Method of Differentiation

FIRST PRINCIPLE OF DIFFERENTIATION

1. The derivative of a given function f at a point x = a in its domain represent the slope of the tangent at that point, and it is defined as:

$\underset{h\to 0}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$, provided the limit exists & is denoted by $f'(a)$.

i.e. $f'(a) = \underset{x\to a}{\text{Limit}} \dfrac{f(x)-f(a)}{x-a}$, provided the limit exists.

2. If x and x + h belong to the domain of a function f defined by y = f(x), then

$\underset{h\to 0}{\text{Limit}} \dfrac{f(x+h)-f(x)}{h}$ if it exists, is called the Derivative of f at x & is denoted by $f'(x)$ or $\dfrac{dy}{dx}$.

i.e., $f'(x) = \underset{h\to 0}{\text{Limit}} \dfrac{f(x+h)-f(x)}{h}$

This method of differentiation is also called ab-initio method or first principle method.

Solved Examples

Ex.1 : Find derivative of following functions by first principle with respect to x.

(i) $f(x) = x^2$ (ii) $f(x) = \tan x$

(iii) $f(x) = e^{\sin x}$

Sol. (i) $f'(x) = \lim\limits_{h\to 0} \dfrac{(x+h)^2 - x^2}{h}$

$= \lim\limits_{h\to 0} \dfrac{2xh + h^2}{h} = 2x.$

(ii) $f'(x) = \lim\limits_{h\to 0} \dfrac{\tan(x+h) - \tan x}{h}$

$= \lim\limits_{h\to 0} \dfrac{\dfrac{\sin(x+h)}{\cos(x+h)} - \dfrac{\sin x}{\cos x}}{h}$

$= \lim\limits_{h\to 0} \dfrac{\sin(x+h-x)}{h\cos x.\cos(x+h)} = \sec^2 x.$

(iii) $f'(x) = \lim\limits_{h\to 0} \dfrac{e^{\sin(x+h)} - e^{\sin x}}{h}$

$= \lim\limits_{h\to 0} e^{\sin x}$

$\dfrac{\left[e^{\sin(x+h)-\sin x} - 1\right]}{\sin(x+h) - \sin x} \left(\dfrac{\sin(x+h) - \sin x}{h}\right)$

$= e^{\sin x} \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$

$= e^{\sin x} \lim_{h \to 0} \frac{2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h}$

$= e^{\sin x} \lim_{h \to 0} \left\{\cos\left(x+\frac{h}{2}\right)\frac{\sin(h/2)}{h/2}\right\}$

$= e^{\sin x} \cos x$

DERIVATIVES OF SOME STANDARD FUNCTIONS

Some Differentiation Formulas

(i) $\frac{d}{dx}(\text{constant}) = 0$

(ii) $\frac{d}{dx}(x^n) = nx^{n-1}$

(iii) $\frac{d}{dx}(e^x) = e^x$

(iv) $\frac{d}{dx}(a^x) = a^x \log_e a$

(v) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(vi) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$

(vii) $\frac{d}{dx}(\sin x) = \cos x$

(viii) $\frac{d}{dx}(\cos x) = -\sin x$

(ix) $\frac{d}{dx}(\tan x) = \sec^2 x$

(x) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(xi) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(xii) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

(xiii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$; $-1 < x < 1$

(xiv) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$; $-1 < x < 1$

(xv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$; $x \in R$

(xvi) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, $\forall\ x \in R$

(xvii) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$

(xviii) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$; $|x| > 1$

(xix) $\frac{d}{dx}(\sinh x) = \cosh x$

(xx) $\frac{d}{dx}(\cosh x) = \sinh x$

(xxi) $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$

(xxii) $\frac{d}{dx}(\coth x) = -\text{cosech}^2 x$

(xxiii) $\frac{d}{dx}(\text{sech } x) = -\text{sech } x \tanh x$

(xxiv) $\frac{d}{dx}(\text{cosech } x) = -\text{cosech } x \coth x$

(xxv) $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$, $\forall\ x \in R$

(xxvi) $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$, $|x| > 1$

(xxvii) $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$, $x \neq \pm 1$

(xxviii) $\frac{d}{dx}\left(\coth^{-1}x\right)=\frac{1}{x^2-1}$, $x \neq \pm 1$

(xxix) $\frac{d}{dx}\left(\text{sech}^{-1}x\right)= -\frac{1}{|x|\sqrt{1-x^2}}$; $|x|<1$

