# Definite Integration

## DEFINITION

Let f(x) be a continuous function defined on a closed interval [a, b] and

$\int f(x)\ dx = F(x) + c$ then

$\int_a^b f(x)\ dx = [F(x)]_a^b$ or $\int_a^b f(x)\,dx = F(b) - F(a)$

is called definite integral of f(x) within limits a and b. The interval [a, b] is called range of integration. Every definite integral has a unique value.

## *Solved Examples*

**Ex.1** $\int_1^2 x^2\ dx$

**Sol.** $\int_1^2 x^2\ dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

**Ex.2** $\int_0^2 \frac{dx}{4 + x^2}$

**Sol.** $\int_0^2 \frac{dx}{4 + x^2} = \frac{1}{2}[\tan^{-1} x/2]_0^2 = \frac{1}{2}(\tan^{-1} 1 - 0) = \pi/8$

**Ex.3** $\int_0^1 xe^{x^2} dx$

**Sol.** $\int_0^1 xe^{x^2} dx = \frac{1}{2}\left[e^{x^2}\right]_0^1 = \frac{1}{2}(e - 1)$

**Ex.4** $\int_0^{\pi/4} \sec^2 x\ dx.$

**Sol.** $\int_0^{\pi/4} \sec^2 x.\ dx = [\tan x]_0^{\pi/4} = \tan \pi/4 - \tan 0 = 1$

**Ex.5** $\int_0^{\pi/2} \sin^2 x\ dx$

**Sol.** $\int_0^{\pi/2} \sin^2 x\ dx = \int_0^{\pi/2} \left(\frac{1 - \cos 2x}{2}\right) dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2}$

$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \pi/4$

**Ex.6** Evaluate $\int_1^2 \frac{dx}{(x+1)(x+2)}$

**Sol.** $\because \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$

(by partial fractions)

$\int_1^2 \frac{dx}{(x+1)(x+2)} = [\ell n(x+1) - \ell n(x+2)]_1^2$

$= \ell n3 - \ell n4 - \ell n2 + \ell n3 = \ell n\left(\frac{9}{8}\right)$

## PROPERTIES OF DEFINITE INTEGRATION

**(P–1)** $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$

Here x is a dummy variable, it can be replaced by any other variable t, u,.....

$\int_0^{\pi/2} \sin(x)\,dx = \int_0^{\pi/2} \sin t\,dt = \int_0^{\pi/2} \sin u\,du =$

Similarly $\sum_{r=1}^{10} r^2 = \sum_{r=1}^{10} t^2 = \sum_{u=1}^{10} u^2 = .....$

**(P–2)** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

i.e. the interchange of limits of a definite integral changes only its sign.

**(P–3)** $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)dx \quad (a < c < b)$

Generally this of property is used when the integrand has two or more rules in the integration interval.

## Solved Examples

**Ex.7** If $f(x) = \begin{cases} x^2, & 0 < x < 2 \\ 3x-4, & 2 \le x < 3 \end{cases}$ then evaluate $\int_0^3 f(x)\,dx$

**Sol.** $\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x-4)\,dx$

$= \left(\frac{x^3}{3}\right)_0^2 + \left(\frac{3x^2}{2} - 4x\right)_2^3 = \frac{8}{3} + \frac{27}{2} - 12 - 6 + 8 = 37/6$

**Ex.8** Evaluate $\int_0^2 |1-x|\,dx$

**Sol.** $\because\ |1-x| = \begin{cases} 1-x, & \text{when } 0 < x < 1 \\ x-1, & \text{when } 1 \le x < 2 \end{cases}$

$\therefore\ I = \int_0^1 (1-x)\,dx + \int_1^2 (x-1)\,dx$

$= \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^2 = (1/2 - 0) + (0 + 1/2) = 1$

**Ex.9** If $f(x) = \begin{cases} x+3 & : x < 3 \\ 3x^2+1 & : x \ge 3 \end{cases}$, then find $\int_2^5 f(x)\,dx$.

**Sol.** $\int_2^5 f(x)\,dx = \int_2^3 f(x)\,dx + \int_3^5 f(x)\,dx = \int_2^3 (x+3)\,dx + \int_3^5 (3x^2+1)\,dx = \left[\frac{x^2}{2} + 3x\right]_2^3 + \left[x^3 + x\right]_3^5$

$= \frac{9-4}{2} + 3\,(3-2) + 5^3 - 3^3 + 5 - 3 = \frac{211}{2}$

**Ex.10** Evaluate $\int_2^8 |x-5|\,dx$.

**Sol.** $\int_2^8 |x-5|\,dx = \int_2^5 (-x+5)\,dx + \int_5^8 (x-5)\,dx = 9$

**Ex.11** Show that $\int_0^2 (2x+1)\,dx = \int_0^5 (2x+1) + \int_5^2 (2x+1)$

**Sol.** L.H.S. $= x^2 + x\,]_0^2 = 4 + 2 = 6$

R.H.S. $= 25 + 5 - 0 + (4 + 2) - (25 + 5) = 6$

**Property (5)** $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Application : $\int_a^b \frac{f(x)}{f(x) + f(a+b-x)}\,dx = \frac{b-a}{2}$

