Indefinite Integration

DEFINITION

If f'(x) is derivative of f(x), then f(x) is perimitive or anti derivative or integration of f'(x). So differentiation and integration are inverse to each other.

For example

$\frac{d}{dx}(\sin x) = \cos x$, so integration of cos x is sin x.

$\frac{d}{dx}(\sin x + c) = \cos x$, so integration of cos x is sin x + c

$\frac{d}{dx}(f(x) + c) = F(x) \Rightarrow f(x) + c$ is perimitive of F(x).

$\Rightarrow \int F(x)\ dx = f(x) + c$

$\left\{\begin{array}{l}\int \text{is integral sign and} \\ \int F(x)dx \text{ means integration of } F(x) \text{ with respect to } x.\end{array}\right.$

where c is constant of integration.

STANDARD FORMULA

(i) $\int (ax + b)^n\ dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C,\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ \ell n\ |ax + b| + C$

(iii) $\int e^{ax+b}\ dx = \frac{1}{a}\ e^{ax+b} + C$

(iv) $\int a^{px+q}\ dx = \frac{1}{p}\ \frac{a^{px+q}}{\ell n\, a} + C;\ a > 0$

(v) $\int \sin(ax + b)\ dx = -\frac{1}{a}\cos(ax + b) + C$

(vi) $\int \cos(ax + b)\ dx = \frac{1}{a}\sin(ax + b) + C$

(vii) $\int \tan(ax + b)\ dx = \frac{1}{a}\ \ell n\ |\sec(ax + b)| + C$

(viii) $\int \cot(ax + b)\ dx = \frac{1}{a}\ \ell n\ |\sin(ax + b)| + C$

(ix) $\int \sec^2(ax + b)\ dx = \frac{1}{a}\tan(ax + b) + C$

(x) $\int \text{cosec}^2(ax + b)\ dx = -\frac{1}{a}\cot(ax + b) + C$

(xi) $\int \sec(ax + b).\tan(ax + b)\ dx = \frac{1}{a}\sec(ax + b) + C$

(xii) $\int \text{cosec}(ax + b).\cot(ax + b)\ dx = -\frac{1}{a}\ \text{cosec}\ (ax + b) + C$

(xiii) $\int \sec x\,dx = \ell n\,|\sec x + \tan x| + C$

OR $\ell n \left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$

(xiv) $\int \text{cosec}\,x\,dx = \ell n\,|\text{cosec}x - \cot x| + C$

OR $\ell n\left|\tan\frac{x}{2}\right| + C$ **OR** $-\ell n\,|\text{cosec}x + \cot x| + C$

(xv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

(xvi) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

(xvii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$

(xviii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ell n\left|x+\sqrt{x^2+a^2}\right| + C$

OR $\sinh^{-1}\frac{x}{a} + C$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ell n\left|x+\sqrt{x^2-a^2}\right| + C$

OR $\cosh^{-1}\frac{x}{a} + C$

(xx) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

(xxii) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

(xxiii) $\int \sqrt{x^2+a^2}\,dx$

$= \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\,\ell n\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + C$

(xxiv) $\int \sqrt{x^2-a^2}\,dx$

$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\,\ell n\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + C$

(xxv) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

(xxvi) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

BASIC THEOREMS ON INTEGRATION

If f(x), g(x) are two functions of a variable x and k is a constant, then

(i) $\int k\,f(x)\,dx = k\int f(x)\,dx$

(ii) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

(iii) $\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$

(iv) $\int \left(\frac{d}{dx}f(x)\right)dx = f(x) + c$

Solved Examples

Ex.1 Evaluate : $\int 4x^5\,dx$

Sol. $\int 4x^5\,dx = \frac{4}{6}x^6 + C = \frac{2}{3}x^6 + C.$

Ex.2 Evaluate : $\int \left(x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right)dx$

Sol. $\int \left(x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right)dx$

$= \int x^3\,dx + \int 5x^2\,dx - \int 4dx + \int \frac{7}{x}dx + \int \frac{2}{\sqrt{x}}dx$

$= \int x^3\,dx + 5.\int x^2\,dx - 4.\int 1.dx + 7.\int \frac{1}{x}dx + 2.\int x^{-1/2}dx$

$= \frac{x^4}{4} + 5.\frac{x^3}{3} - 4x + 7\,\ell n\,|x| + 2\left(\frac{x^{1/2}}{1/2}\right) + C$

$= \frac{x^4}{4} + \frac{5}{3}x^3 - 4x + 7\,\ell n\,|x| + 4\sqrt{x} + C$

Ex.3 Evaluate : $\int\left(e^{x\ell na}+e^{a\ell nx}+e^{a\ell na}\right)dx,\ a>0$

Sol. We have,

$$\int(e^{x\ell na}+e^{a\ell nx}+e^{a\ell na})\,dx$$

$$=\int(e^{\ell na^x}+e^{\ell nx^a}+e^{\ell na^a})\,dx=\int(a^x+x^a+a^a)\,dx$$

$$=\int a^x\,dx+\int x^a\,dx+\int a^a\,dx$$

$$=\frac{a^x}{\ell na}+\frac{x^{a+1}}{a+1}+a^a\,.\,x+C.$$

Ex.4 Evaluate : $\int\frac{2^x+3^x}{5^x}dx$

Sol. $\int\frac{2^x+3^x}{5^x}dx=\int\left(\frac{2^x}{5^x}+\frac{3^x}{5^x}\right)dx$

$$=\int\left[\left(\frac{2}{5}\right)^x+\left(\frac{3}{5}\right)^x\right]dx=\frac{(2/5)^x}{\ell n\frac{2}{5}}+\frac{(3/5)^x}{\ell n\frac{3}{5}}+C$$

Ex.5 Evaluate : $\int\sin^3x\cos^3x\,dx$

Sol. $\int\sin^3x\cos^3x\,dx=\frac{1}{8}\int(2\sin x\cos x)^3\,dx=\frac{1}{8}$

$$\int\sin^3 2x\,dx=\frac{1}{8}\int\frac{3\sin2x-\sin6x}{4}dx$$

$$=\frac{1}{32}\int(3\sin2x-\sin6x)\,dx$$

$$=\frac{1}{32}\left[-\frac{3}{2}\cos2x+\frac{1}{6}\cos6x\right]+C$$

Ex.6 Evaluate : $\int\frac{x^4}{x^2+1}dx$

Sol. $\int\frac{x^4}{x^2+1}dx=\int\frac{x^4-1+1}{x^2+1}dx$

$$=\int\left(\frac{x^4-1}{x^2+1}+\frac{1}{x^2+1}\right)dx$$

$$=\int(x^2-1)\,dx+\int\frac{1}{x^2+1}dx=\frac{x^3}{3}-x+\tan^{-1}x+C$$

Ex.7 Evaluate : $\int\frac{1}{4+9x^2}dx$

Sol. We have

$$\int\frac{1}{4+9x^2}dx=\frac{1}{9}\int\frac{1}{\frac{4}{9}+x^2}dx=\frac{1}{9}\int\frac{1}{(2/3)^2+x^2}dx$$

$$=\frac{1}{9}\cdot\frac{1}{(2/3)}\tan^{-1}\left(\frac{x}{2/3}\right)+C=\frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)+C$$

Ex.8 Evaluate : $\int\cos x\cos2x\,dx$

Sol. $\int\cos x\cos2x\,dx=\frac{1}{2}\int2\cos x\cos2x\,dx$

$$=\frac{1}{2}\int(\cos3x+\cos x)\,dx=\frac{1}{2}\left(\frac{\sin3x}{3}+\sin x\right)+C$$

Self Practice Problems

(1) Evaluate : $\int\tan^2x\,dx$

(2) Evaluate : $\int\frac{1}{1+\sin x}dx$

Answers : (1) $\tan x-x+C$ (2) $\tan x-\sec x+C$

METHOD OF INTEGRATION

Integration by Substitution

(a) When integrand is the product of two factors such that one is the derivative of the other i.e,

$$I=\int f(x)\,f'(x)dx$$

In this case we put f(x) = t to convert it into a standard integral.