(xxx) $\frac{d}{dx}\left(\text{cosech}^{-1}x\right)=-\frac{1}{|x|\sqrt{x^2+1}}$; $\forall x \in R$

(xix) $\frac{d}{dx}(e^{ax}\sin bx)=e^{ax}(a\sin bx+b\cos bx)$

$$=\sqrt{a^2+b^2}\ e^{ax}\sin\left(bx+\tan^{-1}\frac{b}{a}\right)$$

(xx) $\frac{d}{dx}(e^{ax}\cos bx)=e^{ax}(a\cos bx-b\sin bx)$

$$=\sqrt{a^2+b^2}\ e^{ax}\cos\left(bx+\tan^{-1}\frac{b}{a}\right)$$

(xxi) $\frac{d}{dx}|x|=\frac{x}{|x|}$ $(x \neq 0)$

(xxii) $\frac{d}{dx}\log|x|=\frac{1}{x}$ $(x \neq 0)$

Basic theorems :

Sum of two differentiable functions is always differentiable.

Sum of two non-differentiable functions may be differentiable.

There are certain basic theorems in differentiation:

1. $\frac{d}{dx}(f\pm g)=f'(x)\pm g'(x)$

2. $\frac{d}{dx}(k\,f(x))=k\frac{d}{dx}f(x)$

3. $\frac{d}{dx}(f(x)\,.\,g(x))=f(x)\,g'(x)+g(x)\,f'(x)$

4. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\,f'(x)-f(x)g'(x)}{g^2(x)}$

5. $\frac{d}{dx}(f(g(x)))=f'(g(x))\,g'(x)$

This rule is also called the chain rule of differentiation and can be written as

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$$

Note that an important inference obtained from the chain rule is that

$$\frac{dy}{dy}=1=\frac{dy}{dx}\cdot\frac{dx}{dy}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{dx/dy}$$

another way of expressing the same concept is by considering $y=f(x)$ and $x=g(y)$ as inverse functions of each other.

$$\frac{dy}{dx}=f'(x) \quad \text{and} \quad \frac{dx}{dy}=g'(y)$$

$$\Rightarrow \quad g'(y)=\frac{1}{f'(x)}$$

Solved Examples

Ex.2 : Find the derivative of the following functions with respect to x.

(i) $f(x)=\sqrt{\sin(2x+3)}$

(ii) $f(x)=\frac{x}{1+x^2}$

(iii) $f(x)=x\,.\,\sin x$

Sol. (i) $f(x)=\sqrt{\sin(2x+3)}$

$$\Rightarrow \quad f'(x)=\frac{d}{dx}\left(\sqrt{\sin(2x+3)}\right)$$

$$=\frac{1}{2\sqrt{\sin(2x+3)}}\cdot\frac{d}{dx}(\sin(2x+3))$$

(chain rule)

$$=\frac{\cos(2x+3)}{\sqrt{\sin(2x+3)}}$$

(ii) $f(x) = \frac{x}{1+x^2}$

$\Rightarrow f'(x) = \frac{(1+x^2)-x(2x)}{(1+x^2)^2}$ (Quotiant rule)

$= \frac{1-x^2}{(1+x^2)^2}$

(iii) $f(x) = x \sin x$

$\Rightarrow f'(x) = x. \cos x + \sin x$ (Product rule)

Ex.3 : If $f(x) = \sin(x + \tan x)$, then find value of $f'(0)$.

Sol. $\because$ $f(x) = \sin(x + \tan x)$

$\Rightarrow$ $f'(x) = \cos(x + \tan x)(1 + \sec^2 x)$ (chain rule)

Hence, $f'(0) = 2$

Ex.4 : If $f(x) = \ell n(\sin^{-1} x^2)$, then find $f'(x)$.

Sol. $f'(x) = \frac{1}{(\sin^{-1} x^2)} \cdot \frac{1}{\sqrt{1-(x^2)^2}} . 2x$

$= \frac{2x}{(\sin^{-1} x^2)\sqrt{1-x^4}}$

Ex.5 : If $f(x) = 2x \sec^{-1}x - \operatorname{cosec}^{-1}(x)$, then find $f'(-2)$.

