Application of Derivative

DERIVATIVE AS A RATE MEASURE

The meaning of differential coefficient can be interpreted as rate of change of the dependent variable with respect to the independent variable, for example $\frac{dy}{dx}$ is the rate of change of y with respect to x. Similarly $\frac{dv}{dt}$ and $\frac{ds}{dt}$ etc. represent the rate of change of volume and surface area w.r.t. time.

Solved Examples

Ex.1 Displacement 's' of a particle at time 't' is expressed as $s = \frac{1}{2}t^3 - 6t$, find the acceleration at the time when the velocity vanishes (i.e., velocity tends to zero).

Sol. $s = \frac{1}{2}t^3 - 6t$

Thus velocity, $v = \frac{ds}{dt} = \left(\frac{3t^2}{2} - 6\right)$

and acceleration, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 3t$

Velocity vanishes when $\frac{3t^2}{2} - 6 = 0$

$\Rightarrow \quad t^2 = 4 \Rightarrow t = 2$

Thus acceleration when velocity vanishes is $a = 3t = 6$ units.

Ex.2 On the curve $x^3 = 12y$, find the interval of values of x for which the abscissa changes at a faster rate than the ordinate?

Sol. Given $x^3 = 12y$, differentiating with respect to y

$$3x^2 \frac{dx}{dy} = 12$$

$$\therefore \frac{dx}{dy} = \frac{12}{3x^2}$$

The interval in which the abscissa changes at a faster rate than the ordinate, we must have

$$\Rightarrow \quad \left|\frac{dx}{dy}\right| > 1 \quad \text{or} \quad \left|\frac{12}{3x^2}\right| > 1$$

$$\text{or} \quad \frac{4}{x^2} > 1 \Rightarrow \frac{4 - x^2}{x^2} > 0$$

$\Rightarrow x \in (-2, 2) - \{0\}$.

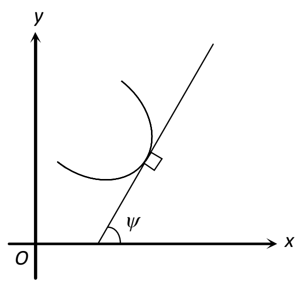
Thus $x \in (-2, 2) - \{0\}$ is the required interval in which abscissa changes at a faster rate than the ordinate.

GEOMETRICAL INTERPRETATION OF THE DERIVATIVE

If $y = f(x)$ be a given function, then the differential coefficient $f'(x)$ or $\frac{dy}{dx}$ at the point $P(x_1, y_1)$ is the trigonometrical tangent of the angle ψ (say) which the positive direction of the tangent to the curve at P makes with the positive direction of x- axis $\left(\frac{dy}{dx}\right)$, therefore represents the slope of the tangent.

Thus

$$f'(x) = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \tan\psi$$



Thus

(i) The inclination of tangent with x- axis.

$$= \tan^{-1}\left(\frac{dy}{dx}\right)$$

(ii) Slope of tangent $= \frac{dy}{dx}$

(iii) Slope of the normal $= -dx/dy$

Solved Examples

Ex.3 Find the following for the curve $y^2 = 4x$ at point (2,–2)

(i) Inclination of the tangent

(ii) Slope of the tangent

(iii) Slope of the normal

Sol. Differentiating the given equation of curve, we get $dy/dx = 2/y = -1$ at (2,–2)

so at the given point.

(i) Inclination of the tangent $= \tan^{-1}(-1) = 135^\circ$

(ii) Slope of the tangent $= -1$

(iii) Slope of the normal $= 1$

EQUATION OF TANGENT

(a) Equation of tangent to the curve $y = f(x)$ at $A(x_1,y_1)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1)$$

(i) If the tangent at $P(x_1,y_1)$ of the curve $y = f(x)$ is parallel to the x- axis (or perpendicular to y- axis) then $\psi = 0$ i.e. its slope will be zero.

$$m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$$

The converse is also true. Hence the tangent at (x_1,y_1) is parallel to x- axis.

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$$

(ii) If the tangent at $P(x_1, y_1)$ of the curve $y = f(x)$ is parallel to y - axis (or perpendicular to x-axis) then $\psi = \pi/2$, and its slope will be infinity i.e.

$$m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \infty$$

The converse is also true. Hence the tangent at (x_1, y_1) is parallel to y- axis

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \infty$$

(iii) If at any point $P(x_1, y_1)$ of the curve $y = f(x)$, the tangent makes equal angles with the axes, then at the point P, $\psi = \pi/4$ or $3\pi/4$, Hence at P, $\tan\psi = dy/dx = \pm 1$. The converse of the result is also true. thus at (x_1,y_1) the tangent line makes equal angles with the axes.

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \pm 1$$

Solved Examples

Ex.4 The equation of tangent to the curve $y^2 = 6x$ at $(2, -3)$.

(A) $x + y - 1 = 0$ (B) $x + y + 1 = 0$

(C) $x - y + 1 = 0$ (D) $x + y + 2 = 0$

Sol. Differentiating equation of the curve with respect to x $\quad 2y\frac{dy}{dx} = 6 \therefore \left(\frac{dy}{dx}\right)_{(2,-3)} = \frac{3}{-3} = -1$

Therefore equation of tangent is

$y + 3 = -(x - 2) \Rightarrow x + y + 1 = 0$ **Ans. [B]**

Ex.5 The equation of tangent at any of the curve $x = at^2$, $y = 2at$ is -

(A) $x = ty + at^2$ (B) $ty + x + at^2 = 0$

(C) $ty = x + at^2$ (D) $ty = x + at^3$

Sol. $dy/dx = (dy/dt)/(dx/dt) = \frac{2a}{2at} = \frac{1}{t}$

$\therefore$ equation of the tangent at (x,y) point is

$(y - 2at) = \frac{1}{t}(x - at^2)$

$\Rightarrow ty = x + at^2$ **Ans.[C]**

Ex.6 The equation of the tangent to the curve $x^2(x - y) + a^2(x + y) = 0$ at origin is-

(A) $x + y + 1 = 0$ (B) $x + y + 2 = 0$

(C) $x + y = 0$ (D) $2x - y = 0$

Sol. The given equation of the curve is

$x^3 - x^2y + a^2x + a^2y = 0$

Differentiating it w.r.t. x

$3x^2 - 2xy - x^2.\frac{dy}{dx} + a^2 + a^2.\frac{dy}{dx} = 0$

$-x^2.\frac{dy}{dx} + a^2.\frac{dy}{dx} = -3x^2 + 2xy - a^2$

Now at origin i.e. $x = 0, y = 0$

$\therefore a^2(1 + dy/dx) = 0$

$\therefore dy/dx = -1$

$\therefore$ the equation of tangent is

$y - 0 = -1(x - 0)$ $y = -x \Rightarrow x + y = 0$ **Ans.[C]**

Ex.7 If a tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is parallel to x- axis, then its point of contact is-

(A) (a, 0) (B) (0,–b)

(C) $(0, \pm b)$ (D) $(\pm a, 0)$

Sol. Differentiating given equation, we have

$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$

If tangent is parallel to x– axis, then

$dy/dx = 0 \Rightarrow -\frac{b^2x}{a^2y} = 0 \Rightarrow x = 0.$

Thus from the given equation $y = \pm b$

$\therefore$ required point $= (0, \pm b)$ **Ans.[C]**

Ex.8 For the curve $xy = c^2$, prove that the portion of the tangent intercepted between the coordinate axes is bisected at the point of contact.

Sol. Let the point at which tangent is drawn be (α, β) on the curve $xy = c^2$.

$\Rightarrow \left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = -\frac{\beta}{\alpha}$ Thus, the equation of tangent is,

$y - \beta = -\frac{\beta}{\alpha}(x - \alpha) \Rightarrow y\alpha - \alpha\beta = -x\beta + \alpha\beta$

$\Rightarrow x\beta + y\alpha = 2\alpha\beta \Rightarrow \frac{x}{2\alpha} + \frac{y}{2\beta} = 1$

It is clear that the tangent line cuts x and y-axis at $A(2\alpha, 0)$ and $B(0, 2\beta)$ and the point (α, β) bisects AB.