Solved Examples

Ex.9 $\int\frac{\log x}{x}dx$

Sol. Let $\log x=t\ \Rightarrow\ \frac{1}{x}dx=dt$

$$\therefore\ I=\int t\,dt=\frac{1}{2}t^2+c=\frac{1}{2}(\log x)^2+c$$

(b) When integrand is a function of function

i.e. $\int f[\phi(x)]\,\phi'(x)\,dx$

Here we put $\phi(x)=t$ so that $\phi'(x)\,dx=dt$ and in that case the integrand is reduced to $\int f(t)\,dt$.

Ex.10 $\int x\cos x^2\,dx$

Sol. Let $x^2=t\ \Rightarrow\ x\,dx=\frac{1}{2}dt$

$$\therefore\ I=\frac{1}{2}\int\cos t\,dt=\frac{1}{2}\sin x^2+c)$$

INTEGRATION BY SUBSTITUTION

If we substitution $\phi(x) = t$ in an integral then

(i) everywhere x will be replaced in terms of new variable t.

(ii) dx also gets converted in terms of dt.

Solved Examples

Ex.11 Evaluate : $\int x^3 \sin x^4 \, dx$

Sol. We have

$$I = \int x^3 \sin x^4 \, dx$$

Let $x^4 = t \Rightarrow d(x^4) = dt$

$\Rightarrow 4x^3\,dx = dt \Rightarrow dx = \frac{1}{4x^3}\,dt$

$$I = \frac{1}{4}\int \sin t\, dt = -\frac{1}{4}\cos t + C = -\frac{1}{4}\cos x^4 + C$$

Ex.12 Evaluate : $\int \frac{(\ell n\, x)^2}{x}\,dx$

Sol. Let $I = \int \frac{(\ell n\, x)^2}{x}\,dx$

Put $\ell nx = t \Rightarrow \frac{1}{x}\,dx = dt$

$$\Rightarrow I = \int t^2 dt = \frac{t^3}{3} + c = \frac{(\ell n x)^3}{3} + C$$

Ex.13 Evaluate : $\int (1+\sin^2 x)\cos x\, dx$

Sol. Let $I = \int (1+\sin^2 x)\cos x\, dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\Rightarrow I = \int (1+t^2)\,dt = t + \frac{t^3}{3} + c = \sin x + \frac{\sin^3 x}{3} + C$$

Ex.14 Evaluate : $\int \frac{x}{x^4+x^2+1}\,dx$

Sol. We have,

$$I = \int \frac{x}{x^4+x^2+1}\,dx = \int \frac{x}{(x^2)^2+x^2+1}\,dx$$

$\{$Put $x^2 = t \Rightarrow x.dx = \frac{dt}{2}\}$

$$\Rightarrow I = \frac{1}{2}\int \frac{1}{t^2+t+1}\,dt = \frac{1}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dt$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)$$

$$+ C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C.$$

Integral of a function of the form (ax+b) dx

Here put ax + b = t and convert it into standard integral. Obviously if

$$\int f(x)\,dx = \phi(x), \text{ then } \int f(ax+b)\,dx = \frac{1}{a}\phi\,(ax+b)$$

Solved Examples

Ex.15 $\int \cos 3x \, \cos 5x\, dx$

Sol. $I = \int \cos 3x\, \cos 5x\, .dx$

$$\Rightarrow \frac{1}{2}\int (\cos 8x + \cos 2x)\,dx$$

$$= \frac{1}{2}\left[\frac{1}{8}\sin 8x + \frac{1}{2}\sin 2x\right] + c$$

Some standard forms of integrals

The following three forms are very useful to write integral directly.

(i) $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$ (provided $n \neq -1$)

(ii) $\int \frac{f'(x)}{f(x)}\,dx = \log[f(x)] + c$

(iii) $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$

Solved Examples

Ex.16 $\int \frac{(\sin^{-1}x)^2}{\sqrt{1-x^2}}dx =$

Sol. $\frac{1}{3}(\sin^{-1}x)^3 + c$

Ex.17 $\int \frac{\sec^2 x}{\sqrt{\tan x}}dx$

Sol. Let $t = \tan x$; $dt = \sec^2 x\,dx$;

$I = \int \frac{dt}{\sqrt{t}} = 2t^{1/2} + c = 2\sqrt{\tan x} + c$

Ex.18 $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$

Sol. Let $e^x + e^{-x} = t$

$(e^x - e^{-x})\,dx = dt$ $\therefore$ $I = \int \frac{dt}{t} = \log t + c$

$\Rightarrow \log(e^x + e^{-x}) + c$

Note:

(i) $\int \frac{dx}{x(x^n+1)}$; $n \in N$

Take x^n common & put $1 + x^{-n} = t$.

(ii) $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}$; $n \in N$,

take x^n common & put $1 + x^{-n} = t^n$

(iii) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$;

take x^n common as x and put $1 + x^{-n} = t$.

Standard Substitution

Following standard substitution will be useful-

Integrand form	**Substitution**
(i) $\sqrt{a^2-x^2}$ or $\frac{1}{\sqrt{a^2-x^2}}$	$x = a\sin\theta$
(ii) $\sqrt{x^2+a^2}$ or $\frac{1}{\sqrt{x^2+a^2}}$	$x = a\tan\theta$ or $x = a\sinh\theta$
(iii) $\sqrt{x^2-a^2}$ or $\frac{1}{\sqrt{x^2-a^2}}$	$x = a\sec\theta$ or $x = a\cosh\theta$
(iv) $\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$ or $\sqrt{x(a+x)}$ or $\frac{1}{\sqrt{x(a+x)}}$	$x = a\tan^2\theta$
(v) $\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$ or $\sqrt{x(a-x)}$ or $\frac{1}{\sqrt{x(a-x)}}$	$x = a\sin^2\theta$
(vi) $\sqrt{\frac{x}{x-a}}$ or $\sqrt{\frac{x-a}{x}}$ or $\sqrt{x(x-a)}$ or $\frac{1}{\sqrt{x(x-a)}}$	$x = a\sec^2\theta$
(vii) $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
(viii) $\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$ $(\beta > \alpha)$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

Ex.19 $\int \frac{1+\sin x}{1-\sin x}\,dx$

Sol. $I = \int \frac{1+\sin x}{1-\sin x}\,dx$. $= \int \left[\frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)}\right]^2 dx$

$= \int \tan^2\left(\frac{\pi}{4}+\frac{x}{2}\right) dx$

$= \int [\sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right) - 1]\,dx \quad = 2\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) - x + c$

Ex.20 $\int \frac{dx}{\sqrt{x(a-x)}} =$

Sol. Let $x = a\sin^2\theta$ then

$dx = 2a\sin\theta\cos\theta\,d\theta$

$\therefore\ I = \int \frac{2a\sin\theta.\cos\theta}{\sqrt{a\sin^2\theta.\,a\cos^2\theta}} = 2\int d\theta = 2\theta + c$