## Solved Examples

**Ex.12** Evaluate : $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

**Sol.** $= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ ....... (1)

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}}\,dx$

[$\because$ here $a + b = \pi/2$]

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx$ ......... (2)

$\therefore\ 2I = \int_{\pi/6}^{\pi/3} 1\,dx == [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \Rightarrow I = \pi/12$

**Ex.13** Prove that $\int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)}\,dx$

$$= \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\sin x)+g(\cos x)}\,dx = \frac{\pi}{4}.$$

**Sol.** Let $I = \int_0^{\frac{\pi}{2}} \frac{g(\sin x)}{g(\sin x)+g(\cos x)}\,dx$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)}{g\left(\sin\left(\frac{\pi}{2}-x\right)\right)+g\left(\cos\left(\frac{\pi}{2}-x\right)\right)}$$

$= \int_0^{\frac{\pi}{2}} \frac{g(\cos x)}{g(\cos x)+g(\sin x)}\,dx$ on adding, we obtain $2I$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{g(\sin x)}{g(\sin x)+g(\cos x)}+\frac{g(\cos x)}{g(\cos x)+g(\sin x)}\right)dx$$

$$= \int_0^{\frac{\pi}{2}} dx \quad \Rightarrow I = \frac{\pi}{4}$$

**(P–4)** $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

With the help of above property following integrals can be obtained .

* $\int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f(\cos x)\,dx$

* $\int_0^{\pi/2} f(\tan x)\,dx = \int_0^{\pi/2} f(\cot x)\,dx$

* $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx$

* $\int_0^{1} f(\log x)dx = \int_0^{1} f\,[\log(1-x)]\,dx$

* $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x+\cos^n x}dx = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x+\sin^n x}dx = \pi/4$

* $\int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x}dx = \int_0^{\pi/2} \frac{\cot^n x}{1+\cot^n x}dx = \pi/4$

* $\int_0^{\pi/2} \frac{1}{1+\tan^n x}dx = \int_0^{\pi/2} \frac{1}{1+\cot^n x}dx = \pi/4$

* $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x+\operatorname{cosec}^n x}dx = \int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\operatorname{cosec}^n x+\sec^n x}dx = \pi/4$

* $\int_0^{\pi/4} \log\ (1+\tan x)dx = \frac{\pi}{8}\log 2$

* $\int_0^{\pi/2} \log\cot x\,dx = \int_0^{\pi/2} \log\tan x\,dx = 0$

## *Solved Examples*

**Ex.14** $\int_0^{\pi/2} \log\tan x\,dx$

**Sol.** $I = \int_0^{\pi/2} \log\tan x\,dx$ ...(1)

$I = \int_0^{\pi/2} \log\tan\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi/2} \log\cot x\,dx$ ...(2)

$$\therefore\ 2I = \int_0^{\pi/2} (\log\tan x+\log\cot x)\,dx$$

$$= \int_0^{\pi/2} \log(\tan x\cot x)\,dx = \int_0^{\pi/2} \log 1\,dx = 0 \quad \Rightarrow \quad I = 0$$

**Ex.15** $\int_0^{1} \log\left(\frac{1}{x}-1\right)dx$

**Sol.** $I = \int_0^{1} \log\left(\frac{1-x}{x}\right)dx$ .....(1)

$$\Rightarrow \quad I = \int_0^{1} \log\left[\frac{1-(1-x)}{1-x}\right]dx$$

$$= \int_0^{1} \log\left(\frac{x}{1-x}\right)dx = -\int_0^{1} \log\left(\frac{1-x}{x}\right)dx = -I$$

$\Rightarrow\ 2I = 0 \qquad \Rightarrow\ I = 0$

**Ex.16** $\int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin x + \cos x} dx$

**Sol.** $I = \int_0^{\pi/2} \frac{a \sin x + b \cos x}{\sin x + \cos x} dx$ ....(1)

$I = \int_0^{\pi/2} \frac{a \sin(\pi/2 - x) + b \cos(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)} dx$

$\int_0^{\pi/2} \frac{a \cos x + b \sin x}{\sin x + \cos x} dx$ .....(2)

$\therefore \quad 2I = \int_0^{\pi/2} \frac{(a+b)(\sin x + \cos x)}{\sin x + \cos x} dx$

$= \int_0^{\pi/2} (a+b)\ dx = (a+b)\,\pi/2 \Rightarrow I = (a+b)\,\pi/4$

**Property (4)** $\int_{-a}^{a} f(x)\,dx = \int_0^a (f(x) + f(-x))\ dx$

$= \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(-x) = f(x) \text{ i.e. } f(x) \text{ is even} \\ 0, & \text{if } f(-x) = -f(x) \text{ i.e. } f(x) \text{ is odd} \end{cases}$

## *Solved Examples*

**Ex.17** Evaluate $\int_{-1}^{1} \frac{e^x + e^{-x}}{1 + e^x} dx$

**Sol.** $\int_{-1}^{1} \frac{e^x + e^{-x}}{1+e^x} dx = \int_0^1 \left( \frac{e^x + e^{-x}}{1+e^x} + \frac{e^{-x} + e^x}{1+e^{-x}} \right) dx$