LENGTH OF INTERCEPTS MADE ON AXES BY THE TANGENT

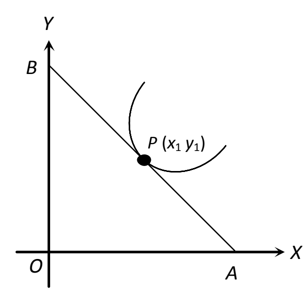
Equation of tangent at any point (x_1, y_1) to the curve $y = f(x)$ is

$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x - x_1)$ ....(1)

Equation of x- axis, $y = 0$ ....(2)

Equation of y – axis, $x = 0$ ....(3)

Solving (1) and (2), we get. $x = x_1 - \left\{\frac{y_1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}\right\}$



$\therefore$ x – intercept = $OA = x_1 - \left\{\frac{y_1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}\right\}$

Similarly solving (1) and (3), we get y– intercept

$OB = y_1 - x_1\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$

Solved Examples

Ex.9 The length of intercepts on coordinate axes made by tangent to the curve

$\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point P (x_1, y_1) are-

(A) $\sqrt{ax_1}, \sqrt{ay_1}$ (B) $\sqrt{a}, \sqrt{a}$

(C) $\sqrt{x_1}, \sqrt{y_1}$ (D) None of these

Sol. $\sqrt{x} + \sqrt{y} = \sqrt{a}$

So $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\sqrt{\frac{y_1}{x_1}}$

$\therefore$ x – intercept $= x_1 - \left\{\frac{y_1}{\left(-\sqrt{\frac{y_1}{x_1}}\right)}\right\}$

$= x_1 + \sqrt{x_1 y_1} \quad = \sqrt{x_1}\left(\sqrt{x_1} + \sqrt{y_1}\right)$

$= \sqrt{a x_1} \quad (\because \sqrt{x_1} + \sqrt{y_1} = \sqrt{a})$

and y – intercept $= y_1 - x_1\left(-\sqrt{\frac{y_1}{x_1}}\right)$

$= y_1 + \sqrt{x_1 y_1} \quad = \sqrt{a y_1}$

Second Method : Equation of tangent at (x_1, y_1)

$y - y_1 = -\sqrt{\frac{y_1}{x_1}}(x - x_1)$

$\Rightarrow \frac{y - y_1}{\sqrt{y_1}} + \frac{x - x_1}{\sqrt{x_1}} = 0$

$\Rightarrow \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{x_1} + \sqrt{y_1}$

$\Rightarrow \frac{x}{\sqrt{ax_1}} + \frac{y}{\sqrt{ay_1}} = 1$

Obviously x – intercept $= \sqrt{ax_1}$

and y – intercept $= \sqrt{ay_1}$ **Ans. [A]**

Ex.10 The length of intercepts on coordinate axes made by tangent to the curve $y = 2x^2 + 3x - 2$ at the point (1,3) are-

(A) 4, –4/7 (B) –4/7, 4

(C) 4/7, –4 (D) 4/7, 4

Sol. The given equation of curve is

$y = 2x^2 + 3x - 2$

$\therefore dy/dx = 4x + 3 \quad \therefore \left(\frac{dy}{dx}\right)_{(1,\ 3)} = 4.1 + 3 = 7$

Now OA = 1–3. (1/7) = 4/7 and OB = 3– 1.7 = –4

$\therefore$ required length of intercepts are 4/7, –4

Here negative sign shows that tangent cuts the y- axis below the origin. **Ans.[C]**

LENGTH OF PERPENDICULAR FROM ORIGIN TO THE TANGENT

The length of perpendicular from origin (0,0) to the tangent drawn at the point (x_1, y_1) of the curve

$y = f(x)$. $\quad p = \left|\frac{y_1 - x_1\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right|$

Explanation :

The equation of tangent at point P (x_1, y_1) of the given curve $\quad y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$

p = perpendicular from origin to tangent

$= \left|\frac{y_1 - x_1\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right|$

Ex.10 The length of perpendicular from (0,0) to the tangent drawn to the curve $y^2 = 4(x+2)$ at point (2,4) is-

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{3}{\sqrt{5}}$

(C) $\frac{6}{\sqrt{5}}$ (D) 1

Sol. Differentiating the given curve w.r.t x $\;2y.\frac{dy}{dx} = 4$

at point (2,4) , $\frac{dy}{dx} = 1/2$

$\because p = \frac{y_1 - x_1\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \quad \therefore p = \frac{4 - 2.\left(\frac{1}{2}\right)}{\sqrt{1+\frac{1}{4}}} = \frac{6}{\sqrt{5}}$ **Ans.[C]**

EQUATION OF NORMAL

The equation of normal at (x_1, y_1) to the curve $y = f(x)$ is

$$(y - y_1) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}(x - x_1)$$

or $(y - y_1)\cdot\left(\frac{dy}{dx}\right)_{(x_1,y_1)} + (x - x_1) = 0$

Some facts about the normal

(i) The slope of the normal drawn at point $P(x_1,y_1)$ to the curve $y = f(x) = -\left(\frac{dx}{dy}\right)_{(x_1,y_1)}$

(ii) If normal makes an angle of θ with positive direction of x– axis then

$-\frac{dx}{dy} = \tan\theta$ or $\frac{dy}{dx} = -\cot\theta$

(iii) If normal is parallel to x– axis then

$-\frac{dx}{dy} = 0$ or $\frac{dy}{dx} = \infty$

(iv) If normal is parallel to y – axis then

$-\left(\frac{dx}{dy}\right) = \infty$ or $\frac{dy}{dx} = 0$

(v) If normal is equally inclined from both the axes or cuts equal intercept then

$-\left(\frac{dx}{dy}\right) = \pm 1$ or $\left(\frac{dy}{dx}\right) = \pm 1$

(vi) The length of perpendicular from origin to normal is

$$P' = \left|\frac{x_1 + y_1\left(\frac{dy}{dx}\right)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right|$$

(vii) The length of intercept made by normal on x- axis is $= x_1 + y_1\left(\frac{dy}{dx}\right)$

and length of intercept on y - axis is

$= y_1 + x_1\left(\frac{dx}{dy}\right)$

Solved Examples

Ex.11 The equation of normal at (1, 6) to the curve $y = 2x^2 + 3x + 1$ is -

(A) $x + 7y - 43 = 0$ (B) $7x + y - 43 = 0$
(C) $7x + y = 0$ (D) None of these

Sol. From the given curve

$\frac{dy}{dx} = 4x + 3$

$\therefore \left(\frac{dy}{dx}\right)_{(1,6)} = 4.1 + 3 = 7 \Rightarrow -\left(\frac{dx}{dy}\right) = -\frac{1}{7}$

$\therefore$ The equation of normal is $y - 6 = -\frac{1}{7}(x - 1)$

$\Rightarrow x + 7y - 43 = 0$ **Ans.[A]**

Ex.12 The equation of normal to the curve $y = x + \sin x \cos x$ at $x = \pi/2$ is -

(A) $x = \pi/2$ (B) $y = \pi/2$
(C) $x + y = \pi/2$ (D) $x - y = \pi/2$

Sol. At $x = \pi/2$, we get $y = \pi/2$

So the point is $(\pi/2, \pi/2)$

Now the equation of curve $y = x + \sin x.\cos x$

$\therefore \frac{dy}{dx} = 1 + \cos^2 x - \sin^2 x = 1 + \cos 2x$

$\therefore \left(\frac{dy}{dx}\right)_{(\pi/2,\pi/2)} = 1 - 1 = 0$

$\therefore$ slope of normal $= \infty$ $\therefore$ The equation of normal

$\Rightarrow (y - \pi/2) = -\frac{1}{0}(x - \frac{\pi}{2}) \Rightarrow x = \frac{\pi}{2}$ **Ans.[A]**

Ex.13 The length of perpendicular from (0,0) to the normal drawn at point (1,6) to the curve $y = 2x^2 + 3x + 1$ is-

(A) 1 (B) 1/2
(C) $43/\sqrt{50}$ (D) 1/6

Sol. $\because P' = \frac{x_1 + y_1\left(\frac{dy}{dx}\right)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$

Since $dy/dx = 7$

$\therefore P' = \frac{1 + 6.7}{\sqrt{1 + 49}} = \frac{43}{\sqrt{50}}$ **Ans.[C]**

Ex.14 The abscissa of a point where the normal drawn to the curve $xy = (x+c)^2$ makes an equal intercept with coordinate axes is-

(A) $\pm\sqrt{2}\,c$ (B) $\pm\frac{c}{\sqrt{2}}$

(C) $\pm\frac{\sqrt{3}}{2}c$ (D) $\pm\sqrt{3}\,c$

Sol. Differentiating the given curve w.r.t. x

$$x\frac{dy}{dx} + y = 2(x+c)$$

$$\therefore \frac{dy}{dx} = \frac{2(x+c)-y}{x}$$

$$\therefore \frac{2(x_1+c)-y_1}{x_1} = \pm 1$$

and $x_1y_1 = (x_1+c)^2$

$$= \frac{2(x_1+c)-\frac{(x_1+c)^2}{x_1}}{x_1} = \pm 1$$

$$= \frac{2x_1^2+2cx_1-x_1^2-2cx_1-c^2}{x_1^2} = \pm 1$$

$$= \frac{x_1^2-c^2}{x_1^2} = \pm 1$$

$\Rightarrow x_1^2 - c^2 = \pm x_1^2 \Rightarrow x_1^2 - c^2 = x_1^2$

and $2x_1^2 - c^2 = 0$

$\therefore 2x_1^2 = c^2 \Rightarrow x_1 = \pm\frac{c}{\sqrt{2}}$ **Ans.[B]**

Ex.15 The coordinate of a point to the curve $y = x\log x$ where the normal is parallel to line $2x - 2y = 3$ is -

(A) $e^{-2}, -2e^{-2}$ (B) $-2e^{-2}, e^{-2}$

(C) e^{-2}, e^{-2} (D) None of these

Sol. Differentiating the equation of line

$2 - 2dy/dx = 0$

$dy/dx = 1$

$\therefore$ slope of tangent $= -1$

$\therefore -1 = 1 + \log x,\ \log x = -2\ \ x = e^{-2}$

$\therefore\ y = e^{-2}\log e^{-2}$

$y = -2e^{-2}$

$\therefore$ point is $(e^{-2}, -2e^{-2})$ **Ans.[A]**

Ex.16 Find the equation of normal to the curve $x + y = x^y$, where it cuts the x-axis.