$= 2\sin^{-1}(\sqrt{x-a}) + c$

Integration of Rational Functions

(a) When denominator can be factorized (Using partial fractions)

If denominator of a rational algebraic function can be factorized, then its integral can easily be obtained by splitting it into partial fractions. The following two standard integrals may be so obtained

- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$
- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$

Solved Examples

Ex.21 $\int \frac{dx}{(x-1)(x-2)}$

Sol. $\int \left(\frac{-1}{x-1}+\frac{1}{x-2}\right) dx \quad = \log\left(\frac{x-2}{x-1}\right) + c$

Ex.22 $\int \frac{2x}{x^2+3x+2}\,dx$

Sol. $\int \frac{2x}{(x+1)(x+2)}\,dx = \int \left(\frac{-2}{x+1}+\frac{4}{x+2}\right) dx$

$= 4\log(x+2) - 2\log(x+1) + c$

$= 2\log\frac{(x+2)^2}{x+1} + c$

(b) When denominator can not be factorized

In this case integral may be in the form

$\int \frac{dx}{ax^2+bx+c}$, $\int \frac{px+q}{ax^2+bx+c}\,dx$

For first integral we express its denominator in the form $(x+\alpha)^2 \pm \beta$ and use the previous results.

For second integral we express its numerator in the form Nr $= A$ (derivative of Dr) $+ B$ and then we integral it easily.

Solved Examples

Ex.23 $\int \frac{dx}{x^2+x+1}$

Sol. $\int \frac{dx}{(x+1/2)^2+3/4} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right) + c$

$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

Ex.24 $\int \frac{x+1}{x^2+x+1}\,dx$

Sol. $\frac{1}{2}\int \frac{(2x+1)+1}{x^2+x+1}\,dx =$

$\frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{1}{2}\int \frac{dx}{x^2+x+1}$

$= \frac{1}{2}\log(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

(c) Integral of rational functions containing only even powers of x

To find integral of such functions, first we divide numerator and denominator by x^2, then express N^r as $d(x \pm 1/x)$ and D^r as a function of $(x \pm 1/x)$. Following examples illustrate it.

Solved Examples

Ex.25 $\int \frac{x^2+1}{x^4-x^2+1}\,dx$

Sol. $\int \frac{1+1/x^2}{x^2-1+1/x^2}\,dx = \int \frac{d(x-1/x)}{(x-1/x)^2+1}$

$= \tan^{-1}\left(x-\frac{1}{x}\right) + c = \tan^{-1}\left(\frac{x^2-1}{x}\right) + c$

Ex.26 $\int \frac{x^2-1}{x^4+1}dx$

Sol. $\int \frac{1-1/x^2}{x^2+1/x^2}dx = \int \frac{d(x+1/x)}{(x+1/x)^2-2}$

$= \frac{1}{2\sqrt{2}}\log\frac{(x+1/x)-\sqrt{2}}{(x+1/x)+\sqrt{2}}+c$

$= \frac{1}{2\sqrt{2}}\log\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}+c$

Integration of irrational functions

If any one term in Nr and Dr is irrational then it is made rational by suitable substitution. Also if integral is of the form

$$\int \frac{dx}{\sqrt{ax^2+bx+c}}, \quad \int \sqrt{ax^2+bx+c}\,dx$$

then we integrate it by expressing

$ax^2 + bx + c = (x + \alpha)^2 + \beta$.

Also for integrals of the form

$$\int \frac{a'x'+b'}{\sqrt{ax^2+bx+c}}dx, \quad \int (a'x+b')\sqrt{ax^2+bx+c}\,dx$$

first we express a'x + b' in the form

$$a'x + b' = A\left\{\frac{d}{dx}(ax^2+bx+c)\right\}+B$$

and then proceed as usual with standard forms.

Solved Examples

Ex.27 $\int \frac{dx}{\sqrt{x^2+2x}}$

Sol. $\int \frac{dx}{\sqrt{(x+1)^2-1}} = \cosh^{-1}(x+1)+c$

Ex.28 $\int \sqrt{x^2+2x}\,dx$

Sol. $\int \sqrt{(x+1)^2-1}\,dx \quad =$

$\frac{1}{2}(x+1)\sqrt{x^2+2x}-\frac{1}{2}\cosh^{-1}(x+1)+c$

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

(i) $\int \frac{dx}{a+b\sin^2x}$ OR $\int \frac{dx}{a+b\cos^2x}$

OR $\int \frac{dx}{a\sin^2x+b\sin x\cos x+c\cos^2x}$

Multiply Nr & Dr by $\sec^2 x$ & put $\tan x = t$.

(ii) $\int \frac{dx}{a+b\sin x}$ OR $\int \frac{dx}{a+b\cos x}$

OR $\int \frac{dx}{a+b\sin x+c\cos x}$

Convert sines & cosines into their respective tangents of half the angles and then, put $\tan\frac{x}{2} = t$

(iii) $\int \frac{a.\cos x+b.\sin x+c}{\alpha\cos x+\beta\sin x+\gamma}dx.$

Express Nr $\equiv$ A(Dr) + B $\frac{d}{dx}$ (Dr) + C & proceed.

Solved Examples

Ex.29 Evaluate : $\int \frac{1}{1+\sin x+\cos x}dx$

Sol. $I = \int \frac{1}{1+\sin x+\cos x}dx$

$= \int \frac{1}{1+\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}dx$

$= \int \frac{1+\tan^2 x/2}{1+\tan^2 x/2+2\tan x/2+1-\tan^2 x/2}dx$

$= \int \frac{\sec^2 x/2}{2+2\tan x/2}dx$

Putting $\tan\frac{x}{2} = t$ and $\frac{1}{2}\sec^2\frac{x}{2}dx = dt$, we get

$I = \int \frac{1}{t+1}dt = \ell n\,|t+1| + C = \ell n\left|\tan\frac{x}{2}+1\right| + C$

Ex.30 Evaluate : $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$

Sol. $I = \int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$

Let $3\sin x + 2\cos x = \lambda.\ (3\cos x + 2\sin x) + \mu \frac{d}{dx}(3\cos x + 2\sin x)$

$\Rightarrow 3\sin x + 2\cos x = \lambda(3\cos x + 2\sin x) + \mu(-3\sin x + 2\cos x)$

Comparing the coefficients of sin x and cos x on both sides, we get

$\lambda = \frac{12}{13}$ and $\mu = -\frac{5}{13}$

$\therefore\ I = \int \frac{\lambda(3\cos x + 2\sin x) + \mu(-3\sin x + 2\cos x)}{3\cos x + \sin x}dx$

$= \lambda \int 1.\,dx + \mu \int \frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$

$= \lambda x + \mu \int \frac{dt}{t}$, where $t = 3\cos x + 2\sin x$

$= \lambda x + \mu\, \ell n\,|t| + C$

$= \frac{12}{13}x - \frac{5}{13}\,\ell n\,|3\cos x + 2\sin x| + C$

Ex.31 Evaluate : $\int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}\,dx$

Sol. We have,

$I = \int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}\,dx$

Let $3\cos x + 2 = \lambda(\sin x + 2\cos x + 3) + \mu(\cos x - 2\sin x) + \nu$

Comparing the coefficients of sin x, cos x and constant term on both sides, we get

