the parabola $x^2 = y$ and line $y = 1$.

Sol. Graph of $y = x^2$

Required area is area OABO

= 2 area (OAEO)



$$= 2\int_0^1 |x|\, dy = 2\int_0^1 \sqrt{y}\, dy = \frac{4}{3}.$$

Ex.9 For any real t, $x = \frac{1}{2}(e^t + e^{-t})$, $y = \frac{1}{2}(e^t - e^{-t})$ is point on the hyperbola $x^2 - y^2 = 1$. Show that the area bounded by the hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$ is t_1.

Sol. It is a point on hyperbola $x^2 - y^2 = 1$.

$$\text{Area (PQRP)} = 2\int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} y\,dx = 2\int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} \sqrt{x^2-1}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ell n\left(x + \sqrt{x^2-1}\right)\right]_1^{\frac{e^{t_1}+e^{-t_1}}{2}}$$

$$= \frac{e^{2t_1} - e^{-2t_1}}{4} - t_1$$

$$\text{Area of } \Delta OPQ = 2 \times \frac{1}{2}\left(\frac{e^{t_1}+e^{-t_1}}{2}\right)\left(\frac{e^{t_1}-e^{-t_1}}{2}\right)$$

$$= \frac{e^{2t_1} + e^{-2t_1}}{4} - t_1.$$

$\therefore$ Required area = area ΔOPQ – area (PQRP)

$= t_1$

(b) If $g(y) \le 0$ for $y \in [c,d]$ then area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and $y = d$ is $-\int_{y=c}^{d} g(y)\,dy$

Note : General formula for area bounded by curve $x = g(y)$ and y–axis between abscissa $y = c$ and $y = d$ is $\int_{y=c}^{d} |g(y)|\,dy$

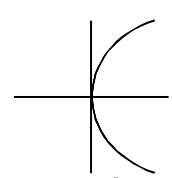
CURVE-TRACING

To find approximate shape of a curve, the following phrases are suggested :

(a) Symmetry:

(i) Symmetry about x-axis :

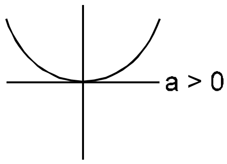
If all the powers of 'y' in the equation are even then the curve (graph) is symmetrical about the x-axis.



E.g. : $y^2 = 4ax$.

(ii) Symmetry about y-axis :

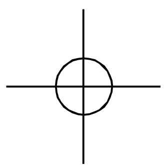
If all the powers of 'x' in the equation are even then the curve (graph) is symmetrical about the y-axis.



E.g. : $x^2 = 4ay$.

(iii) Symmetry about both axis:

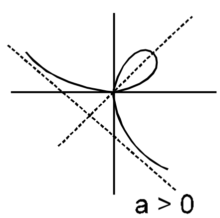
If all the powers of 'x' and 'y' in the equation are even, then the curve (graph) is symmetrical about the axis of 'x' as well as 'y' .



E.g. : $x^2 + y^2 = a^2$.

(iv) Symmetry about the line y = x :

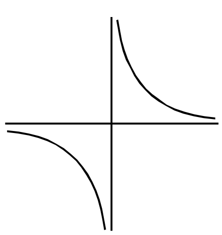
If the equation of the curve remain unchanged on interchanging 'x' and 'y', then the curve (graph) is symmetrical about the line y = x.



E.g. : $x^3 + y^3 = 3axy$.

(v) Symmetry in opposite quadrants :

If the equation of the curve (graph) remain unaltered when 'x' and 'y' are replaced by '–x' and '–y' respectively, then there is symmetry in opposite quadrants.



E.g. : $xy = c^2$

(b) Find the points where the curve crosses the x-axis and the y-axis.

(c) Find $\frac{dy}{dx}$ and equate it to zero to find the points on the curve where you have horizontal tangents.