Sol. Given curve is $x + y = x^y$

at x-axis $y = 0$,

$x + 0 = x^0 \quad \Rightarrow \quad x = 1$

Now to differentiate $x + y = x^y$, take log on both sides

$$\Rightarrow \quad \ln(x+y) = y\ln x$$

$$\therefore \frac{1}{x+y}\left\{1+\frac{dy}{dx}\right\} = y\frac{1}{x} + (\ln x)\frac{dy}{dx}$$

Putting $x = 1, y = 0$, we get

$$\left\{1+\frac{dy}{dx}\right\} = 0 \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_{(1,0)} = -1$$

$\therefore$ slope of normal = 1

Equation of normal is,

$$\frac{y-0}{x-1} = 1 \quad \Rightarrow \quad y = x - 1.$$

ANGLE OF INTERSECTION OF TWO CURVES

Let $y = f(x)$ and $y = g(x)$ be two given intersecting curves. Angle of intersection of these curves is defined as the acute angle between the tangents that can be drawn to the given curves at the point of intersection.

Let (x_1, y_1) be the point of intersection.

$$\Rightarrow \quad y_1 = f(x_1) = g(x_1)$$

Slope of the tangent drawn to the curve $y = f(x)$ at (x_1, y_1) i.e., $m_1 = \left(\frac{df}{dx}\right)_{(x_1,y_1)}$

Similarly slope of the tangent drawn to the curve $y = g(x)$ at (x_1, y_1) i.e., $m_2 = \left(\frac{dg}{dx}\right)_{(x_1,y_1)}$

If α be the angle (acute) of intersection, then $\tan\alpha$

$$= \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|.$$

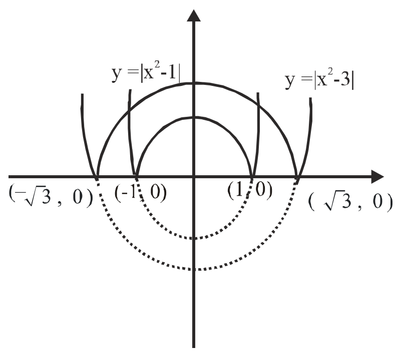
If $\alpha = 0$, then $m_1 = m_2$. Thus the given curves will touch each other at (x_1, y_1).

If $\alpha = \frac{\pi}{2}$, then $m_1 m_2 = -1$. Thus the given curves will meet at right angles at (x_1, y_1) (or curves cut each other orthogonally at (x_1, y_1)).

Solved Examples

Ex.17 Find the acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection.

Sol.



The points of intersection are $(\pm\sqrt{2}, 1)$

Since the curves are symmetrical about y-axis,

the angle of intersection at $(-\sqrt{2}, 1)$

= the angle of intersection at $(\sqrt{2}, 1)$.

At $(\sqrt{2}, 1)$, $m_1 = 2x = 2\sqrt{2}$, $m_2 = -2x = -2\sqrt{2}$.

$$\therefore \tan\theta = \left|\frac{4\sqrt{2}}{1-8}\right| = \frac{4\sqrt{2}}{7} \Rightarrow \theta = \tan^{-1}\frac{4\sqrt{2}}{7}$$

Ex.18 The angle of intersection between the curves $y = x$ and $y^2 = 4x$ at (4,4).

(A) $\tan^{-1}\left(\frac{1}{2}\right)$ (B) $\tan^{-1}\left(\frac{1}{3}\right)$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

Sol. Differentiating given equations, we have

$\left(\frac{dy}{dx}\right)_1 = 1$ and $\left(\frac{dy}{dx}\right)_2 = 2/y$

$\therefore$ at (4,4) $\left(\frac{dy}{dx}\right)_1 = 1$ $\left(\frac{dy}{dx}\right)_2 = 2/4 = 1/2$

Hence angle of intersection

$$= \tan^{-1}\left|\frac{1-\frac{1}{2}}{1+1\left(\frac{1}{2}\right)}\right| = \tan^{-1}(1/3).$$ **Ans. [B]**

Ex.19 The condition that the curves $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $xy = c^2$ mutually intersect orthogonally is -

(A) $a^2 + b^2 = 0$ (B) $a^2 = b^2$

(C) $a^2 - b^2 = 0$ (D) $\frac{a^2}{b^2} = 1$

Sol. The given curves are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{... (1)}$$

$$\text{and } xy = c^2 \quad \text{... (2)}$$

from (1), $\left(\frac{dy}{dx}\right)_1 = \frac{b^2x}{a^2y}$ and

from (2), $\left(\frac{dy}{dx}\right)_2 = -y/x$

(1) and (2) intersect orthogonally if

$$\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2 = -1$$

$$\therefore \left(\frac{b^2x}{a^2y}\right)\left(-\frac{y}{x}\right) = -1$$

$\Rightarrow a^2 = b^2$ which is the required condition.

Ans. [B]

LENGTHS OF THE TANGENT, NORMAL, SUB-TANGENT AND SUB-NORMAL AT ANY POINT OF A CURVE

Let the tangent and the normal at any point (x, y) of the curve $y = f(x)$ meet the x-axis at T and G respectively. Draw the ordinate PM.

Then the lengths TM, MG are called the sub-tangent and sub-normal respectively.

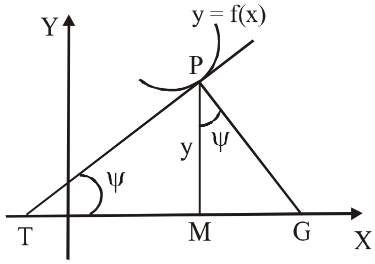
The lengths PT, PG are sometimes referred to as the lengths of the tangent and the normal respectively.

Clearly $\angle MPG = \psi$

Also $\tan\psi = \frac{dy}{dx}$

From the figure, we have

(i) Length of Tangent



$= TP = MP\,|\text{cosec}\,\psi| = |y|\sqrt{(1+\cot^2\psi)}$

$= |y|\sqrt{\left[1+\left(\frac{dx}{dy}\right)^2\right]}$

(ii) Length of Sub-tangent $= TM = MP\,|\cot\psi| = |y\frac{dx}{dy}|$

(iii) Length of Normal $= GP = MP\,|\sec\psi|$

$= |y|\sqrt{(1+\tan^2\psi)} \quad = |y|\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$

(iv) Length of Sub-normal $= MG = MP|\tan\psi| = |y\frac{dy}{dx}|$

Solved Examples

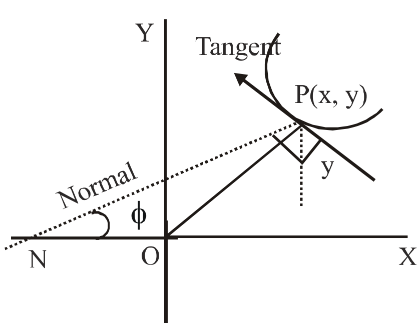
Ex.20 Find the equation of family of curves for which the length of normal is equal to the radius vector.

Sol. Let P(x, y) be the point on the curve.

OP = radius vector = $\sqrt{x^2+y^2}$

PN = length of normal

Now, $\tan\phi = -\frac{1}{\left(\frac{dy}{dx}\right)} \quad \Rightarrow \quad PN = \frac{y}{\sin\phi}$



It is given OP = PN

$\Rightarrow \quad \sqrt{x^2+y^2} = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$\Rightarrow \quad x^2+y^2 = y^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$

$\Rightarrow \quad x^2 = y^2\left(\frac{dy}{dx}\right)^2 \Rightarrow \frac{dy}{dx} = \pm\frac{x}{y}$

$\Rightarrow \quad ydy = \pm\ x\,dx$ integrating both sides,

$y^2 = \pm\, x^2 + c$ is the required family of curves.

Solved Examples

Ex.21 Find the length of tangent, subtangent, normal and subnormal at the point (2,4) of the curve $y^2 = 8x$.

Sol. Differentiating the equation of the curve w.r.t. x, we get

$2y\frac{dy}{dx} = 8 \Rightarrow dy/dx = 8/2y = 4/y$

$\therefore \left(\frac{dy}{dx}\right)_{(2,4)} = 4/4 = 1$

Therefore at the point (2,4):

length of tangent $= \frac{4\sqrt{1+(1)^2}}{1} = 4\sqrt{2}$

length of sub tangent = 4/1 = 4

length of normal = $4\sqrt{1+(1)^2} = 4\sqrt{2}$

length of sub normal = 4.1 = 4

Ex.19 Find the length of tangent, sub-tangent, normal and sub-normal at the point υ of the curve x = a ($\theta + \sin\theta$), and y = a($1-\cos\theta$).