$\lambda - 2\mu = 0,\ 2\lambda + \mu = 3,\ 3\lambda + \nu = 2$

$\Rightarrow \lambda = \frac{6}{5},\ \mu = \frac{3}{5}$ and $\nu = -\frac{8}{5}$

$\therefore\ I = \int \frac{\lambda(\sin x + 2\cos x + 3) + \mu(\cos x - 2\sin x) + \nu}{\sin x + 2\cos x + 3}\,dx$

$\Rightarrow I = \lambda \int dx + \mu \int \frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3}\,dx + \nu \int \frac{1}{\sin x + 2\cos x + 3}\,dx$

$\Rightarrow I = \lambda x + \mu \log|\sin x + 2\cos x + 3| + \nu I_1$

where $I_1 = \int \frac{1}{\sin x + 2\cos x + 3}\,dx$

Putting, $\sin x = \frac{2\tan x/2}{1 + \tan^2 x/2}$, $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$, we get

$I_1 = \int \frac{1}{\frac{2\tan x/2}{1 + \tan^2 x/2} + \frac{2(1 - \tan^2 x/2)}{1 + \tan^2 x/2} + 3}\,dx$

$= \int \frac{1 + \tan^2 x/2}{2\tan x/2 + 2 - 2\tan^2 x/2 + 3(1 + \tan^2 x/2)}\,dx$

$= \int \frac{\sec^2 x/2}{\tan^2 x/2 + 2\tan x/2 + 5}\,dx$

Putting $\tan\frac{x}{2} = t$ and $\frac{1}{2}\sec^2\frac{x}{2} = dt$ or $\sec^2\frac{x}{2}\,dx = 2\,dt$, we get

$I_1 = \int \frac{2dt}{t^2 + 2t + 5}$

$= 2\int \frac{dt}{(t+1)^2 + 2^2} = \frac{2}{2}\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{2}\right)$

Hence, $I = \lambda x + \mu \log|\sin x + 2\cos x + 3| + \nu \tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{2}\right) + C$

where $\lambda = \frac{6}{5},\ \mu = \frac{3}{5}$ and $\nu = -\frac{8}{5}$

Ex.32 Evaluate : $\int \frac{dx}{1+3\cos^2 x}$

Sol. Multiply Nr. & Dr. of given integral by $\sec^2 x$, we get

$$I = \int \frac{\sec^2 x\, dx}{\tan^2 x + 4} = \frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right) + C$$

Ex.33 Evaluate $\int \frac{dx}{1+3\sin^2 x}$

Sol. $I = \int \frac{\sec^2 x\, dx}{\sec^2 x + 3\tan^2 x}$

(Dividing Num^r and Den^r by $\cos^2 x$)

$$= \int \frac{\sec^2 x\, dx}{1+4\tan^2 x} = \frac{1}{2}\tan^{-1}(2\tan x) + c$$

(i) $\int \frac{dx}{a\sin x + b}$,

(ii) $\int \frac{dx}{a\cos x + b}$,

(iii) $\int \frac{dx}{a\sin x + b\cos x}$,

(iv) $\int \frac{dx}{a\sin x + b\cos x + c}$.

For such types of integration first we express them in terms of tan x/2 by replacing

$\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$ and $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ and the put $\tan x/2 = t$.

Solved Examples

Ex.34 $\int \frac{dx}{5+4\cos x}$

Sol. $\int \frac{dx}{5+4\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)} = \int \frac{\sec^2 x/2\, dx}{9+\tan^2 x/2}$

$$= 2.\frac{1}{3}\tan^{-1}\left(\frac{\tan x/2}{3}\right) + c$$

Some integrals of different expressions of e^x

(i) $\int \frac{ae^x}{b+ce^x}dx$ [put $e^x = t$]

(ii) $\int \frac{1}{1+e^x}dx$

[multiplying and divide by e^{-x} and put $e^{-x} = t$]

(iii) $\int \frac{1}{1-e^x}dx$

[multiplying and divide by e^{-x} and put $e^{-x} = t$]

(iv) $\int \frac{1}{e^x - e^{-x}}dx$ [multiply and divided by e^x]

(v) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$ $\left[\frac{f'(x)}{f(x)}\text{form}\right]$

(vi) $\int \frac{e^x + 1}{e^x - 1}dx$ [multiply and divide by $e^{-x/2}$]

(vii) $\int \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 dx$ [integrand = $\tanh^2 x$]

(viii) $\int \left(\frac{e^{2x} + 1}{e^{2x} - 1}\right)^2 dx$ [integrand = $\coth^2 x$]

(ix) $\int \frac{1}{(e^x + e^{-x})^2}dx$ [integrand = $\frac{1}{4}\text{sech}^2 x$]

(x) $\int \frac{1}{(e^x - e^{-x})^2}dx$ [integrand = $\frac{1}{4}\text{cosech}^2 x$]

(xi) $\int \frac{1}{(1+e^x)(1-e^{-x})}dx$

[multiply and divide by e^x and put $e^x = t$]

(xii) $\int \frac{1}{\sqrt{1-e^x}}dx$ [multiply and divide by $e^{-x/2}$]

(xiii) $\int \frac{1}{\sqrt{1+e^x}}dx$ [multiply and divide by $e^{-x/2}$]

(xiv) $\int \frac{1}{\sqrt{e^x - 1}}dx$ [multiply and divide by $e^{-x/2}$]

(xv) $\int \frac{1}{\sqrt{2e^x - 1}}dx$ [multiply and divide by $\sqrt{2}e^{-x/2}$]

(xvi) $\int \sqrt{1-e^x}\,dx$ [integrand = $(1-e^x)/\sqrt{1-e^x}$]

(xvii) $\int \sqrt{1-e^x}\,dx$ [integrand = $(1+e^x)/\sqrt{1+e^x}$]

(xviii) $\int \sqrt{e^x - 1}\,dx$ [integrand = $(e^x - 1)/\sqrt{e^x - 1}$]

(xix) $\int \sqrt{\frac{e^x + a}{e^x - a}}\,dx$ [integrand = $(e^x + a)/\sqrt{e^{2x} - a^2}$]

Solved Examples

Ex.35 $\int \frac{1}{e^x - 1}\,dx$

Sol. Here $I = \int \frac{1}{e^x - 1}\,dx$

$\Rightarrow \int \frac{e^{-x}}{1 - e^{-x}}\,dx = \log(1 - e^{-1}) + C$

Ex.36 $\int \sqrt{e^x - 1}\,dx$

Sol. Here $I = \int \sqrt{e^x - 1}\,dx$

$\Rightarrow \int \frac{e^x - 1}{\sqrt{e^x - 1}}\,dx = \int \frac{e^x}{\sqrt{e^x - 1}}\,dx - \int \frac{1}{\sqrt{e^x - 1}}\,dx$

Let $e^x - 1 = t^2$, then $e^x\,dx = 2t\,dt$

$\therefore\ I = 2\int dt - \int \frac{2}{t^2 + 1}\,dt = 2t - 2\tan^{-1}(t) + C$

$= 2\left[\sqrt{e^x - 1} - \tan^{-1}\sqrt{e^x - 1}\right] + C$

Integration of type $\int \sin^m x . \cos^n x\,dx$

Case - I

If m and n are even natural number then converts higher power into higher angles.

Case - II

If at least one of m or n is odd natural number then if m is odd put cosx = t and vice-versa.

Case - III

When m + n is a negative even integer then put tan x = t.