(d) Examine intervals when f(x) is increasing or decreasing

(e) Examine what happens to 'y' when $x \to \infty$ or $x \to -\infty$

(f) Asymptotes :

Asymptote(s) is (are) line (s) whose distance from the curve tends to zero as point on curve moves towards infinity along branch of curve.

(i) If $\lim\limits_{x\to a} f(x) = \infty$ or $\mathop{\mathrm{Lt}}\limits_{x\to a} f(x) = -\infty$, then x = a is asymptote of y = f(x)

(ii) If $\lim\limits_{x\to\infty} f(x) = k$ or $\mathop{\mathrm{Lt}}\limits_{x\to-\infty} f(x) = k$ then y = k is asymptote of y = f(x)

(iii) If $\lim\limits_{x\to\infty} \frac{f(x)}{x} = m_1$, $\mathop{\mathrm{Lt}}\limits_{x\to\infty} (f(x) - m_1x) = c$, then $y = m_1x + c_1$ is an asymptote (inclined to right).

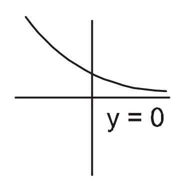
(iv) If $\lim\limits_{x\to-\infty} \frac{f(x)}{x} = m_2$, $\lim\limits_{x\to-\infty} (f(x) - m_2x) = c_2$, then $y = m_2x + c_2$ is an asymptote (inclined to left).

Solved Examples

Ex.10 Find asymptote of $y = e^{-x}$

Sol. $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} e^{-x} = 0$

$\therefore$ y = 0 is asymptote.



Graph of $y = e^{-x}$

Ex.11 Find asymptotes of xy = 1 and draw graph.

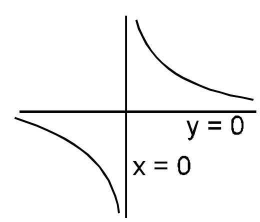
Sol. $y = \frac{1}{x}$

$\lim\limits_{x\to0} y = \lim\limits_{x\to0} \frac{1}{x} = \infty$

$\Rightarrow$ x = 0 is asymptote.

$\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} \frac{1}{x} = 0$

$\Rightarrow$ y = 0 is asymptote.



Ex.12 Find asymptotes of $y = x + \frac{1}{x}$ and sketch the curve (graph).

Sol. $\lim\limits_{x\to 0} y = \lim\limits_{x\to 0}\left(x+\frac{1}{x}\right) = +\infty$ or $-\infty$

$\Rightarrow$ x = 0 is asymptote.

$\lim\limits_{x\to 0} y = \lim\limits_{x\to 0}\left(x+\frac{1}{x}\right) = \infty$

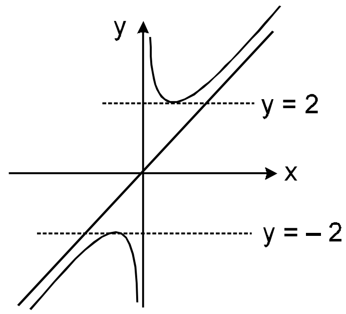
$\Rightarrow$ there is no asymptote of the type y = k

$$\lim_{x\to\infty}\frac{y}{x} = \lim_{x\to\infty}\left(1+\frac{1}{x^2}\right) = 1$$

$$\lim_{x\to\infty}(y-x) = \lim_{x\to\infty}\left(x+\frac{1}{x}-x\right) = \lim_{x\to\infty}\frac{1}{x} = 0$$

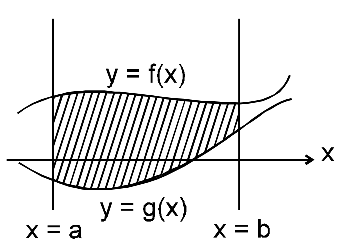
$\therefore$ y = x + 0 $\Rightarrow$ y = x is asymptote.