Sol. $\because \left(\frac{dy}{dx}\right) = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{a\sin\theta}{a(1+\cos\theta)}$

$\Rightarrow \frac{dy}{dx} = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$

$\therefore \frac{dy}{dx} = \tan\frac{\theta}{2}$

$\therefore$ length of tangent

$$= \frac{a\,(1-\cos\theta)\sqrt{1+\tan^2\theta/2}}{\tan\theta/2}$$

$$= \frac{a\,\sin^2\theta/2\,\sec\theta/2}{\frac{\sin\theta/2}{\cos\theta/2}} = a.\sin\theta/2$$

length of normal

$= a\,(1-\cos\theta)\sqrt{1+\tan^2\theta/2}$

$= a.2\sin^2\theta/2\,.\,\sec\theta/2$

length of sub tangent

$$= \frac{a\,(1-\cos\theta)}{\tan\theta/2} = \frac{2a\,\sin^2\theta/2}{\frac{\sin\theta/2}{\cos\theta/2}} = a\sin\theta$$

length of sub normal $= a\,(1-\cos\theta)\tan\theta/2 = 2a\sin^2\theta/2\tan\theta/2$

POINT OF INFLEXION

If at any point P, the curve is concave on one side and convex on other side with respect to x- axis, then the point P is called the point of inflexion. Thus P is a point of inflexion if at P,

$$\frac{d^2y}{dx^2} = 0\text{, but }\frac{d^3y}{dx^3} \neq 0$$

Also point P is a point of inflexion if $f''(x) = f'''(x) = \ldots = f^{n-1}(x) = 0$ and $f^n(x) \neq 0$ for odd n.

Solved Examples

Ex.20 Prove that origin for the curve $y = x^3$ is a point of inflexion.

Sol. $\because y = x^3$

$$\therefore \frac{dy}{dx} = 3x^2,\ \frac{d^2y}{dx^2} = 6x,\ \frac{d^3y}{dx^3} = 6$$

clearly at (0,0)

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} \neq 0$$

$\therefore$ There is a point of inflexion at (0, 0).

ROLLE'S THEOREM

If a function f defined on the closed interval [a, b], is

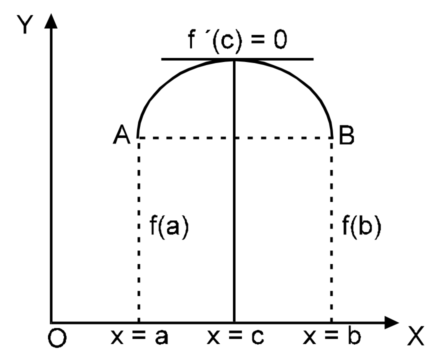
(i) Continuous on [a, b],

(ii) Derivable on (a, b) and

(iii) f(a) = f(b), then there exists atleast one real number c between a and b (a < c < b) such that $f'(c) = 0$

Geometrical interpretation

Let the curve y = f(x), which is continuous on [a, b] and derivable on (a, b), be drawn.



The theorem states that between two points with equal ordinates on the graph of f, there exists atleast one point where the tangent is parallel to x-axis.

Algebraic interpretation

Between two zeros a and b of f (x) (i.e., between two roots a and b of f(x) = 0) there exists atleast one zero of $f'(x)$.

Solved Examples

Ex.21 Let $f(x) = x^2 - 3x + 4$. Verify Rolle's theorem in [1, 2].

Sol. f(1) = f(2) = 2

Now, $f'(x) = 0 \Rightarrow 2x - 3 = 0$

$$\Rightarrow x = \frac{3}{2} \in (1, 2).$$

Hence, Rolle's theorem is verified.

Ex. 22 Let $f(x) = (x-a)(x-b)(x-c)$, $a < b < c$, show that $f'(x) = 0$ has two roots one belonging to (a, b) and other belonging to (b, c).

Sol. Here, $f(x)$ being a polynomial is continuous and differentiable for all real values of x. We also have $f(a) = f(b) = f(c)$. If we apply Rolle's theorem to $f(x)$ in $[a, b]$ and $[b,c]$ we would observe that $f'(x) = 0$ would have at least one root in (a, b) and at least one root in (b, c). But $f'(x)$ is a polynomial of degree two, hence $f'(x) = 0$ can not have more than two roots. It implies that exactly one root of $f'(x) = 0$ would lie in (a, b) and exactly one root of $f'(x) = 0$ would lie in (b, c).

Remarks:

Let $y = f(x)$ be a polynomial function of degree n. If $f(x) = 0$ has real roots only, then $f'(x) = 0$, $f''(x) = 0$, ... , $f^{n-1}(x) = 0$ would have only real roots. It is so because if $f(x) = 0$ has all real roots, then between two consecutive roots of $f(x) = 0$, exactly one roots of $f'(x) = 0$ would lie.

Solved Examples

Ex.23 Prove that if $a_0, a_1, a_2, ..., a_n$ are real numbers such that $\frac{a_0}{n+1} + \frac{a_1}{n} + ... + \frac{a_{n-1}}{2} + a_n = 0$ then there exists at least one real number x between 0 and 1 such that $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n = 0$,

Sol. Consider a function f defined as

$$f(x) = \frac{a_0}{n+1}x^{n+1} + \frac{a_1}{n}x^n + ... + \frac{a_{n-1}}{2}x^2 + a_n x,\ x \in [0,1]$$

f being a polynomial satisfies the following conditions.

(i) f is continuous in [0, 1]

(ii) f is derivable in (0, 1)

(iii) Since $f(0) = 0$ and $f(1) = 0$ by hypothesis,

$\therefore\ f(0) = f(1)$

Hence there is some $x \in (0, 1)$ such that $f'(x) = 0$

$$\Rightarrow \frac{a_0}{n+1}(n+1)x^n + \frac{a_1}{n}nx^{n-1} + ... + \frac{a_{n-1}}{2}.2x + a_n = 0$$

$$\Rightarrow \quad a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$$

LAGRANGE'S MEAN VALUE THEOREM

If a function f defined on the closed interval [a, b], is

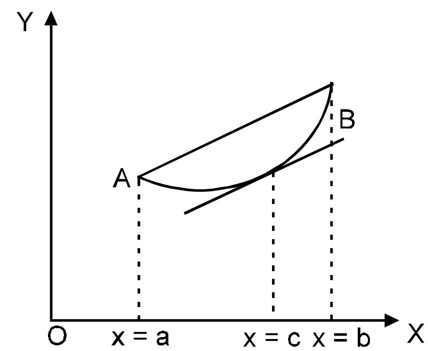
(i) Continuous on [a, b] and

(ii) Derivable on (a, b), then there exists atleast one real number c between a and b ($a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical interpretation

The theorem states that between two points A and B on the graph of f there exists atleast one point where the tangent is parallel to the chord AB.



Solved Examples

Ex. 24 If $f(x)$ and $g(x)$ be differentiable functions in (a, b), continuous at a and b and $g(x) \neq 0$ in [a, b], then prove that

$$\frac{g(a)f(b) - f(a)g(b)}{g(c)f'(c) - f(c)g'(c)} = \frac{(b-a)g(a)g(b)}{(g(c))^2}$$

for atleast one $c \in (a, b)$.

Sol. We have to prove (after rearranging the terms)

$$\frac{\frac{f(b)}{g(b)} - \frac{f(a)}{g(a)}}{(b-a)} = \frac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}$$

Let $\quad F(x) = \frac{f(x)}{g(x)}$

As $f(x)$ and $g(x)$ are differentiable function in (a, b), $F(x)$ will also be differentiable in (a, b). Further F is continuous at a and b. So according to LMVT, there exist one $c \in (a, b)$ such that $F'(c) = \frac{F(b) - F(a)}{b - a}$, which proves the required result.

Ex. 25 If the function $f : [0, 4] \to R$ is differentiable, then show that, $(f(4))^2 - (f(0))^2 = 8 f'(a) f(b)$ for some $a, b \in (0, 4)$

Sol. Since, f is differentiable $\Rightarrow$ f is continuous also.

Thus by Lagrange's mean value theorem, $a \in (0,4)$ such that

$$f'(a) = \frac{f(4)-f(0)}{4-0} = \frac{f(4)-f(0)}{4} \quad ... (1)$$

Also, by Intermediate value theorem there exists $b \in (0,4)$ such that

$$f(b) = \frac{f(4)+f(0)}{2} \quad ... (2)$$

$$f'(a) f(b) = \frac{(f(4))^2 - (f(0))^2}{8}$$

$\Rightarrow (f(4))^2 - (f(0))^2 = 8f'(a) f(b)$

for some $a, b \in (0, 4)$.