$= \int_0^1 \left( \frac{e^x + e^{-x}}{1+e^x} + \frac{e^x(e^{-x} + e^x)}{e^x + 1} \right) dx$

$= \int_0^1 (e^x + e^{-x})\ dx = e - 1 + \frac{(e^{-1} - 1)}{-1} = \frac{e^2 - 1}{e}$

**Ex.18** Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx.$

**Sol.** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx = 2\int_0^{\frac{\pi}{2}} \cos x\ dx = 2$

($\because$ cos x is even function)

**Ex.19** Evaluate $\int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx.$

**Sol.** Let $f(x) = \log_e\left(\frac{2-x}{2+x}\right)$

$\Rightarrow f(-x) = \log_e\left(\frac{2+x}{2-x}\right) = -\log_e\left(\frac{2-x}{2+x}\right) = -f(x)$

i.e. f(x) is odd function

$\therefore \int_{-1}^{1} \log_e\left(\frac{2-x}{2+x}\right) dx = 0$

**Ex.20** $\int_{-\pi/2}^{\pi/2} \cos^2 x\,dx$ is equal to

(1) $\pi/4$ (2) $\pi/2$

(3) $\pi/6$ (4) $\pi/3$

**Sol.** Here $I = 2\int_0^{\pi/2} \cos^2 x\ dx$ $\{\because f(-x) = f(x)\}$

$\int_0^{\pi/2} (1 + \cos 2x)\,dx = \left(x + \frac{\sin 2x}{2}\right)_0^{\pi/2} = \pi/2$

**Ex.21** $\int_{-1}^{1} \frac{x^3 \sin(1+x^2)}{1+x^2} dx$ is equal to

(1) 1 (2) 2

(3) 0 (4) none of these

**Sol.** Here $f(x) = \frac{x^3 \sin(1+x^2)}{1+x^2}$

& $f(-x) = -\frac{x^3 \sin(1+x^2)}{1+x^2}$

$\because f(x) = -f(x)$

$\therefore I = 0$

**Property (6)** $\int_0^{2a} f(x)\ dx = \int_0^a (f(x) + f(2a - x))\ dx$

$= \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

## Solved Examples

**Ex.22** Evaluate $\int_0^{\pi} \sin^3 x \cos^3 x \; dx$.

**Sol.** Let $f(x) = \sin^3 x \cos^3 x \quad \Rightarrow f(\pi - x) = -f(x)$

$$\therefore \int_0^{\pi} \sin^3 x \cos^3 x \; dx = 0$$

**Ex.23** Evaluate $\int_0^{\pi} \frac{dx}{1+2\sin^2 x}$ dx.

**Sol.** Let $f(x) = \frac{1}{1+2\sin^2 x} \quad \Rightarrow f(\pi - x) = f(x)$

$$\Rightarrow \int_0^{\pi} \frac{dx}{1+2\sin^2 x} = 2\int_0^{\frac{\pi}{2}} \frac{dx}{1+2\sin^2 x} = 2$$

$$\int_0^{\frac{\pi}{2}} \frac{\sec^2 x \, dx}{1+\tan^2 x + 2\tan^2 x}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 x \, dx}{1+3\tan^2 x} = \frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\frac{\pi}{2}}$$

$\because \tan \frac{\pi}{2}$ is undefined, we take limit

$$= \frac{2}{\sqrt{3}} \left[ \underset{x \to \frac{\pi}{2}^-}{\text{Lt}} \tan^{-1}\left(\sqrt{3}\tan x\right) - \tan^{-1}\left(\sqrt{3}\tan 0\right) \right]$$

$$= \frac{2}{\sqrt{3}} \; \frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$$

**Alternatively :** $\int_0^{\pi} \frac{dx}{1+2\sin^2 x} = \int_0^{\pi} \frac{\text{cosec}^2 x}{\text{cosec}^2 x + 2} dx$

$$= \int_0^{\pi} \frac{\text{cosec}^2 x \, dx}{\cot^2 x + 3}$$

Observe that we are not converting in terms of tan x as it is not continuous in $(0, \pi)$

$$= -\frac{1}{\sqrt{3}} \left[\tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right)\right]_0^{\pi}$$

$$= -\frac{1}{\sqrt{3}} \left[ \underset{x \to \pi^-}{\text{Lt}} \tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right) - \underset{x \to 0^+}{\text{Lt}} \tan^{-1}\left(\frac{\cot x}{\sqrt{3}}\right) \right]$$

$$= -\frac{1}{\sqrt{3}} \left[ -\frac{\pi}{2} - \frac{\pi}{2} \right] = \frac{\pi}{\sqrt{3}}$$

**Ex.24** Prove that $\int_0^{\frac{\pi}{2}} \ell n \sin x \; dx = \int_0^{\frac{\pi}{2}} \ell n \cos x \; dx$

$$= \int_0^{\frac{\pi}{2}} \ell n\,(\sin 2x)\; dx = -\frac{\pi}{2}\, \ell n\, 2 .$$