A rough sketch is as follows



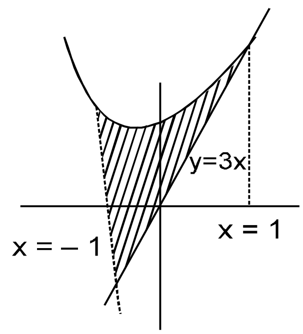
AREA BETWEEN TWO CURVES

If f(x) $\geq$ g(x) for x $\in$ [a,b] then area bounded by curves (graph) y = f(x) and y = g(x) between ordinates x = a and x = b is $\int_a^b (f(x)-g(x))dx$.



Ex.13 Find the area enclosed by curve (graph) y = x² + x + 1 and its tangent at (1,3) between ordinates x = – 1 and x = 1.

Sol. $\frac{dy}{dx} = 2x + 1$



$\frac{dy}{dx} = 3$ at x = 1

Equation of tangent is

y – 3 = 3 (x – 1)

y = 3x

Required area $= \int_{-1}^{1}(x^2+x+1-3x)\,dx$

$$= \int_{-1}^{1}(x^2-2x+1)\,dx = \left[\frac{x^3}{3}-x^2+x\right]_{-1}^{1}$$

$$= \left(\frac{1}{3}-1+1\right) - \left(-\frac{1}{3}-1-1\right) = \frac{2}{3}+2 = \frac{8}{3}$$

Note :

Area bounded by curves y = f(x) and y = g (x) between ordinates x = a and x = b is

$$\int_a^b |f(x)-g(x)|\,dx.$$

Solved Examples

Ex.14 Find the area of the region bounded by y = sin x, y = cos x and ordinates x = 0, x = $\pi/2$

Sol. $\int_0^{\pi/2}|\sin x-\cos x|\,dx$

$$\int_0^{\pi/4}(\cos x-\sin x)\,dx \quad + \quad \int_{\pi/4}^{\pi/2}(\sin x-\cos x)\,dx$$

$= 2(\sqrt{2}-1)$

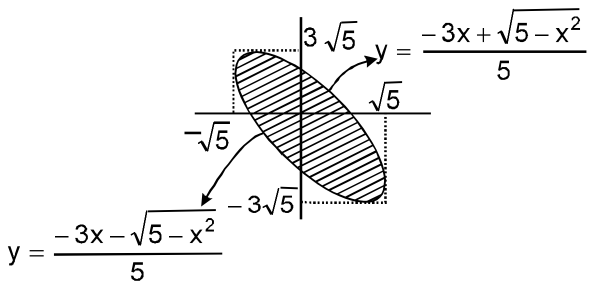
Ex.15 Find area contained by ellipse $2x^2 + 6xy + 5y^2 = 1$

Sol. $5y^2 + 6xy + 2x^2 - 1 = 0$

$$y = \frac{-6x \pm \sqrt{36x^2 - 20(2x^2 - 1)}}{10}$$

$$y = \frac{-3x \pm \sqrt{5 - x^2}}{5}$$

$\because$ y is real $\Rightarrow$ R.H.S. is also real.



$\Rightarrow -\sqrt{5} \le x \le \sqrt{5}$

If $x = -\sqrt{5}$, $\quad y = 3\sqrt{5}$

If $x = \sqrt{5}$, $\quad y = -3\sqrt{5}$

If $x = 0$, $\quad y = \pm \frac{1}{\sqrt{5}}$

If $y = 0$, $\quad x = \pm \frac{1}{\sqrt{2}}$

Required area

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \left(\frac{-3x + \sqrt{5 - x^2}}{5} - \frac{-3x - \sqrt{5 - x^2}}{5} \right) dx$$

$$= \frac{2}{5} \int_{-\sqrt{5}}^{\sqrt{5}} \sqrt{5 - x^2}\, dx \quad = \frac{4}{5} \int_{0}^{\sqrt{5}} \sqrt{5 - x^2}\, dx$$

Put $x = \sqrt{5} \sin\theta : dx = \sqrt{5} \cos\theta\, d\theta$

L.L : $x = 0 \Rightarrow \theta = 0$

U.L : $x = \sqrt{5} \Rightarrow \theta = \frac{\pi}{2}$

$$= \frac{4}{5} \int_{\theta=0}^{\frac{\pi}{2}} \sqrt{5 - 5\sin^2\theta}\ \sqrt{5}\cos\theta d\theta = 4 \int_{0}^{\frac{\pi}{2}} \cos^2\theta d\theta$$

$$= 4\ \frac{1}{2}\ \frac{\pi}{2} = \pi$$

Ex.16 Find the area contained between the two arms of curves $(y - x)^2 = x^3$ between $x = 0$ and $x = 1$.