Ex. 26 If f(x) is continuous in [a, b] and differentiable in (a, b) then prove that there exists atleast one $c \in (a,b)$ such that $\frac{f'(c)}{3c^2} = \frac{f(b)-f(a)}{b^3 - a^3}$.

Sol. We have to prove

$(b^3 - a^3) f'(c) - (f(b) - f(a)) (3c^2) = 0$

Let us assume a function

$F(x) = (b^3 - a^3) f(x) - (f(b) - f(a)) x^3$

which will be continuous in [a, b] and differentiable in (a, b) as f(x) and x^3 both are continuous.

Also $F(a) = b^3 f(a) - a^3 f(b) = F(b)$

So, according to Rolle's theorem, there exists atleast one $c \in (a,b)$ such that, $F'(c) = 0$, which proves the required result.

MONOTONICITY

INTRODUCTION

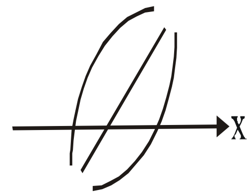
In this chapter, we shall study the nature of a function which is governed by the sign of its derivative. If the graph of a function is in upward going direction or in downward coming direction then it is called as monotonic function, and this property of the function is called **Monotonicity**. If a function is defined in any interval, and if in some part of the interval, graph moves upwards and in the remaining part moves downward then function is not monotonic in that interval.

1. Increasing Function

f(x) is said to be increasing in D_1 if for every $x_1, x_2 \in D_1$,

$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

It means that there is a certain increase in the value of f(x)



Increasing function

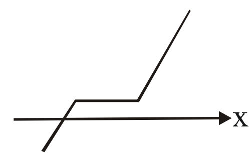
with an increase in the value of x (Refer to the adjacent figure).

2. Non-Decreasing Function

f(x) is said to be non-decreasing in D_1 if for every $x_1, x_2 \in D_1$,

$x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$. It means that the value of f(x)

would never decrease with an increase in the value of x



Non-decreasing function

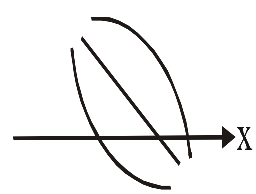
(Refer to the adjacent figure).

3. Decreasing Function

f(x) is said to be decreasing in D_1

if for every $x_1, x_2 \in D_1, x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$

it means that there is a certain decrease

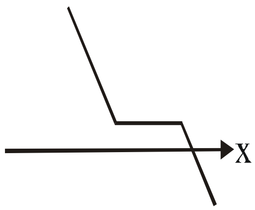


Decreasing function

in the value of f(x) with an increase in the value of x (Refer to the adjacent figure).

4. Non-increasing Function

$f(x)$ is said to be non-increasing in D_1
if for every $x_1, x_2 \in D_1$, $x_1 > x_2 \Rightarrow f(x_1) \le f(x_2)$.
It means that the value of $f(x)$ would never increase with an increase in the value of x

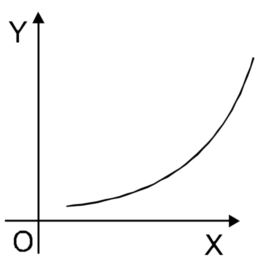


Non-decreasing function

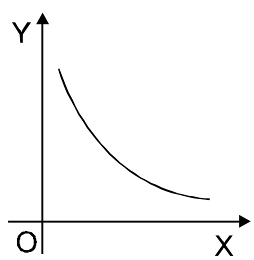
(Refer to the adjacent figure).

NOTE :

If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in D$, then $f(x)$ is called strictly increasing in domain D.



Similarly if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2), \forall\ x_1, x_2 \in D$ then it is called strictly decreasing in domain D.



For Example

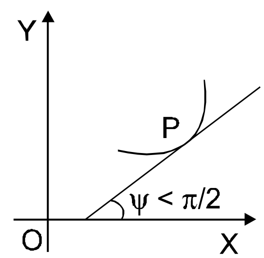
(i) $f(x) = e^x$ is a monotonic increasing function where as $g(x) = 1/x$ is monotonic decreasing function.

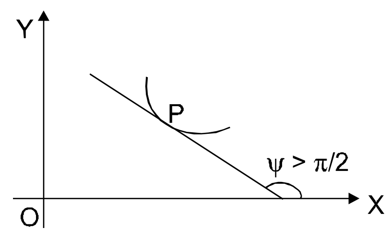
(ii) $f(x) = x^2$ and $g(x) = |x|$ are monotonic increasing for $x > 0$ and monotonic decreasing for $x < 0$. In general they are not monotonic functions.

(iii) Sin x, cos x are not monotonic function whereas tan x, cot x are monotonic.

METHOD OF TESTING MONOTONICITY

(i) At a Point : A function $f(x)$ is said to be monotonic increasing (decreasing) at a point $x = a$ of its domain if it is monotonic increasing (decreasing) in the interval $(a - h, a + h)$ where h is a small positive number. Hence we may observe that if $f(x)$ is monotonic increasing at $x = a$, then at this point tangent to its graph will make an acute angle with the x–axis where as if the function is monotonic decreasing these tangent will make an obtuse angle with x–axis. Consequently $f'(a)$ will be positive or negative according as $f(x)$ is monotonic increasing or decreasing at $x = a$.





So at $x = a$, function $f(x)$ is
Monotonic increasing $\Rightarrow f'(a) > 0$
Monotonic deacreasing $\Rightarrow f'(a) < 0$

Ex. The function $f(x) = \cos x$ is decreasing at $x = \pi/3$ and increasing at $x = 4\pi/3$ since
$f'(\pi/3) = -\sqrt{3}/2 < 0$
and $f'(4\pi/3) = +\sqrt{3}/2 > 0$

(ii) In an interval : A function $f(x)$ defined in the intervel $[a, b]$ will be

Monotonic increasing	$\Rightarrow f'(x) \ge 0$
Monotonic decreasing	$\Rightarrow f'(x) \le 0$
Constant	$\Rightarrow f'(x) = 0\ \ \forall x \in (a,b)$
Strictly increasing	$\Rightarrow f'(x) > 0$
Strictly decreasing	$\Rightarrow f''(x) < 0$

NOTE :

(i) In the above result f'(x) should not be zero for all value of x otherwise f(x) will be a constant function.

(ii) If in [a, b], f'(x) < 0, for atleast one value of x and f'(x) > 0 for atleast one value of x then f(x) will not be monotonic in [a, b].

Ex. Function f(x) = sin x is monotonic increasing in [0, π/2] because

$f'(x) = \cos x > 0\ \forall\ x \in (0, \pi/2)$

Ex. Function $f(x) = e^{-x}$ is nonotonically decreasing in [–1, 0], since

$f'(x) = -e^{-x} < 0,\ \forall\ x \in (-1, 0)$

Ex. Function $f(x) = x^2 + 1$ is monotonically decreasing in [–1, 0] because

$f'(x) = 2x < 0,\ \forall\ x \in (-1, 0)$

Ex. Function $f(x) = x^2$ is not a monotonic function in the interval [–1, 1] because

$f'(x) > 0$, when x = 1/2

$f'(x) = 2x \Rightarrow$

$f'(x) < 0$, when x = –1/2

Ex. Function $f(x) = \sin^2 x + \cos^2 x$ is constant function in [0, π/2] because

$f'(x) = 2\sin x \cos x - 2\sin x \cos x = 0\ \forall\ x \in (0, \pi/2)$

EXAMPLES OF MONOTONIC FUNCTION

If a function is monotonic increasing (decreasing) at every point of its domain, then it is said to be monotonic increasing(decreasing) function.

In the following table we have examples of some monotonic / not monotonic functions.

Monotonic Increasing	**Monotonic Decreasing**	**Not Monotonic**
x^3	1/x	x^2
x | x |	1 – 2x	| x |
e^x	e^{-x}	$e^x + e^{-x}$
$\log_a x, a > 1$	$\log_a x, a < 1$	sin x
tan x	cot x	cos x
sinh x	cosech x	cosh x
[x]	coth x	sech x

PROPERTIES OF MONOTONIC FUNCTIONS

(i) If f(x) is strictly increasing in some interval, then in that interval, f' exists and that is also strictlly increasing function.

(ii) If f(x) is continuous in [a, b] and differentiable in (a, b), then

$f'(c) \geq 0\ \forall\ c \in (a, b) \Rightarrow$ f(x) is monotonic increasing in [a, b]

$f'(c) \leq 0\ \forall\ c \in (a, b) \Rightarrow$ f(x) is monotonic decreasing in [a, b]

(iii) If both f(x) and g(x) are increasing (or decreasing) in [a,b] and gof is defined in [a, b] then gof is increasing.

(iv) If f(x) and g(x) are two monotonic functions in [a, b] such that one is increasing and other is decreasing then gof, if it is defined, is decreasing function.

Solved Examples

Ex. 27

(i) Find the critical points and the intervals of increase and decrease for $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 7$.