**Sol.** Let $I = \int_0^{\frac{\pi}{2}} \ell n \sin x \, dx$ ........(i)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \ell n\left(\sin\left(\frac{\pi}{2} - x\right)\right) dx \quad \text{(by property P – 5)}$$

$I = \int_0^{\frac{\pi}{2}} \ell n\,(\cos x)\; dx$ ..........(ii)

Adding (i) and (ii)

$$2I = \int_0^{\frac{\pi}{2}} \ell n\,(\sin x \,.\, \cos x)\; dx = \int_0^{\frac{\pi}{2}} \ell n\left(\frac{\sin 2x}{2}\right) dx$$

$$2I = \int_0^{\frac{\pi}{2}} \ell n\,(\sin 2x)\; dx - \int_0^{\frac{\pi}{2}} \ell n\, 2\, dx$$

$2I = I_1 - \frac{\pi}{2}\, \ell n\, 2$ ..........(iii)

where $I_1 = \int_0^{\frac{\pi}{2}} \ell n\,(\sin 2x)\; dx$

put $2x = t \quad \Rightarrow \quad dx = \frac{1}{2}\, dt$

L . L : $x = 0 \quad \Rightarrow \quad t = 0$

U . L : $x = \frac{\pi}{2} \quad \Rightarrow \quad t = \pi$

$$\Rightarrow I_1 = \int_0^{\pi} \ell n\,(\sin t) \,.\, \frac{1}{2}\, dt = \frac{1}{2} \times 2 \int_0^{\frac{\pi}{2}} \ell n\,(\sin t)\; dt$$

(by using property P – 6)

$\Rightarrow I_1 = I$

$\therefore$ (iii) gives $I = -\frac{\pi}{2}\, \ell n\, 2$

**Property (7)**

If f(x) is a periodic function with period T, then

(i) $\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx,\ n \in z$

(ii) $\int_a^{a+nT} f(x)\,dx = n\int_0^{T} f(x)\,dx,\ n \in z,\ a \in R$

(iii) $\int_{mT}^{nT} f(x)\,dx = (n-m)\int_0^{T} f(x)\,dx,\ m, n \in z$

(iv) $\int_{nT}^{a+nT} f(x)\,dx = \int_0^{a} f(x)\,dx,\ n \in z,\ a \in R$

(v) $\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^{b} f(x)\,dx,\ n \in z,\ a, b \in R$

## Solved Examples

**Ex.25** Evaluate $\int_{-1}^{2} e^{\{x\}}\,dx$.

**Sol.** $\int_{-1}^{2} e^{\{x\}}\,dx = \int_{-1}^{-1+3} e^{\{x\}}\,dx = 3\int_0^{1} e^{\{x\}}\,dx = 3\int_0^{1} e^{\{x\}}\,dx$

$= 3(e-1)$

**Ex.26** Evaluate $\int_0^{n\pi+v} |\cos x|\,dx$, $\frac{\pi}{2} < v < \pi$ and $n \in z$.

**Sol.** $\int_0^{n\pi+v} |\cos x|\,dx = \int_0^{v} |\cos x|\,dx + \int_v^{n\pi+v} |\cos x|\,dx$

$= \int_0^{\frac{\pi}{2}} \cos x - \int_{\pi/2}^{v} \cos x\,dx + n\int_0^{\pi} |\cos x|\,dx$

$= (1-0) - (\sin v - 1) + 2n\int_0^{\frac{\pi}{2}} \cos x\,dx$

$= 2 - \sin v + 2n(1-0) = 2n + 2 - \sin v$

**Property (8)**

If $\psi(x) \le f(x) \le \phi(x)$ for $a \le x \le b$,

then $\int_a^b \psi(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b \phi(x)\,dx$

**Property (9)**

If $m \le f(x) \le M$ for $a \le x \le b$, then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Further if f(x) is monotonically decreasing in (a, b), then $f(b)(b-a) < \int_a^b f(x)\,dx < f(a)(b-a)$ and if f(x) is monotonically increasing in (a, b), then $f(a)(b-a) < \int_a^b f(x)\,dx < f(b)(b-a)$

**Property (10)** $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

**Property (11)** If $f(x) \ge 0$ on [a, b], then $\int_a^b f(x)\,dx \ge 0$

## Solved Examples

**Ex.27** For $x \in (0, 1)$ arrange $f_1(x) = \frac{1}{\sqrt{4-x^2}}$, $f_2(x) = \frac{1}{\sqrt{4-2x^2}}$ and $f_3(x) = \frac{1}{\sqrt{4-x^2-x^3}}$ in ascending order and hence prove that $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$.