Sol. $(y - x)^2 = x^3 \Rightarrow y = x \pm x^{3/2}$

For arm

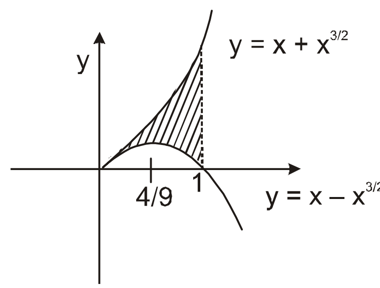
$y = x + x^{3/2} \Rightarrow \frac{dy}{dx} = 1 + \frac{3}{2} x^{1/2} > 0 \qquad x \ge 0.$

y is increasing function.

For arm

$y = x - x^{3/2} \qquad \Rightarrow \qquad \frac{dy}{dx} = 1 - \frac{3}{2} x^{1/2}$

$\frac{dy}{dx} = 0 \quad \Rightarrow x = \frac{4}{9}, \frac{d^2y}{dx^2} = -\frac{3}{4} x^{-\frac{1}{2}} < 0$ at $x = \frac{4}{9}$



Figure

$\therefore$ at $x = \frac{4}{9}$, $y = x - x^{3/2}$ has maxima.

Required area $\quad = \int_0^1 (x + x^{3/2} - x + x^{3/2})dx$

$$= 2\int_0^1 x^{3/2}dx = \left[\frac{2\,x^{5/2}}{5/2}\right]_0^1 \qquad = \frac{4}{5}$$

Ex.17 Let A (m) be area bounded by parabola $y = x^2 + 2x - 3$ and the line $y = mx + 1$. Find the least area A(m).

Sol. Solving we obtain

$x^2 + (2 - m)x - 4 = 0$

Let α, β be roots $\Rightarrow \alpha + \beta = m - 2$, $\alpha\beta = -4$

$$A(m) = \left| \int_\alpha^\beta (mx + 1 - x^2 - 2x + 3)dx \right|$$

$$= \left| \int_\alpha^\beta (-x^2 + (m - 2)x + 4)dx \right|$$

$$= \left| \left(-\frac{x^3}{3} + (m-2)\frac{x^2}{2} + 4x \right)_{\alpha}^{\beta} \right|$$

$$= \left| \frac{\alpha^3 - \beta^3}{3} + \frac{m-2}{2}(\beta^2 - \alpha^2) + 4(\beta - \alpha) \right|$$

$$= |\beta - \alpha| . \left| -\frac{1}{3}(\beta^2 + \beta\alpha + \alpha^2) + \frac{(m-2)}{2}(\beta + \alpha) + 4 \right|$$

$$= \sqrt{(m-2)^2 + 16} \left| -\frac{1}{3}\left((m-2)^2 + 4\right) + \frac{(m-2)}{2}(m-2) + 4 \right|$$

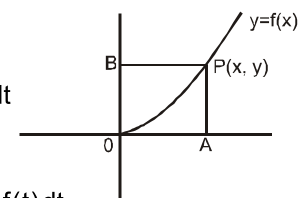
$$= \sqrt{(m-2)^2 + 16} \left| \frac{1}{6}(m-2)^2 + \frac{8}{3} \right|$$

$$A(m) = \frac{1}{6}\left((m-2)^2 + 16\right)^{3/2}$$

$$\text{Least } A(m) = \frac{1}{6}(16)^{3/2} = \frac{32}{3}.$$

Ex.18 A curve $y = f(x)$ passes through the origin and lies entirely in the first quadrant. Through any point $P(x, y)$ on the curve, lines are drawn parallel to the coordinate axes. If the curve divides the area formed by these lines and coordinate axes in m : n, then show that $f(x) = cx^{m/n}$ or $f(x) = cx^{n/m}$ (c-being arbitrary).