(ii) Find the intervals of monotonicity of the following functions:

(a) $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7$

(b) $f(x) = x \ln x$

Sol.

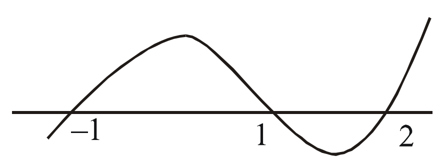
(i) $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 7$

$f'(x) = 12x^3 - 24x^2 - 12x + 24 = 0$

sign scheme for $f'(x)$:

$\Rightarrow 12(x^3 - 2x^2 - x + 2) = 0$

$\Rightarrow 12(x - 1)(x - 2)(x + 1) = 0$



Critical points are –1, 1 and 2.

The wavy curve of the derivative is given in the figure.

Hence function increases in the interval [–1, 1] $\cup$ [2, ∞) and

decreases in the interval $(-\infty, -1] \cup [1, 2]$.

(ii) (a) we have

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7,\ x \in R \Rightarrow$

$f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x-1)(x-2)(x-3)$

From the sign scheme for $f'(x)$, we can see that f(x)

decreases in $(-\infty, 1]$ increases in $[1, 2]$

decreases in $[2, 3]$ and increases in $[3, \infty)$.

(b) we have $f(x) = x \ln x,\ x > 0$

$\Rightarrow f'(x) = \ln x + 1 < 0\ \forall x < e^{-1}$

$\Rightarrow$ f(x) decreases in $(0, e^{-1}]$ increases in $[e^{-1}, \infty)$.

Ex. 28 Prove the following inequalities :

(a) $\ln(1+x) > x - \frac{x^2}{2}\ \forall x \in (0, \infty)$

(b) $\sin x \le x \le \tan x\ \forall x \in \left(0, \frac{\pi}{2}\right)$

Sol. (a) Consider the function

$$f(x) = \ln(1+x) - x + \frac{x^2}{2},\ x \in (0, \infty)$$

Then $f'(x) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x} > 0\ \forall x \in (0, \infty)$

$\Rightarrow$ f(x) increases in $(0, \infty)$ $\Rightarrow$ $f(x) > f(0^+) = 0$

i.e., $\ln(1+x) > x - \frac{x^2}{2}$ which is the desired result.

(b) Consider the function

$$f(x) = \tan x - x,\ x \in \left(0, \frac{\pi}{2}\right)$$

$$f'(x) = \sec^2 x - 1 > 0\ \forall x \in \left(0, \frac{\pi}{2}\right)$$

Thus f(x) increases in $\left(0, \frac{\pi}{2}\right)$

$\Rightarrow f(x) \ge f(0) = 0$ i.e., $\tan x \ge x$

Now, consider the function, $g(x) = x - \sin x$,

$x \in \left(0, \frac{\pi}{2}\right)$

Then $g'(x) = 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right) \ge 0\ \forall x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ g(x) increases in $\left(0, \frac{\pi}{2}\right)$ $\Rightarrow g(x) \ge g(0) = 0$

i.e., $\sin x \le x$

Ex.29 Function $f(x) = \lambda x - \sin x$ is incereasing, when -

(A) $\lambda < 1$ (B) $\lambda > 1$

(C) $\lambda = 0$ (D) None of these

Sol. $f'(x) = \lambda - \cos x$

Now function f(x) is increasing, if

$f'(x) > 0 \Rightarrow \lambda - \cos x > 0$

If $\lambda > 1$, then $\lambda - \cos x$ is always positive.

Therefore f(x) in increasing when $\lambda > 1$.

Ans.[B]

Ex.30 Function $f(x) = \cos x - 2\lambda x$ is decreasing when -

(A) $\lambda > 1/2$ (B) $\lambda < 1/2$

(C) $\lambda < 2$ (D) $\lambda > 2$

Sol. f(x) is monotonic decreasing when

$f'(x) < 0\ \forall x$

$\Rightarrow -\sin x - 2\lambda < 0$

$\Rightarrow 2\lambda > -\sin x$

$\Rightarrow 2\lambda > 1$

$\Rightarrow \lambda > 1/2$

($\because$ maximum value of $-\sin x = 1$) **Ans.[A]**

Ex.31 In which interval the function f(x) = $2x^3 - 15x^2 + 36x + 1$ is monotonically decreasing-

(A) (2, 3) (B) $(-\infty, 2)$

(C) $(3, \infty)$ (D) None of these

Sol. $f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3)$

Since f(x) is decreasing $\Rightarrow f'(x) < 0$

$\Rightarrow (x-2)(x-3) < 0$

$\Rightarrow x - 2 > 0$ and $x - 3 < 0$

or $x - 2 < 0$ and $x - 3 > 0$

$\Rightarrow x > 2$ and $x < 3$

or $x < 2$ and $x > 3$

Here $x < 2$ and $x > 3$ is not possible.

Hence $x > 2$ and $x < 3$

$\Rightarrow x \in (2, 3)$ **Ans.[A]**

Ex.32 In which interval the function $f(x) = x^2 - x + 1$ is not a monotonic function –

(A) (0, 1/2) (B) $(1/2, \infty)$

(C) (0, 1) (D) None of these

Sol. Here $f'(x) = 2x - 1$

Obviously f(x) is monotonic decreasing function in the interval (0, 1/2) (since $x < 1/2 \Rightarrow 2x - 1 < 0$) and is monotonically increasing in the interval (1/2, 1)

Thus the function is neither a decreasing function nor an increasing function in (0, 1) **Ans.[C]**

Ex.33 In the interval (1, 2), function

$f(x) = 2|x - 1| + 3|x - 2|$ is –

(A) Monotonic increasing

(B) Monotonic decreasing

(C) Not monotonic

(D) Constant

Sol. $x \in (1, 2)$

$\Rightarrow f(x) = 2(x - 1) - 3(x - 2) = -x + 4$

$\Rightarrow f'(x) = -1 < 0 \ \forall \ x$

$\therefore$ f(x) is monotonic decreasing in (1, 2) **Ans.[B]**

Ex.34 The function $f(x) = [x(x - 3)]^2$ is increasing when –

(A) $0 < x < \infty$ (B) $-\infty < x < 0$

(C) $0 < x < 3/2$ (D) $1 < x < 3$

Sol. We have

$f(x) = [x(x - 3)]^2$

$f'(x) = 2x(x - 3)[(x - 3) + x]$

$= 2x(x - 3)(2x - 3)$

If f(x) is an increasing function, then

$f'(x) > 0 \Rightarrow x(x - 3)(2x - 3) > 0$

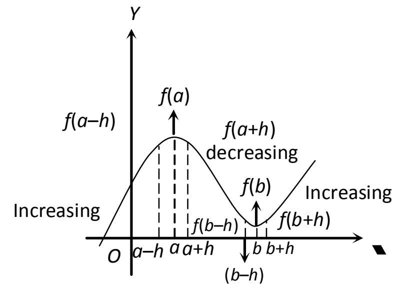
$\Rightarrow 0 < x < 3/2$ or $x > 3$ **Ans.[C]**

MAXIMUM & MINIMUM POINTS

The value of a function f (x) is said to be maximum at x = a, if there exists a very small positive number h, such that

$f(x) < f(a) \ \forall \ x \in (a - h, a + h), x \neq a$

In this case the point x = a is called a point of maxima for the function f(x).



(i) x = a is a maximum point of f(x)

$\begin{cases} f(a) - f(a+h) > 0 \\ f(a) - f(a-h) > 0 \end{cases}$

(ii) x = b is a minimum point of f(x)

$\begin{cases} f(b) - f(b+h) < 0 \\ f(b) - f(b-h) < 0 \end{cases}$

(iii) x = c is neither a maximum point nor a minimum point

$\left.\begin{matrix} f(c) - f(c+h) \\ \text{and} \\ f(c) - f(c-h) > 0 \end{matrix}\right\}$ have opposite signs.

Note :

(i) The maximum and minimum points are also known as extreme points.

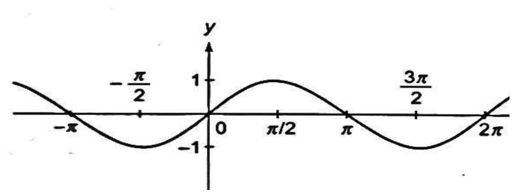
(ii) A function may have more than one maximum and minimum points.

(iii) A maximum value of a function f(x) in an interval [a,b] is not necessarily its greatest value in that interval. Similarly, a minimum value may not be the least value of the function. A minimum value may be greater than some maximum value for a function.

(iv) If a continuous function has only one maximum (minimum) point, then at this point function has its greatest (least) value.

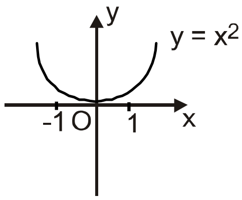
(v) Monotonic functions do not have extreme points.

Ex. Function $y = \sin x, x \in (0, \pi)$ has a maximum point at $x = \pi/2$ because the value of $\sin \pi/2$ is greatest in the given interval for sin x.