Sol. $f'(x) = 2\sec^{-1}(x) + \frac{2x}{|x|\sqrt{x^2-1}} + \frac{1}{|x|\sqrt{x^2-1}}$

Hence, $f'(-2) = 2.\sec^{-1}(-2) - \frac{2}{\sqrt{3}} + \frac{1}{2\sqrt{3}}$

$f'(-2) = \frac{4\pi}{3} - \frac{\sqrt{3}}{2}$

LOGARITHMIC DIFFERENTIATION

The process of taking logarithm of the function first and then differentiate is called **Logarithmic differentiation**. It is often useful in situations when

(i) a function is the product or quotient of a number of functions OR

(ii) a function is of the form $[f(x)]^{g(x)}$ where f & g are both derivable,

Solved Examples

Ex.6 : If $y = (\sin x)^{\ell n x}$, find $\frac{dy}{dx}$

Sol. $\ell n\, y = \ell n\, x . \ell n(\sin x)$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\ell n(\sin x) + \ell n\, x. \frac{\cos x}{\sin x}$

$\Rightarrow \frac{dy}{dx} = (\sin x)^{\ell n x}\left[\frac{\ell n(\sin x)}{x} + \cot x \;\; \ell n\, x\right]$

Ex.7 : If $y = \frac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/5}}$, then find $\frac{dy}{dx}$.

Sol. $\because$ $y = \frac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/5}}$

taking $\log_e$ on both side

$\ell n\, y = \frac{1}{2}\ell n\, x + \frac{2}{3}\ell n(1-2x) - \frac{3}{4}\ell n(2-3x) - \frac{4}{5}\ell n(3-4x)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}$

$\Rightarrow \frac{dy}{dx} = y \left(\frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{5(3-4x)}\right)$

Ex.8 Find the derivative of x^x, w.r.t. x.

Sol. Let $y = x^x$ OR $y = x^x$

$\log y = x \log x$

$= e^{x \ell n x}$

$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \log x$

$$\frac{dy}{dx} = \frac{d}{dx}e^{x\ln x} = e^{x\ln x}\frac{d}{dx}(x\ln x)$$

$$\frac{dy}{dx} = x^x(1 + \log x)$$

$$= e^{x\ln x}\left\{x \cdot \frac{1}{x} + \ln x.1\right\}$$

$$\therefore \quad \boxed{\frac{d}{dx}x^x = x^x \log ex}$$

$$= x^x(1 + \ln x)$$

Hence $\frac{d}{dx}(x^x) = x^x(1 + \ln x)$

Short method : we can directly write the derivative of $[f(x)]^{g(x)}$ as following :

$$\boxed{\frac{d}{dx}(f(x))^{g(x)} = f(x)^{g(x)}\left[\frac{d}{dx}\{g(x)\log f(x)\}\right]}$$

For the above function

$$\frac{d}{dx}(x^x) = x^x\left[\frac{d}{dx}(x\log x)\right] = x^x[1 + \log x]$$

Ex.9 Find derivative of $(\sin x)^{\cos x}$

Sol. $\frac{d}{dx}(\sin x)^{\cos x}$

$$= (\sin x)^{\cos x}\left[\frac{d}{dx}\{\cos x \log(\sin x)\}\right]$$

$$= (\sin x)^{\cos x}[\cos x . \cot x - \sin x . \log \sin x]$$

Ex.10 Find derivative of $(x)^{\sin x} + (\sin x)^x$

Sol. Derivative $= \frac{d}{dx}(x)^{\sin x} + \frac{d}{dx}(\sin x)^x$

$$= (x)^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right) +$$

$$(\sin x)^x(\log \sin x + x\cot x)$$

DIFFERENTIATION OF IMPLICIT FUNCTIONS

If in an implicit function $f(x, y) = 0$, y cannot be expressed in terms of x, then we differentiate both sides of the given equation w.r.t. x and collect all terms containing $\frac{dy}{dx}$ on L.H.S.

NOTE : In the above process we obtain dy/dx in terms of both x and y. If we want dy/dx in terms of x only, then let us first express y in terms of x .

Solved Examples

Ex.11 Find dy/dx when $x^2 + y^2 = 6x - 2y$

Sol. Differentiating both sides w.r.t.x, we have

$$2x + 2y\frac{dy}{dx} = 6 - 2\frac{dy}{dx}$$

$$\Rightarrow (y+1)\frac{dy}{dx} = 3 - x \Rightarrow \frac{dy}{dx} = \frac{3-x}{y+1}$$

Short method of differentiation for implicit functions

If $f(x,y)$ = constant, then

$$\frac{dy}{dx} = -\left(\frac{\partial f}{\partial x}\right)/\left(\frac{\partial f}{\partial y}\right)$$

where $\left(\frac{\partial f}{\partial x}\right)$ and $\left(\frac{\partial f}{\partial y}\right)$ are partial derivatives of f (x ,y) with respect to x and y respectively.