Sol. Area (OAPB) = xy



Figure

$$\text{Area (OAPO)} = \int_0^x f(t)\,dt$$

$$\text{Area (OPBO)} = xy - \int_0^x f(t)\,dt$$

$$\frac{\text{Area (OAPO)}}{\text{Area (OPBO)}} = \frac{m}{n}$$

$$n\int_0^x f(t)\,dt = m\left(xy - \int_0^x f(t)\,dt \right)$$

$$n\int_0^x f(t)\,dt = mxf(x) - m\int_0^x f(t)\,dt$$

Differentiating w.r.t. x

$$nf(x) = m f(x) + mx f'(x) - m f(x)$$

$$\frac{f'(x)}{f(x)} = \frac{n}{m}\frac{1}{x}$$

$f(x) = cx^{n/m}$

similarly $f(x) = cx^{m/n}$

Self practice problems :

(1) Find the area between curve $y = x^2 - 3x + 2$ and x–axis

(i) bounded between x = 1 and x = 2.

(ii) bound between x = 0 and x = 2.

(2) Find the area included between curves $y = 2x - x^2$ and $y + 3 = 0$.

(3) Find area between curves $y = x^2$ and $y = 3x - 2$ from x = 0 to x = 2.

(4) What is geometrical significance of

(i) $\int_0^{\pi} |\cos x|\,dx$, (ii) $\int_0^{\frac{3\pi}{2}} \cos x\,dx$

(5) Find the area of the region bounded by the x-axis and the curves defined by

$y = \tan x, \left(\text{where} \frac{-\pi}{3} \le x \le \frac{\pi}{3} \right)$ and

$y = \cot x \left(\text{where} \frac{\pi}{6} \le x \le \frac{2\pi}{3} \right)$.

(6) Curves y = sinx and y = cosx intersect at infinite number of points forming regions of equal area between them calculate area of one such region.

(7) Find the area of the region bounded by the parabola $(y-2)^2 = (x-1)$ and the tangent to it at ordinate y = 3 and x–axis.

(8) Find the area included between $y = \tan^{-1}x$, $y = \cot^{-1}x$ and y–axis.

(9) Find area common to circle $x^2 + y^2 = 2$ and the parabola $y^2 = x$.

(10) Find the area included between curves $y = \frac{4 - x^2}{4 + x^2}$ and $5y = 3|x| - 6$.

(11) Find the area bounded by the curve $|y| + \frac{1}{2} = e^{-|x|}$.

(12) Find the area of loop $y^2 = x(x-1)^2$.

(13) Find the area enclosed by $|x| + |y| \le 3$ and $xy \ge 2$.

(14) Find are bounded by $x^2 + y^2 \le 2ax$ and $y^2 \ge ax$, $x \ge 0$.

Ans.

1. (i) $\frac{1}{6}$ (ii) 1
2. $\frac{32}{3}$
3. 1
4. (i) Area bounded by y = cos x, x-axis between x = 0, x = π.
   (ii) Difference of area bounded by y = cos x, x-axis between x = 0, x = $\frac{\pi}{2}$ and area bounded by y = cos x, x-axis between x = $\frac{\pi}{2}$, x = $\frac{3\pi}{2}$.
5. $\ell n \frac{3}{2}$
6. $2\sqrt{2}$
7. 9
8. $\ell n2$
9. $\frac{\pi}{3} - \frac{\sqrt{3}}{2} - \frac{2}{3}$
10. $2\pi - \frac{8}{5}$
11. $2(1-\ell n2)$
12. $\frac{8}{15}$
13. $3-4\ell n2$
14. $\left(\frac{3\pi-8}{6}\right)a^2$