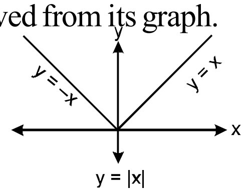
Clearly function y = sin x is increasing in the interval (0, $\pi/2$) and decreasing in the interval ($\pi/2$, π) for that reason also it has maxima at x = $\pi/2$. Similarly we can see from the graph of cos x which has a minimum point at x = π .

Ex. $f(x) = x^2$, $x \in (-1,1)$ has a minimum point at x = 0 because at x = 0, the value of x^2 is 0, which is less than the all the values of function at different points of the interval.



Clearly function $y = x^2$ is decreasing in the interval (–1, 0) and increasing in the interval (0,1) So it has minima at x = 0.

Ex. f(x) = |x| has a minimum point at x = 0. It can be easily observed from its graph.



CONDITIONS FOR MAXIMA & MINIMA OF A FUNCTION

A. Necessary Condition : A point x = a is an extreme point of a function f(x) if f'(a) = 0, provided f'(a) exists. Thus if f' (a) exists, then

x = a is an extreme point $\Rightarrow$ f'(a) = 0

or

f' (a) $\neq$ 0 $\Rightarrow$ x = a is not an extreme point.

But its converse is not true i.e.

f' (a) = 0 x = a is an extreme point.

For example if $f(x) = x^3$, then f' (0) = 0 but x = 0 is not an extreme point.

B. Sufficient Condition :

(i) The value of the function f(x) at x = a is maximum, if f' (a) = 0 and f'' (a) < 0.

(ii) The value of the function f(x) at x = a in minimum if f' (a) = 0 and f'' (a) > 0.

Note:

(i) If f' (a) = 0, f'' (a) = 0, f''' (a) $\neq$ 0 then x = a is not an extreme point for the function f(x).

(ii) If f'(a) = 0, f'' (a) = 0, f''' (a) = 0 then the sign of $f^{(iv)}$ (a) will determine the maximum and minimum value of function i.e. f(x) is maximum, if $f^{(iv)}$ (a) < 0 and minimum if $f^{(iv)}$ (a) > 0.

Tests for Local Maxima/Minima

I. Test for Local Maximum/Minimum at x = a if f(x) is Differentiable at x = a.

If f(x) is differentiable at x = a and if it is a critical point of the function (i.e., $f'(a) = 0$) then we have the following three tests to decide whether f(x) has a local maximum or local minimum or neither at x = a.

First Derivative Test :

If $f'(a) = 0$ and $f'(x)$ changes it's sign while passing through the point x = a, then

(i) f(x) would have a local maximum at x = a if $f'(a-0) > 0$ and $f'(a+0) < 0$. It means that $f'(x)$ should change it's sign from positive to negative.

(ii) f(x) would have a local minimum at x = a if $f'(a-0) < 0$ and $f'(a+0) > 0$. It means that $f'(x)$ should change it's sign from negative to positive.

(iii) If f(x) doesn't change it's sign while passing through x = a, then f(x) would have neither a maximum nor a minimum at x = a.

Second Derivative Test:

This test is basically the mathematical representation of the first derivative test. It simply says that,

(i) If $f'(a) = 0$ and $f''(a) < 0$, then f(x) would have a local maximum at x = a.

(ii) If $f'(a) = 0$ and $f''(a) > 0$, then f(x) would have a local minimum at x = a.

(iii) If $f'(a) = 0$ and $f''(a) = 0$, then this test fails and the existence of a local maximum/minimum at x = a is decided on the basis of the nth derivative test.

nth Derivative Test

It is nothing but the general version of the second derivative test, It says that if, $f'(a) = f''(a) = f'''(a) = ...f^n(a) = 0$ and $f^{n+1}(a) \neq 0$ (all derivatives of the function up to order n vanishes and (n + 1)th order derivative does not vanish at x = a), then f(x) would have a local maximum or local minimum at x = a if n is odd natural number and that x = a would be a point of local maxima if $f^{n+1}(a) < 0$ and would be a point of local minima if $f^{n+1}(a) > 0$. However if n is even, then f has neither a maxima nor a minima at x = a.

It is clear that the last two tests are basically the mathematical representation of the first derivative test. But that shouldn't diminish the importance of these tests. Because at that times it becomes very difficult to decide whether $f'(x)$ changes it's sign or not while passing through point x = a, and the remaining tests may come handy in these kind of situations.

Solved Examples

Ex.35 Find the points of maxima and minima for the function $f(x) = x^3 - 9x^2 + 15x - 11$.

Sol. Let $f(x) = x^3 - 9x^2 + 15x - 11$

then $f'(x) = 3x^2 - 18x + 15 = 3(x^2 - 6x + 5)$

For maxima and minima

$f'(x) = 0 \Rightarrow x^2 - 6x + 5 = 0$

$\Rightarrow (x-1)(x-5) = 0 \Rightarrow x = 1, 5$

Again $f''(A) = -12 < 0$

$\Rightarrow$ x = 1 is a point of maxima

and $f''(5) = 12 > 0$

$\Rightarrow$ x = 5 is a point of minima **Ans.**

Ex.36 Determine maximum and minimum points of sin x.

Sol. Let f(x) = sin x, then

$f'(x) = \cos x$, $f''(x) = -\sin x$

Now $f'(x) = 0 \Rightarrow \cos x = 0$

$x = \pm \pi/2, \ \pm 3\pi/2, ...$

Also $f''(\pi/2) = -1 < 0 \Rightarrow x = \pi/2$ is a maximum point

$f''(-\pi/2) = 1 > 0 \Rightarrow x = -\pi/2$ is minimum point

$f''(3\pi/2) = 1 > 0 \Rightarrow x = 3\pi/2$ is a minimum point

$f''(-3\pi/2) = -1 < 0 \Rightarrow x = -3\pi/2$ is a maximum point.

Thus we shall find that-

$x = \pi/2,\ 5\pi/2, ... -3\pi/2, -7\pi/2$ are maximum points and

$x = 3\pi/2,\ 7\pi/2, ... -\pi/2, -5\pi/2$ are minimum points. **Ans.**

Ex.37 Find the maximum value of $x^5 - 5x^4 + 5x^3 - 10$.

Sol. Let $f(x) = x^5 - 5x^4 + 5x^3 - 10$

$f'(x) = 5x^4 - 20x^3 + 15x^2$

$f''(x) = 20x^3 - 60x^2 + 30x$

Now $f'(x) = 0 \Rightarrow 5x^2(x^2 - 4x + 3) = 0$

$\Rightarrow x = 0, 1, 3$

But $f''(0) = 0$, $f'''(0) \neq 0$

$\therefore$ x = 0 is not an extreme point

Also $f''(1) = 20 - 60 + 30 = -10 < 0$

$\therefore$ f(1) is max. value and maximum value is

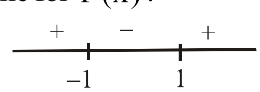
$= 1 - 5 + 5 - 10 = -9$ **Ans.**

Ex.38 Let $f(x) = x + \frac{1}{x}, x \neq 0$. Discuss the maximum and minimum values of f(x).

Sol. Here, $f'(x) = 1 - \frac{1}{x^2} \Rightarrow$

$$f'(x) = \frac{x^2 - 1}{x^2} = \frac{(x-1)(x+1)}{x^2}$$

sign scheme for $f'(x)$:



Using number line rule, we have maximum at x = –1 and minimum at x = 1

$\therefore$ at x = –1 we have local maximum $\Rightarrow f_{max}(x) = -2$

and at x = 1 we have local minimum $\Rightarrow f_{min}(x) = 2$

II. Test for Local Maximum/Minimum at x = a if f(x) is not differentiable at x = a

Case 1 :

When f(x) is continuous at x = a and $f'(a-h)$ and $f'(a+h)$ exist and are non-zero, then f(x) has a local maximum or minimum at x = a if $f'(a-h)$ and $f'(a+h)$ are of opposite signs.

If $f'(a-h) > 0$ and $f'(a+h) < 0$ then x = a will be a point of local maximum.

If $f'(a-h) < 0$ and $f'(a+h) > 0$ then x = a will be a point of local minimum.

Case 2:

When f(x) is continuous and $f'(a-h)$ and $f'(a+h)$ exist but one of them is zero, we should infer the information about the existence of local maxima/minima from the basic definition of local maxima/minima.

Case 3:

If f(x) is not continuous at x = a and $f'(a-h)$ and/or $f'(a+h)$ are not finite, then compare the values of f(x) at the neighboring points of x = a.

Remark:

It is advisable to draw the graph of the function in the vicinity of the point x = a because the graph would given us the clear picture about the existence of local maxima/minima at x = a.

Ex.39 Let $f(x) = \begin{cases} x^3 + x^2 + 10x, & x < 0 \\ -3\sin x, & x \geq 0 \end{cases}$. Investigate x = 0 for local maxima/ minima.