**Sol.** $\because\ 0 < x^3 < x^2 \quad \Rightarrow \quad x^2 < x^2 + x^3 < 2x^2$

$\Rightarrow -2x^2 < -x^2 - x^3 < -x^2$

$\Rightarrow 4 - 2x^2 < 4 - x^2 - x^3 < 4 - x^2$

$\Rightarrow \sqrt{4-2x^2} < \sqrt{4-x^2-x^3} < \sqrt{4-x^2}$

$\Rightarrow f_1(x) < f_3(x) < f_2(x) \quad$ for $x \in (0, 1)$

$\Rightarrow \int_0^1 f_1(x)\,dx < \int_0^1 f_3(x)\,dx < \int_0^1 f_2(x)\,dx$

$\sin^{-1}\left(\frac{x}{2}\right)\Big]_0^1 < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{\sqrt{2}}\Big]_0^1$

$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$

**Ex.28** Estimate the value of $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}$ dx.

**Sol.** Let $f(x) = \frac{\sin x}{x}$

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{(\cos x)(x - \tan x)}{x^2} < 0$$

$\Rightarrow$ f(x) is monotonically decreasing function.

f(0) is not defined, so we evaluate

$$\underset{x\to 0^+}{\text{Lt}} f(x) = \underset{x\to 0^+}{\text{Lt}} \frac{\sin x}{x} = 1. \text{ Take } f(0) = \underset{x\to 0^+}{\text{Lt}} f(x) = 1$$

$$f\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$$

$$\frac{2}{\pi} \cdot \left(\frac{\pi}{2} - 0\right) < \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx < 1 \cdot \left(\frac{\pi}{2} - 0\right)$$

$$\Rightarrow 1 < \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx < \frac{\pi}{2}$$

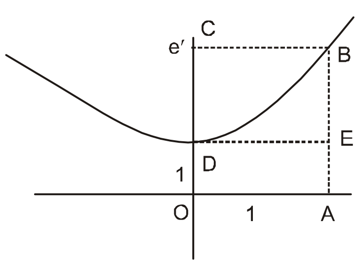
**Ex.29** Estimate the value of $\int_0^1 e^{x^2} dx$ using (i) rectangle, (ii) triangle.

**Sol.** (i) By using rectangle

$$\text{Area OAED} < \int_0^1 e^{x^2} dx < \text{Area OABC}$$

$$1 < \int_0^1 e^{x^2} dx < 1 \cdot e$$

$$1 < \int_0^1 e^{x^2} dx < e$$




(ii) By using triangle

Area OAED < $\int_0^1 e^{x^2} dx$ < Area OAED + Area of triangle DEB

$$1 < \int_0^1 e^{x^2} dx < 1 + \frac{1}{2} \cdot 1 \cdot (e - 1)$$

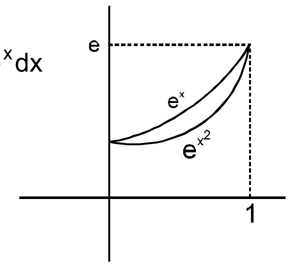
$$1 < \int_0^1 e^{x^2} dx < \frac{e+1}{2}$$

**Ex.30** Estimate the value of $\int_0^1 e^{x^2} dx$ by using $\int_0^1 e^x dx$.

**Sol.** For $x \in (0, 1)$, $e^{x^2} < e^x$

$$\Rightarrow 1 \times 1 < \int_0^1 e^{x^2} dx < \int_0^1 e^x dx$$

$$1 < \int_0^1 e^{x^2} dx < e - 1$$




**(P–9)** $\frac{d}{dx} \int_{g(x)}^{h(x)} f(t)\ dt = h'(x) f(h(x)) - g'(x) f(g(x))$

In particular $\frac{d}{dx} \int_a^{h(x)} f(t)\, dt = h'(x) f(h(x))$

[a is any constant independent of x]

or $\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$

## Solved Examples

**Ex.31** $\frac{d}{dt} \int_{t^2}^{t^3} \frac{1}{\log x} dx$

**Sol.** $\frac{d}{dt} \int_{t^2}^{t^3} \frac{1}{\log x} dx = \frac{1}{\log t^3} \cdot \frac{d}{dt}(t^3) - \frac{1}{\log t^2} \cdot \frac{d}{dt}(t^2)$

$$= \frac{3t^2}{3\log t} - \frac{2t}{2\log t} = \frac{t(t-1)}{\log t}$$

**Ex.32** If $F(x) = \int_x^{x^2} \sqrt{\sin t}\, dt$, then find F′(x).

**Sol.** $F'(x) = 2x \cdot \sqrt{\sin x^2} - 1 \cdot \sqrt{\sin x}$

**Ex.33** If $F(x) = \int_{e^{2x}}^{e^{3x}} \frac{t}{\log_e t} dt$, then find first and second derivative of F(x) with respect to $\ell n\, x$ at $x = \ell n\, 2$.