[By partial derivative of f(x , y) with respect to x, we mean the derivative of f(x, y) with respect to x when y is treated as a constant.]

Ex.12 Find $\frac{dy}{dx}$ when $x^2 + y^2 = 6x - 2y$

Sol. writing the given equation as

$f(x, y) = x^2 + y^2 - 6x + 2y = 0$

Now $\frac{\partial f}{\partial x} = 2x - 6$, $\frac{\partial f}{\partial y} = 2y + 2$

$$\therefore \quad \frac{dy}{dx} = -\left(\frac{\partial f}{\partial x}\right)\Big/\left(\frac{\partial f}{\partial y}\right)$$

$$= -\frac{2(x-3)}{2(y+1)} = \left(\frac{3-x}{y+1}\right)$$

Ex.13 If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$

Sol. Here $f(x, y) = x^y + y^x - a^b = 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{yx^y - 1 + y^x \log y}{x^y \log x + xy^{x-1}}$$

Ex.14 : If $x^3 + y^3 = 3xy$, then find $\frac{dy}{dx}$.

Sol. Differentiating both sides w.r.t.x, we get

$$3x^2 + 3y^2 \frac{dy}{dx} = 3x \frac{dy}{dx} + 3y$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

Note that above result holds only for points where $y^2 - x \neq 0$

Ex.15 : If $x^y + y^x = 2$ then find $\frac{dy}{dx}$

Sol. $u + v = 2$

$$\Rightarrow \quad \frac{du}{dx} + \frac{dv}{dx} = 0 \qquad \text{........(i)}$$

where $u = x^y$ & $v = y^x$

$\Rightarrow \quad \ell n\, u = y\, \ell n\, x$ & $\ell n\, v = x\, \ell n\, y$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \frac{y}{x} + \ell n\, x \frac{dy}{dx} \quad \&$$

$$\frac{1}{v}\frac{dv}{dx} = \ell n\, y + \frac{x}{y}\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = x^y \left(\frac{y}{x} + \ell n\, x \frac{dy}{dx}\right) \quad \&$$

$$\frac{dv}{dx} = y^x \left(\ell n\, y + \frac{x}{y}\frac{dy}{dx}\right)$$

Now, equation (i) becomes

$$x^y \left(\frac{y}{x} + \ell n\, x \frac{dy}{dx}\right) + y^x \left(\ell n\, y + \frac{x}{y}\frac{dy}{dx}\right) = 0.$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\left(y^x \ell n\, y + x^y . \frac{y}{x}\right)}{\left(x^y \ell n\, x + y^x . \frac{x}{y}\right)}$$

DIFFERENTIATION USING TRIGONOMETRICAL SUBSTITUTIONS

Some times before differentiation we reduce the given function in a simple form using suitable trigonometrical or algebraic transformations. For this following formulae and substitutions should be remembered.

(i) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1} [x\sqrt{1-y^2} \pm y\sqrt{1-x^2}]$

(ii) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1} [xy \mp \sqrt{(1-x^2)(1-y^2)}]$

(iii) $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \left[\frac{x \pm y}{1 \mp x\, y}\right]$

(iv) $2\sin^{-1} x = \sin^{-1} (2x\sqrt{1-x^2})$

(v) $2\cos^{-1} x = \cos^{-1} (2x^2 - 1)$

(vi) $2\tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2}\right)$

(vii) $2\tan^{-1} x = \sin^{-1} \left(\frac{2x}{1+x^2}\right)$

(viii) $2\tan^{-1} x = \cos^{-1} \left(\frac{1-x^2}{1+x^2}\right)$

(ix) $\frac{\pi}{4} - \tan^{-1} x = \tan^{-1} \left(\frac{1-x}{1+x}\right)$

(x) $3\sin^{-1} x = \sin^{-1} (3x - 4x^3)$

(xi) $3\cos^{-1} x = \cos^{-1} (4x^3 - 3x)$

(xii) $3\tan^{-1} x = \tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2}\right)$

(xiii) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(xiv) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

(xv) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$

(xvi) $\tan\left(\frac{\pi}{4} + \theta\right) = \left(\frac{1 + \tan\theta}{1 - \tan\theta}\right)$

SOME SUITABLE SUBSTITUTIONS

(i) If the function involve the term $\sqrt{a^2 - x^2}$, then put $x = a\sin\theta$ or $x = a\cos\theta$

(ii) If the function involve the term $\sqrt{x^2 + a^2}$, then put $x = a\tan\theta$ or $x = a\cot\theta$