Solved Examples

Ex.37 Evaluate : $\int \sin^5 x \cos^4 x\,dx$

Sol. Let $I = \int \sin^5 x \cos^4 x\,dx$ put $\cos x = t$

$\Rightarrow -\sin x\,dx = dt$

$\Rightarrow I = -\int (1 - t^2)^2 . t^4 . dt = -\int (t^4 - 2t^2 + 1)\,t^4\,dt$

$= -\int (t^8 - 2t^6 + t^4)\,dt$

$= -\frac{t^9}{9} + \frac{2t^7}{7} - \frac{t^5}{5} + C = -\frac{\cos^9 x}{9} + 2\,\frac{\cos^7 x}{7}$

$-\frac{\cos^5 x}{5} + C$

Ex.38 Evaluate : $\int (\sin x)^{1/3} (\cos x)^{-7/3}\,dx$

Sol. Here $m + n = \frac{1}{3} - \frac{7}{3} = -2$ (a negative integer)

$\therefore \int (\sin x)^{1/3} (\cos x)^{-7/3}\,dx = \int (\tan x)^{1/3} \frac{1}{\cos^2 x}$

dx {put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$}

$= \int t^{1/3}\,dt = \frac{3}{4}\,t^{4/3} + C = \frac{3}{4}\,(\tan x)^{4/3} + C$

Ex.39 Evaluate : $\int \sin^2 x \cos^4 x\,dx$

Sol. $\frac{1}{8} \int \sin^2 2x (1 + \cos 2x)\,dx$

$= \frac{1}{8} \int \sin^2 2x\,dx + \frac{1}{8} \int \sin^2 2x \cos 2x\,dx$

$= \frac{1}{16} \int (1 - \cos 4x)\,dx + \frac{1}{16} \left(\frac{\sin^3 2x}{3}\right)$

$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C$

INTEGRATION OF RATIONAL ALGEBRAIC FUNCTIONS BY USING PARTIAL FRACTIONS

PARTIAL FRACTIONS :

If f(x) and g(x) are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function of x.

If degree of f(x) < degree of g(x), then $\frac{f(x)}{g(x)}$ is called a proper rational function.

If degree of f(x) ≥ degree of g(x) then $\frac{f(x)}{g(x)}$ is called an improper rational function.

If $\frac{f(x)}{g(x)}$ is an improper rational function, we divide f(x) by g(x) so that the rational function $\frac{f(x)}{g(x)}$ is expressed in the form $\phi(x) + \frac{\Psi(x)}{g(x)}$, where $\phi(x)$ and $\Psi(x)$ are polynomials such that the degree of $\Psi(x)$ is less than that of g(x). Thus, $\frac{f(x)}{g(x)}$ is expressible as the sum of a polynomial and a proper rational function.

Any proper rational function $\frac{f(x)}{g(x)}$ can be expressed as the sum of rational functions, each having a simple factor of g(x). Each such fraction is called a partial fraction and the process of obtained them is called the resolutions or decomposition of $\frac{f(x)}{g(x)}$ into partial fractions.

The resolution of $\frac{f(x)}{g(x)}$ into partial fractions depends mainly upon the nature of the factors of g(x) as discussed below :

CASE I

When denominator is expressible as the product of non-repeating linear factors.

Let $g(x) = (x - a_1)(x - a_2) .....(x - a_n)$. Then, we assume that

$$\frac{f(x)}{g(x)} \equiv \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + ... + \frac{A_n}{x-a_n}$$

where $A_1, A_2, ... A_n$ are constants and can be determined by equating the numerator on R.H.S. to the numerator on L.H.S. and then substituting $x = a_1, a_2, ..., a_n$.

Solved Examples

Ex.40 Resolve $\frac{3x+2}{x^3-6x^2+11x-6}$ into partial fractions.

Sol. We have, $\frac{3x+2}{x^3-6x^2+11x-6} = \frac{3x+2}{(x-1)(x-2)(x-3)}$

Let $\frac{3x+2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$.

Then,

$$\Rightarrow \frac{3x+2}{(x-1)(x-2)(x-3)}$$

$$= \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$\Rightarrow 3x + 2 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$ ....(i)

Putting $x - 1 = 0$ or $x = 1$ in (i), we get

$5 = A(1 - 2)(1 - 3) \Rightarrow A = \frac{5}{2}$,

Putting $x - 2 = 0$ or, $x = 2$ in (i), we obtain

$8 = B(2 - 1)(2 - 3) \Rightarrow B = -8$.

Putting $x - 3 = 0$ or, $x = 3$ in (i), we obtain

$11 = C(3 - 1)(3 - 2) \Rightarrow C = \frac{11}{2}$.

$$\therefore \frac{3x+2}{x^3-6x^2+11x-6} = \frac{3x+2}{(x-1)(x-2)(x-3)}$$

$$= \frac{5}{2(x-1)} - \frac{8}{x-2} + \frac{11}{2(x-3)}$$

Note : In order to determine the value of constants in the numerator of the partial fraction corresponding to the non-repeated linear factor (px + q) in the denominator of a rational expression, we may proceed as follows :

Replace x by $-\frac{q}{p}$ (obtained by putting $px + q = 0$) everywhere in the given rational expression except in the factor px + q itself. For example, in the above illustration the value of A is obtained by replacing x by 1 in all factors of $\frac{3x+2}{(x-1)(x-2)(x-3)}$ except (x – 1) i.e.

$$A = \frac{3\times1+2}{(1-2)(1-3)} = \frac{5}{2}$$

Similarly, we have

$$B = \frac{3\times2+1}{(1-2)(2-3)} = -8 \text{ and, } C = \frac{3\times3+2}{(3-1)(3-2)} = \frac{11}{2}$$

Solved Examples

Ex.41 Resolve $\dfrac{x^3-6x^2+10x-2}{x^2-5x+6}$ into partial fractions.

Sol. Here the given function is an improper rational function. On dividing we get

$$\frac{x^3-6x^2+10x-2}{x^2-5x+6}=x-1+\frac{(-x+4)}{(x^2-5x+6)} \text{ ....(i)}$$

we have, $\dfrac{-x+4}{x^2-5x+6}=\dfrac{-x+4}{(x-2)(x-3)}$

So, let $\dfrac{-x+4}{(x-2)(x-3)}=\dfrac{A}{x-2}+\dfrac{B}{x-3}$, then

$-x+4=A(x-3)+B(x-2)$ ....(ii)

Putting $x-3=0$ or $x=3$ in (ii), we get

$1=B(1) \Rightarrow B=1.$

Putting $x-2=0$ or $x=2$ in (ii), we get

$2=A(2-3)\Rightarrow A=-2$

$$\therefore \frac{-x+4}{(x-2)(x-3)}=\frac{-2}{x-2}+\frac{1}{x-3}$$

Hence $\dfrac{x^3-6x^2+10x-2}{x^2-5x+6}=x-1-\dfrac{2}{x-2}+\dfrac{1}{x-3}$

CASE II

When the denominator g(x) is expressible as the product of the linear factors such that some of them are repeating.

Example $\dfrac{1}{g(x)}=\dfrac{1}{(x-a)^k(x-a_1)(x-a_2).......(x-a_r)}$
this can be expressed as

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+....+\frac{A_k}{(x-a)^k}+\frac{B_1}{(x-a_1)}+\frac{B_2}{(x-a_2)}+......+\frac{B_r}{(x-a_r)}$$

Now to determine constants we equate numerators on both sides. Some of the constants are determined by substitution as in case I and remaining are obtained by equating the coefficient of same power of x.

The following example illustrate the procedure.