Sol. Clearly f(x) is continuous at x = 0 but not differentiable at x = 0 as $f(0) = f(0-0) = f(0+0) = 0$

$$f'_{-}(0) = \lim_{h\to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{-h^3 + h^2 - 10h - 0}{-h} = 10$$

But $f'_{+}(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{-3\sinh}{h} = -3$

Since $f'_{-}(0) > 0$ and $f'_{+}(0) < 0, x = 0$ is the point of local maximum.

CONCEPT OF GLOBAL MAXIMUM MINIMUM

Let y = f(x) be a given function with domain D. Let $[a, b] \subseteq D$. Global maximum/minimum of f(x) in [a, b] is basically the greatest/least value of f(x) in [a, b].

Global maximum and minimum in [a, b] would always occur at critical points of f(x) within [a, b] or at the end points of the interval, if f is continuous in [a, b].

I. Global Maximum/Minimum in [a, b]

In order to find the global maximum and minimum of a continuous function f(x) in [a, b], find out all the critical points of f(x) in (a, b). Let $c_1, c_2, \ldots, c_n$ be the different critical points. Find the value of the function at these critical points. Let $f(c_1), f(c_2), \ldots, f(c_n)$ be the values of the function at critical points.

Say, $M_1 = \max \{f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)\}$

and $M_2 = \min \{f(a), f(c), f(c_2), \ldots, f(c_n), f(b)\}$

Then M_1 is the greatest value of f(x) in [a, b] and M_2 is the least value of f(x) in [a, b].

II. Global Maximum/Minimum in (a, b)

Method for obtaining the greatest and least values of f(x) in (a, b) is almost same as the method used for obtaining the greatest and least values in [a, b] however with a caution.

Let y = f(x) be a continuous function and $c_1, c_2 \ldots c_3$ be the different critical points of the function in (a, b).

Let $M_1 = \max. \{f(c_1), f(c_2), f(c_3) \ldots f(c_n)\}$

and $M_2 = \min \{f(c_1), f(c_2), f(c_3) \ldots f(c_n)\}$

Now if $\lim\limits_{\substack{x\to a+0 \\ (\text{or } x\to b-0)}} f(x) > M_1$ or $< M_2$, f(x) would not have global maximum (or global minimum) in (a, b).

This means that if the limiting values at the end points are greater than M_1 or less than M_2, then f(x) would not have global maximum/minimum in (a, b).

On the other hand if $M_1 > \lim\limits_{\substack{x\to a-0 \\ (\text{and } x\to b+0)}} f(x)$ and $M_2 < \lim\limits_{\substack{x\to a+0 \\ (\text{and } x\to b-0)}} f(x)$, then M_1 and M_2 would respectively be the global maximum and global minimum of f(x) in (a, b).

Solved Examples

Ex.40 let $f(x) = 2x^3 - 9x^2 + 12x + 6$. Discuss the global maximum and minimum of f(x) in [0, 2] and in (1, 3).

Sol. $f(x) = 2x^3 - 9x^2 + 12x + 6$

$\Rightarrow f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$

First of all let us discuss [0, 2].

Clearly the critical point of f(x) in [0, 2] is x = 1.

f(0) = 6, f(1) = 11, f(2) = 10

Thus x = 0 is the point of global minimum of f(x) in [0, 2] and x = 1 is the point of global maximum.

Now let us consider (1, 3)

Clearly, x = 2 is the only critical point in (1, 3),

$f(2) = 10$, $\lim_{x\to 1+0} f(x) = 11$ and $\lim_{x\to 3-0} f(x) = 15$

Thus x = 2 is the point of global minimum in (1, 3) and the global maximum in (1, 3) does not exist.

Greatest & least value in an interval

Ex.41 Find the greatest value of $x^3 - 12x^2 + 45x$ in the interval [0, 7].

Sol. Let $f(x) = x^3 - 12x^2 + 45x$, then

$f'(x) = 3x^2 - 24x + 45$

$= 3(x-3)(x-5)$ and $f''(x) = 6x - 24$

Now for maximum and minimum values

$f'(x) = 0 \Rightarrow 3(x-3)(x-5) = 0$

$\Rightarrow x = 3, 5$

Again $f''(3) = -6 < 0 \Rightarrow$ The function if maximum at x = 3 and $f''(5) = 6 > 0$

$\Rightarrow$ The function is minimum at x = 5

Now f(0) = 0, f(3)= 54, f(5) = 50, f(7) = 70

$\Rightarrow$ The greatest value in [0,7]

= max. {0,54, 50, 70} = 70 **Ans.**

PROPERTIES OF MAXIMA & MINIMA

If f(x) is a continuous function and the graph of this function is drawn, then-

(i) Between two equal values of f(x), there lie at least one maxima or minima.

(ii) Maxima and minima occur alternately. For example if x = – 1, 0,2,3 are extreme points of a continuous function and if x = 0 is a maximum point then x = –1,2 will be minimum points.

(iii) When x passes a maximum point, the sign of f'(x) changes from + ve to – ve, whereas x passes through a minimum point, the sign of f'(x) changes from – ve to + ve.

(iv) If there is no change in the sign of dy/dx on two sides of a point, then such a point is not an extreme point.

(v) If f(x) is a maximum (minimum) at a point x = a, then 1/f (x), $[f(x) \neq 0]$ will be minimum (maximum) at that point.

(vi) If f(x) is maximum (minimum) at a point x = a, then for any $\lambda \in R$, $\lambda + f(x)$, log f(x) and for any k > 0, k f(x), $[f(x)]^k$ are also maximum (minimum) at that point.

MAXIMA & MINIMA OF FUNCTIONS OF TWO VARIABLES

If a function is defined in terms of two variables and if these variables are associated with a given relation then by eliminating one variable, we convert function in terms of one variable and then find the maxima and minima by known methods.

Solved Examples

Ex.42 If x+y = 8 then find the maximum value of xy.

Sol. Let z = xy

$\therefore z = x(8-x)$ or $z = 8x - x^2$

$dz/dx = 8 - 2x = 0$

$\Rightarrow x = 4$ $d^2z/dx^2 = -2 < 0$

$\Rightarrow$ x = 4 is a maximum point. So maximum value is $z = 8.4 - 4^2 = 16$. **Ans.**

SOME STANDARD GEOMETRICAL RESULTS RELATED TO MAXIMA & MINIMA

The following results can easily be established.

(i) The area of rectangle with given perimeter is greatest when it is a square.

(ii) The perimeter of a rectangle with given area is least when it is a square.

(iii) The greatest rectangle inscribed in a given circle is a square.

(iv) The greatest triangle inscribed in a given circle is equilateral.

(v) The semi vertical angle of a cone with given slant height and maximum volume is $\tan^{-1}\sqrt{2}$.

(vi) The height of a cylinder of maximum volume inscribed in a sphere of radius a is a $2a/\sqrt{3}$.

SOME IMPORTANT RESULTS

(i) Equilateral triangle :

Area = $(\sqrt{3}/4)\,x^2$, where x is its side.

(ii) Square :

Area = a^2 , perimeter = 4a, where a is its side.

(iii) Rectangle:

Area = ab, perimeter = 2 (a+b) where a,b are its sides

(iv) Trapezium :

Area = 1/2 (a+ b) h

Where a,b are lengths of parallel sides and h be the distance between them.

(v) Circle :

Area = πa^2 , perimeter = $2\pi a$, where a is its radius.

(vi) Sphere :

Volume = $4/3\ \pi a^3$, surface $4\pi a^2$ where a is its radius

(vii) Right Circular cone :

Volume = $1/3\ \pi r^2 h$, curved surface = $\pi r \ell$

Where r is the radius of its base, h be its height and ℓ be its slant heights

(viii) Cylinder :

Volume = $\pi r^2 h$

whole surface = $2\pi r\,(r+h)$

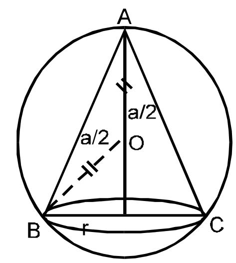
where r is the radius of the base and h be its height.

Solved Examples

Ex.43 Find the height of a right circular cone of maximum volume inscribed in a sphere of diameter a.

Sol. Let r be the radius of the base and x be the height of the inscribed cone. Then

$r^2 = a^2/4 - (x-a/2)^2 = ax - x^2$



If V be the volume of the cone, then

$V = 1/3\ \pi r^2 x = 1/3\ \pi\ (ax - x^2)\,x$

$= \pi/3\ (ax^2 - x^3)$

$\Rightarrow \frac{dV}{dx} = \frac{\pi}{3}(2ax - 3x^2)\ ,\ \frac{d^2V}{dx^2} = \frac{2\pi a}{3} - 2\pi x$

Now $\frac{dv}{dx} = 0 \Rightarrow x = 0$ or $x = 2a/3$

But $x \neq 0$ and $x = 2a/3$, $\frac{d^2V}{dx^2} = \frac{-2\pi a}{3} < 0$.

so V is maximum when height of cone = (2/3) a.**Ans.**