**Sol.** $\dfrac{dF(x)}{d(\ell n\, x)} = \dfrac{dF(x)}{dx}\,\dfrac{dx}{d(\ell n\, x)}$

$= \left[3\,.\,e^{3x}\,.\,\dfrac{e^{3x}}{\ell n\, e^{3x}} - 2\,.\,e^{2x}\,\dfrac{e^{2x}}{\ell n\, e^{2x}}\right] x \;= e^{6x} - e^{4x}.$

$\dfrac{d^2F(x)}{d(\ell n\, x)^2} = \dfrac{d}{d(\ell n\, x)}\,(e^{6x} - e^{4x}) = \dfrac{d}{dx}\,(e^{6x} - e^{4x}) \times$

$\dfrac{1}{d\ell nx/dx} = (6\,e^{6x} - 4\,e^{4x})\,x$

First derivative of F(x) at $x = \ell n\, 2$

(i.e. $e^x = 2$) is $2^6 - 2^4 = 48$

Second derivative of F(x) at $x = \ell n\, 2$

(i.e. $e^x = 2$) is $(6\,.\,2^6 - 4\,.\,2^4)\;.\;\ell n\, 2 = 5\,.\,2^6\,.\,\ell n\, 2.$

**Ex.34** Evaluate $\underset{x\to\infty}{\text{Lt}} \dfrac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x e^{2t^2}\,dt}$.

**Sol.** $\underset{x\to\infty}{\text{Lt}} \dfrac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x e^{2t^2}\,dt}$ $\left(\dfrac{\infty}{\infty}\text{ from}\right)$

Applying L′ Hospital rule

$= \underset{x\to\infty}{\text{Lt}} \dfrac{2\,.\int_0^x e^{t^2}dt\,.\,e^{x^2}}{1\,.\,e^{2x^2}} = \underset{x\to\infty}{\text{Lt}} \dfrac{2\,.\int_0^x e^{t^2}dt}{e^{x^2}}$

$= \underset{x\to\infty}{\text{Lt}} \dfrac{2\,.\,e^{x^2}}{2x\,.\,e^{x^2}} = 0$

**Ex.35** If $f(x) = \int_{\log_e^x}^{x} \dfrac{dt}{x+t}$, then find f′(x).

**Sol.** $f'(x) = \int_{\ell n\, x}^{x} \dfrac{-1}{(x+t)^2}\,dt + 1\,.\,\dfrac{1}{2x} - \dfrac{1}{x}\,\dfrac{1}{(x+\ell n\, x)} =$

$\left.\dfrac{1}{(x+t)}\right]_{\ell n\, x}^{x} + \dfrac{1}{2x} - \dfrac{1}{x\,(x+\ell n\, x)}$

$= \dfrac{1}{2x} - \dfrac{1}{x+\ell n\, x} + \dfrac{1}{2x} - \dfrac{1}{x\,(x+\ell n\, x)}$

$= \dfrac{1}{x} - \dfrac{x+1}{x\,(x+\ell n\, x)} = \dfrac{\ell n\, x - 1}{x\,(x+\ell n\, x)}$

**Alternatively :**

$f(x) = \int_{\ell n\, x}^{x} \dfrac{dt}{x+t} = \Big[\;\ell n\,(x+t)\Big]_{\ell n\, x}^{x}$

(treating 't' as constant)

$f(x) = \ell n\, 2x - \ell n\,(x + \ell n\, x)$

$f'(x) = \dfrac{1}{x} - \dfrac{1}{(x+\ell n\, x)}\left(1+\dfrac{1}{x}\right) = \dfrac{\ell n\, x - 1}{x(x+\ell n\, x)}$

**Ex.36** Evaluate $\int_0^1 \dfrac{x^b - 1}{\ell nx}$, 'b' being parameter.

**Sol.** Let $I(b) = \int_0^1 \dfrac{x^b - 1}{\ell nx}\,dx$

$\dfrac{dI(b)}{db} = \int_0^1 \dfrac{x^b \ell nx}{\ell nx}\,dx + 0 - 0$

(using modified Leibnitz Theorem)

$= \int_0^1 x^b\,dx = \left.\dfrac{x^{b+1}}{b+1}\right]_0^1 = \dfrac{1}{b+1}$

$I(b) = \ell n\,(b+1) + c$

$b = 0 \quad \Rightarrow \quad I(0) = 0$

$\therefore\; c = 0 \quad \therefore\; I(b) = \ell n\,(b+1)$

**Ex.37** Evaluate $\int_0^1 \dfrac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}$ dx, 'a' being parameter.

**Sol.** Let $I(a) = \int_0^1 \dfrac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}\,dx$

$\dfrac{dI(a)}{da} = \int_0^1 \dfrac{x}{(1+a^2x^2)}\,\dfrac{1}{x\sqrt{1-x^2}}\,dx$

$= \int_0^1 \dfrac{dx}{(1+a^2x^2)\sqrt{1-x^2}}$

Put $x = \sin t \quad \Rightarrow \quad dx = \cos t\,dt$

L.L. : $x = 0 \quad \Rightarrow \quad t = 0$

U.L. : $x = 1 \quad \Rightarrow \quad t = \dfrac{\pi}{2}$

$$\frac{dI(a)}{da} = \int_0^{\frac{\pi}{2}} \frac{1}{1+a^2\sin^2 t}\,\frac{1}{\cos t}\cos t\,dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{dt}{1+a^2\sin^2 t} \qquad = \int_0^{\frac{\pi}{2}} \frac{\sec^2 t\,dt}{1+(1+a^2)\tan^2 t}$$

$$= \frac{1}{\sqrt{1+a^2}}\tan^{-1}\left(\sqrt{1+a^2}\,\tan t\right)\Big]_0^{\frac{\pi}{2}} = \frac{1}{\sqrt{1+a^2}}\cdot\frac{\pi}{2}$$

$$\Rightarrow I(a) = \frac{\pi}{2}\,\ell n\left(a+\sqrt{1+a^2}\right)+c$$

But $I(0) = 0 \qquad \Rightarrow \qquad c = 0$

$$\Rightarrow I(a) = \frac{\pi}{2}\,\ell n\left(a+\sqrt{1+a^2}\right)$$

## REDUCTION FORMULAE IN DEFINITE INTEGRALS

**1.** If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$, then show that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