Solved Examples

Ex.42 Resolve $\dfrac{3x-2}{(x-1)^2(x+1)(x+2)}$ into partial fractions, and evaluate $\displaystyle\int\frac{(3x-2)dx}{(x-1)^2(x+1)(x+2)}$

Sol. Let $\dfrac{3x-2}{(x-1)^2(x+1)(x+2)}$

$$=\frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{A_3}{x+1}+\frac{A_4}{x+2}$$

$\Rightarrow 3x-2=A_1(x-1)(x+1)(x+2)+A_2(x+1)(x+2)$
$+A_3(x-1)^2(x+2)+A_4(x-1)^2(x+1)$ ....(i)

Putting $x-1=0$ or, $x=1$ in (i) we get

$1=A_2(1+1)(1+2)\Rightarrow A_2=\dfrac{1}{6}$

Putting $x+1=0$ or, $x=-1$ in (i) we get

$-5=A_3(-2)^2(-1+2)\Rightarrow A_3=-\dfrac{5}{4}$

Putting $x+2=0$ or, $x=-2$ in (i) we get

$-8=A_4(-3)^2(-1)\Rightarrow A_4=\dfrac{8}{9}$

Now equating coefficient of x^3 on both sides, we get $0=A_1+A_3+A_4$

$\Rightarrow A_1=-A_3-A_4=\dfrac{5}{4}-\dfrac{8}{9}=\dfrac{13}{36}$

$$\therefore \frac{3x-2}{(x-1)^2(x+1)(x+2)}$$

$$=\frac{13}{36(x-1)}+\frac{1}{6(x-1)^2}-\frac{5}{4(x+1)}+\frac{8}{9(x+2)}$$

and hence $\displaystyle\int\frac{(3x-2)dx}{(x-1)^2(x+1)(x+2)}$

$$=\frac{13}{36}\ \ell n|x-1|-\frac{1}{6(x-1)}-\frac{5}{4}\ \ell n|x+1|+\frac{8}{9}\ \ell n|x+2|+C$$

CASE III

When some of the factors of denominator g(x) are quadratic but non-repeating. Corresponding to each quadratic factor $ax^2 + bx + c$, we assume partial fraction of the type $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of similar powers of x in the numerator of both sides. In practice it is advisable to assume partial fractions of the type

$$\frac{A(2ax+b)}{ax^2+bx+c} + \frac{B}{ax^2+bx+c}$$

The following example illustrates the procedure

Solved Examples

Ex.43 Resolve $\frac{2x-1}{(x+1)(x^2+2)}$ into partial fractions and evaluate $\int \frac{2x-1}{(x+1)(x^2+2)}dx$

Sol. Let $\frac{2x-1}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2}$. Then,

$$\frac{2x-1}{(x+1)(x^2+2)} = \frac{A(x^2+2)+(Bx+C)(x+1)}{(x+1)(x^2+2)}$$

$\Rightarrow 2x - 1 = A(x^2 + 2) + (Bx + C)(x + 1)$ ....(i)

Putting $x + 1 = 0$ or, $x = -1$ in (i),

we get $-3 = A(3) \Rightarrow A = -1$.

Comparing coefficients of the like powers of x on both sides of (i), we get

$A + B = 0, C + 2A = -1$ and $C + B = 2$

$\therefore$ $-1 + B = 0, C - 2 = -1$ (Putting $A = -1$)

$\Rightarrow$ $B = 1, C = 1$

$$\therefore \quad \frac{2x-1}{(x+1)(x^2+2)} = -\frac{1}{x+1} + \frac{x+1}{x^2+2}$$

Hence $\int \frac{2x-1}{(x+1)(x^2+2)}dx$

$$= -\ell n|x+1| + \frac{1}{2}\ell n|x^2+2| + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

CASE IV

When some of the factors of the denominator g(x) are quadratic and repeating fractions of the form

$$\left\{\frac{A_0(2ax+b)}{ax^2+bx+c} + \frac{A_1}{ax^2+bx+c}\right\} +$$

$$\left\{\frac{A_1(2ax+b)}{(ax^2+bx+c)^2} + \frac{A_2}{(ax^2+bx+c)^2}\right\}$$

$$+ \ldots\ldots. + \left\{\frac{A_{2k-1}(2ax+b)}{(ax^2+bx+c)^k} + \frac{A_{2k}}{(ax^2+bx+c)^k}\right\}$$

The following example illustrates the procedure.

Solved Examples

Ex.44 Resolve $\frac{2x-3}{(x-1)(x^2+1)^2}$ into partial fractions.

Sol. Let $\frac{2x-3}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$.

Then,

$2x - 3 = A(x^2 + 1)^2 + (Bx + C)(x - 1)(x^2 + 1) + (Dx + E)(x - 1)$ ......(i)

Putting $x = 1$ in (i), we get $-1 = A(1 + 1)^2$

$\Rightarrow A = -\frac{1}{4}$

Comparing coefficients of like powers of x on both side of (i), we have

$A + B = 0, C - B = 0, 2A + B - C + D = 0,$

$C + E - B - D = 2$ and $A - C - E = -3$.

Putting $A = -\frac{1}{4}$ and solving these equations, we get

$B = \frac{1}{4} = C, D = \frac{1}{2}$ and $E = \frac{5}{2}$

$$\therefore \frac{2x-3}{(x-1)(x^2+1)^2} = \frac{-1}{4(x-1)} + \frac{x+1}{4(x^2+1)} + \frac{x+5}{2(x^2+1)^2}$$

Ex.45 Resolve $\frac{2x}{x^3-1}$ into partial fractions.

Sol. We have, $\frac{2x}{x^3-1} = \frac{2x}{(x-1)(x^2+x+1)}$

So, let $\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$. Then,

$2x = A(x^2+x+1) + (Bx+C)(x-1)$ ....(i)

Putting $x - 1 = 0$ or, $x = 1$ in (i), we get $2 = 3A$

$\Rightarrow A = \frac{2}{3}$

Putting $x = 0$ in (i), we get $A - C = 0 \Rightarrow C = A = \frac{2}{3}$

Putting $x = -1$ in (i), we get $-2 = A + 2B - 2C$.

$\Rightarrow -2 = \frac{2}{3} + 2B - \frac{4}{3} \Rightarrow B = -\frac{2}{3}$

$\therefore \frac{2x}{x^3-1} = \frac{2}{3} \cdot \frac{1}{x-1} + \frac{(-2/3)x + 2/3}{x^2+x+1}$ or $\frac{2x}{x^3-1}$

$= \frac{2}{3}\frac{1}{x-1} + \frac{2}{3}\frac{1-x}{x^2+x+1}$

Integration by Parts

If u and v are the differentiable functions of x, then

$$\int u.v\ dx = u\int v dx - \int\left[\left(\frac{d}{dx}(u)\right)\left(\int v dx\right)\right] dx.$$

i.e. Integral of the product of two functions

= first function x integral of second function – $\int$ [derivative of first) x (Integral of second)]

(i) How to choose Ist and IInd function : If two functions are of different types take that function as Ist which comes first in the word ILATE, where I stands for inverse circular function, L stands for logrithmic function, A stands for algebric functions, T stands for trigonometric and E for exponential functions.

(ii) For the integration of logarthmic or inverse trigonometric functions alone, take unity (1) as the second function

Solved Examples

Ex.46 Evaluate $\int x^2 e^x dx$

Sol. $I = \int x^2 e^x dx = x^2 e^x - \int 2x.e^x dx$

$= x^2 e^x - 2[x.e^x - \int 1.e^x dx]$ (taking x as first function)

$= x^2 e^x - 2xe^x + 2e^x + c$

If the integral is of the form $\int e^x[f(x) + f'(x)]\,dx$ then by breaking this integral into two integrals, integrate one integral by parts and keep other integral as it is, By doing so, we get -

$$\int e^x[f(x) + f'(x)]dx = e^x f(x) + c$$

Solved Examples

Ex.47 Evaluate $\int e^x(\sin x + \cos x)dx$

Sol. $I = \int e^x(\sin x + \cos x)dx$

This is of the form

$\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + c$ $= e^x f(x) + c$

Now here $f(x) = \sin x$ $\therefore \Rightarrow e^x \sin x + c$

If the integral is of the form $\int [x f'(x) + f(x)]\,dx$ then by breaking this integral into two integrals integrate one integral by parts and keep other integral as it is, by doing so, we get

$$\int [x f'(x) + f(x)]\,dx = x f(x) + c$$

Solved Examples

Ex.48 Evaluate $\int (x \sec^2 x + \tan x)\,dx$

Sol. Here $I = \int (x\sec^2 x + \tan x)\,dx = \int [x f'(x) + f(x)]\,dx$

where $f(x) = \tan x$ $= x f(x) + c$ $= x.\tan x + c$

Integration by Parts :

Product of two functions f(x) and g(x) can be integrate using formula :

$$\int (f(x)\, g(x))\, dx$$

$$= f(x) \int (g(x))dx - \int\left(\frac{d}{dx}(f(x))\int (g(x))\, dx\right) dx$$

(i) when you find integral $\int g(x)dx$ then it will **not** contain arbitarary constant.

(ii) $\int g(x)dx$ should be taken as same at both places.

(iii) The choice of f(x) and g(x) can be decided by ILATE guideline.

the function will come later is taken an integral function (g(x)).

I → Inverse function

L → Logarithmic function

A → Algebraic function

T → Trigonometric function

E → Exponential function

Solved Examples

Ex.49 Evaluate : $\int x \tan^{-1} x\, dx$

Sol. Let $I = \int x \tan^{-1} x\, dx$

$$= (\tan^{-1} x)\ \frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2}\ dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2+1-1}{x^2+1}\, dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int\left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + C.$$

Ex.50 Evaluate : $\int x\, \ell n(1+x)\, dx$

Sol. Let $I = \int x\, \ell n(1+x)\, dx$

$$= \ell n\,(x+1) \cdot \frac{x^2}{2} - \int \frac{1}{x+1} \cdot \frac{x^2}{2}\ dx$$

$$= \frac{x^2}{2}\ \ell n\,(x+1) - \frac{1}{2} \int \frac{x^2}{x+1}\, dx$$

$$= \frac{x^2}{2}\ \ell n\,(x+1) - \frac{1}{2} \int \frac{x^2-1+1}{x+1}\, dx$$

$$= \frac{x^2}{2}\ \ell n\,(x+1) - \frac{1}{2} \int\left(\frac{x^2-1}{x+1} + \frac{1}{x+1}\right) dx$$

$$= \frac{x^2}{2}\ \ell n\,(x+1) - \frac{1}{2} \int\left((x-1) + \frac{1}{x+1}\right) dx$$

$$= \frac{x^2}{2}\ \ell n\,(x+1) - \frac{1}{2}\left[\frac{x^2}{2} - x + \ell n\,|x+1|\right] + C$$

Ex.51 Evaluate : $\int e^{2x} \sin 3x\ dx$

Sol. Let $I = \int e^{2x} \sin 3x\ dx$

$$= e^{2x}\left(-\frac{\cos 3x}{3}\right) - \int 2e^{2x}\left(-\frac{\cos 3x}{3}\right) dx$$

$$= -\frac{1}{3}\ e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x\ dx$$

$$= -\frac{1}{3}\ e^{2x} \cos 3x + \frac{2}{3}\left[e^{2x}\frac{\sin 3x}{3} - \int 2e^{2x}\frac{\sin 3x}{3}\, dx\right]$$

$$= -\frac{1}{3}\ e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} \int e^{2x} \sin 3x\ dx$$

$$\Rightarrow I = -\frac{1}{3}\ e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9}\, I$$

$$\Rightarrow I + \frac{4}{9}\, I = \frac{e^{2x}}{9}\ (2 \sin 3x - 3 \cos 3x)$$

$$\Rightarrow \frac{13}{9}\, I = \frac{e^{2x}}{9}\ (2 \sin 3x - 3 \cos 3x)$$

$$\Rightarrow I = \frac{e^{2x}}{13}\ (2 \sin 3x - 3 \cos 3x) + C$$

Note :

(i) $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + C$

(ii) $\int [f(x) + x f'(x)]\, dx = x\, f(x) + C$

Ex.52 Evaluate : $\int e^x \dfrac{x}{(x+1)^2}\,dx$

Sol. Given integral $= \int e^x \dfrac{x+1-1}{(x+1)^2}\,dx$

$$= \int e^x \left(\frac{1}{(x+1)} - \frac{1}{(x+1)^2}\right) dx = \frac{e^x}{(x+1)} + C$$

Ex.53 Evaluate : $\int e^x \left(\dfrac{1-\sin x}{1-\cos x}\right) dx$

Sol. Given integral $= \int e^x \left(\dfrac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right) dx$

$$= \int e^x \left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right)$$

$$dx = -e^x \cot\frac{x}{2} + C$$

Ex.54 Evaluate : $\int \left[\ell n(\ell nx) + \dfrac{1}{(\ell nx)^2}\right] dx$

Sol. Let $I = \int \left[\ell n(\ell nx) + \dfrac{1}{(\ell nx)^2}\right] dx$

{put $x = e^t$ $\Rightarrow$ $e^t\,dt$}

$$\therefore\ I = \int e^t \left(\ell nt + \frac{1}{t^2}\right) dt = \int e^t \left(\ell nt - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right) dt$$

$$= e^t\left(\ell nt - \frac{1}{t}\right) + C = x\left[\ell n(\ell nx) - \frac{1}{\ell nx}\right] + C$$

Integration of type

$\int \dfrac{x^2 \pm 1}{x^4 + Kx^2 + 1}$ **dx where K is any constant.**

Divide Nr & Dr by x² & put $x \mp \dfrac{1}{x} = t$.

Solved Examples

Ex.55 Evaluate : $\int \dfrac{1-x^2}{1+x^2+x^4}\,dx$

Sol. Let $I = \int \dfrac{1-x^2}{1+x^2+x^4}\,dx = -\int \dfrac{\left(1-\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}+1}$

{put $x + \dfrac{1}{x} = t$ $\Rightarrow \left(1-\dfrac{1}{x^2}\right)dx = dt$}

$$\therefore\ I = -\int \frac{dt}{t^2-1} = -\frac{1}{2}\,\ell n\left|\frac{t-1}{t+1}\right| + C$$

$$= -\frac{1}{2}\,\ell n\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C$$

Ex.56 Evaluate : $\int \dfrac{1}{x^4+1}\,dx$

Sol. We have,

$$I = \int \frac{1}{x^4+1}\,dx = \int \frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \frac{1}{2}\int \frac{\frac{2}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} - \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\,dx$$

Putting $x - \dfrac{1}{x} = u$ in 1st integral and $x + \dfrac{1}{x} = \nu$ in 2nd integral, we get

$$I = \frac{1}{2}\int \frac{du}{u^2+\left(\sqrt{2}\right)^2} - \frac{1}{2}\int \frac{d\nu}{\nu^2-\left(\sqrt{2}\right)^2}$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\,\frac{1}{2\sqrt{2}}\,\ell n\left|\frac{\nu-\sqrt{2}}{\nu+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\,\ell n\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) - \frac{1}{4\sqrt{2}}\,\ell n\left|\frac{x^2-\sqrt{2}\,x+1}{x^2+x\sqrt{2}+1}\right| + C$$

Integration of type

$$\int \frac{dx}{(ax+b)\sqrt{px+q}} \textbf{ OR } \int \frac{dx}{\left(ax^2+bx+c\right)\sqrt{px+q}}.$$

Put px + q = t².