**Proof :** $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$

$$I_n = \left[-\sin^{n-1}x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}x\,.\cos^2 x\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,.(1-\sin^2 x)\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\,dx$$

$$I_n + (n-1)I_n = (n-1)I_{n-2}$$

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

**Note :**

**1.** $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$

**2.** $I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)\,.....\,I_0$ or $I_1$

according as n is even or odd. $I_0 = \frac{\pi}{2}$, $I_1 = 1$

Hence

$$I_n = \begin{cases}\left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)........\left(\frac{1}{2}\right).\frac{\pi}{2}, & \text{if n is even}\\[2mm] \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right)........\left(\frac{2}{3}\right).1, & \text{if n is odd}\end{cases}$$

## *Solved Examples*

**Ex.38** If $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$, then show that $I_n + I_{n-2} = \frac{1}{n-1}$

**Sol.** $I_n = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2}\,.\tan^2 x\,dx = \int_0^{\frac{\pi}{4}} (\tan x)^{n-2}(\sec^2 x - 1)\,dx$

$$= \int_0^{\frac{\pi}{4}} (\tan x)^{n-2}\sec^2 x\,dx - \int_0^{\frac{\pi}{4}} (\tan x)^{n-2}\,dx$$

$$= \left[\frac{(\tan x)^{n-1}}{n-1}\right]_0^{\frac{\pi}{4}} - I_{n-2}$$

$$I_n = \frac{1}{n-1} - I_{n-2}$$

$$\therefore\ I_n + I_{n-2} = \frac{1}{n-1}$$

**Ex.39** If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cdot \cos^n x\, dx$, then show that $I_{m,n} = \frac{m-1}{m+n} I_{m-2,\,n}$

**Sol.** $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{m-1} x\,(\sin x \cos^n x)\, dx$

$= \left[-\frac{\sin^{m-1} x \cdot \cos^{n+1} x}{n+1}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos^{n+1} x}{n+1}$

$(m-1)\sin^{m-2} x \cos x\, dx$

$= \left(\frac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} \sin^{m-2} x \cdot \cos^n x \cdot \cos^2 x\, dx$

$= \left(\frac{m-1}{n+1}\right) \int_0^{\frac{\pi}{2}} \left(\sin^{m-2} x \cdot \cos^n x - \sin^m x \cdot \cos^n x\right) dx$

$= \left(\frac{m-1}{n+1}\right) I_{m-2,n} - \left(\frac{m-1}{n+1}\right) I_{m,n} \Rightarrow \left(1 + \frac{m-1}{n+1}\right) I_{m,n}$

$= \left(\frac{m-1}{n+1}\right) I_{m-2,n}$

$I_{m,n} = \left(\frac{m-1}{m+n}\right) I_{m-2,n}$

**Note :**

$I_{m,n} = \left(\frac{m-1}{m+n}\right)\left(\frac{m-3}{m+n-2}\right)\left(\frac{m-5}{m+n-4}\right) \ldots\ldots\ldots I_{0,n}$ or $I_{1,n}$ according as m is even or odd.

$I_{0,n} = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$ and $I_{1,n} = \int_0^{\frac{\pi}{2}} \sin x \cdot \cos^n x\, dx$

$= \frac{1}{n+1}$

## IMPORTANT INTEGRALS

**(i) Walli's farmula**

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \frac{(n-1)(n-3)\ldots 2}{n(n-2)\ldots 1}$$

(if n is odd positive interger)

$$= \frac{(n-1)(n-3)\ldots 1}{n(n-2)\ldots 2}\left(\frac{\pi}{2}\right)$$ (if n is even positive integer)

## *Solved Examples*

**Ex.40** Evaluate $\int_0^{\pi/2} \sin^6 x\, dx$

**Sol.** $I = \frac{5.3.1}{6.2.4} \times \frac{\pi}{2} = \frac{5}{32}\pi$

**Ex.41** Evaluate $\int_0^{\pi/2} \cos^7 x\, dx$

**Sol.** $I = \frac{6.4.2}{7.5.3} = \frac{48}{105}$

**(ii)** $$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

where $\Gamma(n)$ is called **gamma function**

OR

$$\int_0^{\pi/2} \sin^m x \cos^n x dx =$$

$$= \frac{((m-1)(m-3)\ldots(2 \text{ or } 1))\,((n-1)(n-3)..(2 \text{ or } 1))}{(m+n)(m+n-2)\ldots.(2 \text{ or } 1)}$$