Solved Examples

Ex.57 Evaluate : $\int \frac{1}{(x-3)\sqrt{x+1}}\, dx$

Sol. Let $I = \int \frac{1}{(x-3)\sqrt{x+1}}\, dx$ {Put $x + 1 = t^2$

$\Rightarrow dx = 2t\, dt$}

$\therefore \; I = \int \frac{1}{(t^2-1-3)} \frac{2t}{\sqrt{t^2}}\, dt$

$\Rightarrow I = 2\int \frac{dt}{t^2-2^2} = 2 \cdot \frac{1}{2(2)} \ell n \left|\frac{t-2}{t+2}\right| + C$

$\Rightarrow \; I = \frac{1}{2} \ell n \left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right| + C.$

Ex.58 Evaluate : $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\, dx$

Sol. Let $I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\, dx$

Putting $x + 1 = t^2$, and $dx = 2t\, dt$,

we get $I = \int \frac{(t^2+1)\, 2t\, dt}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}$

$\Rightarrow 2\int \frac{(t^2+1)}{t^4+t^2+1}\, dt = 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\, dt$

{put $t - \frac{1}{t} = u$}

$= 2\int \frac{du}{u^2+\left(\sqrt{3}\right)^2} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left\{\frac{t-\frac{1}{t}}{\sqrt{3}}\right\} + C$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t^2-1}{t\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}} \tan^{-1}\left\{\frac{x}{\sqrt{3(x+1)}}\right\} + C$

Integration of type

$$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}, \qquad \textbf{put } ax+b=\frac{1}{t};$$

$$\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}, \qquad \textbf{put } x=\frac{1}{t}$$

Solved Examples

Ex.59 Evaluate $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$

Sol Let $I = \int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$

{put $x + 1 = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\, dt$ }

$\Rightarrow I = \int \frac{-dt}{t^2\left(\frac{1}{t}\right)\sqrt{\left(\frac{1}{t}-1\right)^2+\frac{1}{t}}} = \int \frac{-dt}{t\sqrt{\frac{1}{t^2}-\frac{1}{t}+1}}$

$= \int \frac{-dt}{\sqrt{t^2-t+1}}$

$= \int \frac{-dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}} = -\ell n\left|t-\frac{1}{2}+\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}\right| + C,$

where $t = \frac{1}{x+1}$

Ex.60 Evaluate $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Sol. Put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\, dt$

$\Rightarrow \; I = \int \frac{dt}{(t^2+1)\sqrt{t^2-1}}$

{put $t^2 - 1 = y^2 \Rightarrow t\,dt = y\,dy$ }

$\Rightarrow I = -\int \frac{y\, dy}{(y^2+2)\, y} = -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + C$

$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + C$

Integration of type

$$\int \sqrt{\frac{x-\alpha}{\beta-x}}\,dx \text{ or} \int \sqrt{(x-\alpha)(\beta-x)}\,dx;$$

put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$$\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx \text{ or} \int \sqrt{(x-\alpha)(x-\beta)}\,dx;$$

put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}};$$

put $x - \alpha = t^2$ or $x - \beta = t^2$.

Reduction formula of $\int \tan^n x\,dx$, $\int \cot^n x\,dx$, $\int \sec^n x\,dx$, $\int \text{cosec}^n x\,dx$

1. $I_n = \int \tan^n x\,dx = \int \tan^2 x \tan^{n-2} x\,dx$

$= \int (\sec^2 x - 1)\tan^{n-2} x\,dx$

$\Rightarrow I_n = \int \sec^2 x \tan^{n-2} x\,dx \quad - I_{n-2}$

$\Rightarrow I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$, $n \geq 2$

2. $I_n = \int \cot^n x\,dx = \int \cot^2 x\,.\cot^{n-2} x\,dx$

$= \int (\text{cosec}^2 x - 1)\cot^{n-2} x\,dx$

$\Rightarrow I_n = \int \text{cosec}^2 x \cot^{n-2} x\,dx - I_{n-2}$

$\Rightarrow I_n = -\dfrac{\cot^{n-1} x}{n-1} - I_{n-2}$, $n \geq 2$

3. $I_n = \int \sec^n x\,dx = \int \sec^2 x \sec^{n-2} x\,dx$

$\Rightarrow I_n = \tan x \sec^{n-2} x - \int (\tan x)(n-2)\sec^{n-3} x.$

$\sec x \tan x\,dx.$

$\Rightarrow I_n = \tan x \sec^{n-2} x - (n-2) \int (\sec^2 x - 1)$

$\sec^{n-2} x\,dx$

$\Rightarrow (n-1)\,I_n = \tan x \sec^{n-2} x + (n-2)\,I_{n-2}$

$I_n = \dfrac{\tan x \sec^{n-2} x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}$

4. $I_n = \int \text{cosec}^n x\,dx = \int \text{cosec}^2 x\ \text{cosec}^{n-2} x\,dx$

$\Rightarrow I_n = -\cot x\ \text{cosec}^{n-2} x + \int (\cot x)(n-2)$

$(-\text{cosec}^{n-3} x\ \text{cosec}\,x \cot x)\,dx$

$\Rightarrow -\cot x\ \text{cosec}^{n-2} x - (n-2)\int \cot^2 x\,\text{cosec}^{n-2} x\,dx$

$\Rightarrow I_n = -\cot x\ \text{cosec}^{n-2} x - (n-2) \int (\text{cosec}^2 x - 1)$

$\text{cosec}^{n-2} x\,dx$

$\Rightarrow (n-1)\,I_n = -\cot x\ \text{cosec}^{n-2} x + (n-2)\,I_{n-2}$

$I_n = \dfrac{\cot x\,\text{cosec}^{n-2} x}{-(n-1)} + \dfrac{n-2}{n-1} I_{n-2}$

Solved Examples

Ex.61 Obtain reducation formula for $I_n = \int \sin^n x\,dx$.

Hence evaluate $\int \sin^4 x\,dx$

Sol. $I_n = \int (\sin x)(\sin x)^{n-1}\,dx$

II I

$= -\cos x(\sin x)^{n-1} + (n-1)\int (\sin x)^{n-2}\cos^2 x\,dx$

$= -\cos x(\sin x)^{n-1} + (n-1)\int (\sin x)^{n-2}(1 - \sin^2 x)\,dx$

$I_n = -\cos x(\sin x)^{n-1} + (n-1)\,I_{n-2} - (n-1)\,I_n$

$\Rightarrow I_n = -\dfrac{\cos x(\sin x)^{n-1}}{n} + \dfrac{(n-1)}{n} I_{n-2} \qquad (n \geq 2)$

Hence $I_4 = -\dfrac{\cos x(\sin x)^3}{4} + \dfrac{3}{4}$

$\left(-\dfrac{\cos x(\sin x)}{2} + \dfrac{1}{2}x\right) + C$