(if m and n both are not simultaneously even positive integers)

$$\frac{((m-1)(m-3)\ldots(1))((n-1)(n-3)\ldots(1))}{(m+n)(m+n-2)\ldots(2)}\left(\frac{\pi}{2}\right)$$

(if m and n are both even positive integers)

## Solved Examples

**Ex.42** Evaluate $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$.

**Sol.** Using gamma function formula

$$I = \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\Gamma\left(\frac{4+5+2}{2}\right)} = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{2\ \Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{\frac{3}{2}.\frac{1}{2}\sqrt{\pi}2.1}{2\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}} = \frac{8}{315}$$

**Ex.43** Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^2 x(\sin x + \cos x) \, dx$.

**Sol.** Given integral

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^3 x \, dx$$

$$= 0 + 2\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x \, dx$$

($\because \sin^3 x \cos^2 x$ is odd and $\sin^2 x \cos^3 x$ is even)

$$= 2.\ \frac{1.2}{5.3.1} = \frac{4}{15}$$

**Ex.44** Evaluate $\int_0^{\pi} x \sin^5 x \cos^6 x \, dx$.

**Sol.** Let $I = \int_0^{\pi} x \sin^5 x \cos^6 x \, dx$

$$I = \int_0^{\pi} (\pi - x)\sin^5(\pi - x)\cos^6(\pi - x)\, dx$$

$$= \pi\int_0^{\pi} \sin^5 .\cos^6 x \, dx - \int_0^{\pi} x \sin^5 x . \cos^6 x \, dx$$

$$\Rightarrow 2I = \pi . 2\int_0^{\frac{\pi}{2}} \sin^5 x . \cos^6 x \, dx$$

$$I = \pi\frac{4.2.5.3.1}{11.9.7.5.3.1} \qquad I = \frac{8\pi}{693}$$

**Ex.45** Evaluate $\int_0^1 x^3(1-x)^5 dx$.

**Sol.** Put $x = \sin^2\theta \quad \Rightarrow \quad dx = 2 \sin\theta \cos\theta \, d\theta$

L . L : $x = 0 \quad \Rightarrow \quad \theta = 0$

U.L. : $x = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$

$$\therefore \int_0^1 x^3(1-x)^5 dx = \int_0^{\frac{\pi}{2}} \sin^6\theta(\cos^2\theta)^5\, 2.\sin\theta.\cos\theta\, d\theta$$

$$= 2 . \int_0^{\frac{\pi}{2}} \sin^7\theta \cos^{11}\theta \, d\theta$$

$$= 2 . \frac{6.4.2.10.8.6.4.2}{18.16.14.12.10.8.6.4.2} = \frac{1}{504}$$

## SUMMATION OF SERIES BY INTEGRATION

For finding sum of an infintie series with the help of definite integration, following formula is used-

$$\lim_{n\to\infty}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right).\frac{1}{n} = \int_0^1 f(x)\, dx$$

The following methode is used to solve the questions on summation of series.

(i) After writing (r – 1)th or rth term of the series, express it in the form $\frac{1}{n}f\left(\frac{r}{n}\right)$. Therefore the given series will take the form

$$\lim_{n\to\infty}\sum_{r=0}^{n-1} f\left(\frac{r}{b}.\frac{1}{n}\right)$$

(ii) Now writing $\int$ in place of $\left(\lim_{n\to\infty}\sum\right)$, x in place of $\frac{1}{n}$, we get the integral $\int$ f(x) dx in place of above series.

(iii) The lower limit of this integral

$$= \lim_{n\to\infty} \left(\frac{r}{n}\right)_{r=0}$$

where r = 0 is taken corresponding of first term of the series and uppen limit

$$= \lim_{n\to\infty} \left(\frac{r}{n}\right)_{r=n-1}$$

where r = n – 1 is taken corresponding to the last term.

## Solved Examples

**Ex.46** Evaluate $\lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}\right]$

**Sol.** Limit $= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r} = \lim_{n\to\infty} \sum \frac{1}{1+\frac{r}{n}} = \frac{1}{n}$

$$= \int_0^1 \frac{1}{1+x} dx = [\log(1+x)]_0^1 = \log 2.$$

**Ex.47** Find the value of

$$\lim_{n\to\infty} \left[\frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + .... + \frac{1}{4n}\right]$$

**Sol.** Here $t_r = \frac{n}{(n+r)^2} = \frac{1}{n} . \frac{1}{[1+(r/n)]^2}$

Therefore given series

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{[1+(r/n)]^2} . \frac{1}{n} = \int \frac{1}{(1+x)^2} dx$$

Now lower limit $= \lim_{n\to\infty} \left(\frac{r}{n}\right)_{r=1} = 1$

upper limit $= \lim_{n\to\infty} \left(\frac{r}{n}\right)_{r=n} = 0$

∴ Given series

$$= \int_0^1 \frac{1}{(1+x)^2} dx = \left[-\frac{1}{1+x}\right]_0^1 = \frac{-1}{2} + 1 = 1/2$$