(iii) If the function involve the term $\sqrt{x^2 - a^2}$, then put $x = a\sec\theta$ or $x = a\,\mathrm{cosec}\theta$

(iv) If the function involve the term $\left(\sqrt{\dfrac{a-x}{a+x}}\right)$, then put $x = a\cos\theta$ or $x = a\cos2\theta$

Solved Examples

Ex.16 If $y = \cot^{-1}\left(\dfrac{\sqrt{1+x^2}+1}{x}\right)$, then find $\dfrac{dy}{dx}$

Sol. Putting $x = \tan\theta$, we have

$$y = \cot^{-1}\left(\frac{\sec\theta + 1}{\tan\theta}\right) = \cot^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right)$$

$$= \cot^{-1}(\cot\theta/2) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{1+x^2}$$

Ex.17 : Differentiate $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ with respect to x.

Sol. Let $x = \tan\theta$, where $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) - \{0\}$

$$y = \tan^{-1}\left(\frac{|\sec\theta| - 1}{\tan\theta}\right)$$

$$\left\{ |\sec\theta| = \sec\theta \ \forall \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow y = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\Rightarrow y = \frac{\theta}{2}$$

$$\left\{ \tan^{-1}(\tan x) = x \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}$$

$$\Rightarrow y = \frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

Ex.18 : Find $\dfrac{dy}{dx}$, where $y = \tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$.

Sol. Let $x = \cos\theta$, where $\theta \in [0, \pi]$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\sqrt{2}\cos\frac{\theta}{2} - \sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2} + \sqrt{2}\sin\frac{\theta}{2}}\right)$$

$$\left[\because \ \sqrt{1+\cos\theta} = \left|\sqrt{2}\cos\frac{\theta}{2}\right| \text{ but for } \frac{\theta}{2} \in \left(0, \frac{\pi}{2}\right), \ \left|\sqrt{2}\cos\frac{\theta}{2}\right| = \sqrt{2}\cos\frac{\theta}{2}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right)$$

$$\Rightarrow \quad y = \frac{\pi}{4} - \frac{\theta}{2} \quad \text{as } -\frac{\pi}{4} \le \frac{\pi}{4} - \frac{\theta}{2} \le \frac{\pi}{4}$$

$$\Rightarrow \quad y = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$$

Ex.19 : If $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then find

(i) $f'(2)$ (ii) $f'\left(\frac{1}{2}\right)$ (iii) $f'(1)$

Sol. $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\Rightarrow y = \sin^{-1}(\sin 2\theta)$

$$y = \begin{cases} \pi - 2\theta & , \frac{\pi}{2} < 2\theta < \pi \\ 2\theta & , -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2} \\ -(\pi + 2\theta) & , -\pi < 2\theta < -\frac{\pi}{2} \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} \pi - 2\tan^{-1}x & x > 1 \\ 2\tan^{-1}x & -1 \le x \le 1 \\ -(\pi + 2\tan^{-1}x) & x < -1 \end{cases}$$

$$\Rightarrow \quad f'(x) = \begin{cases} -\frac{2}{1+x^2} & x > 1 \\ \frac{2}{1+x^2} & -1 < x < 1 \\ \frac{-2}{1+x^2} & x < -1 \end{cases}$$

(i) $f'(2) = -\frac{2}{5}$ (ii) $f'\left(\frac{1}{2}\right) = \frac{8}{5}$

(iii) $f'(1^+) = -1$ and $f'(1^-) = +1$

$\Rightarrow$ $f'(1)$ does not exist.

Ex.20 : If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then prove that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

Sol. Put $x = \sin\alpha$, where $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ and

$y = \sin\beta$, where $-\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$.

$\Rightarrow \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$

$\Rightarrow 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$= 2a\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$\Rightarrow \cot\left(\frac{\alpha-\beta}{2}\right) = a$

$\Rightarrow \alpha - \beta = 2\cot^{-1}(a)$ or $-\pi + 2\cot^{-1}a$

$\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$

or $-\pi + 2\cot^{-1}a$

differentiating w.r.t to x.

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

PARAMETRIC DIFFERENTIATION

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

Solved Examples

Ex.21 : If $x = a\cos^3 t$ and $y = a\sin^3 t$, then find the value of $\frac{dy}{dx}$.

Sol. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3a\sin^2 t\cos t}{3a\cos^2 t\sin t} = -\tan t$

Ex.22 : If $y = a\cos t$ and $x = a(t - \sin t)$, then find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{2}$.

Sol. $\frac{dy}{dx} = \frac{-a\sin t}{a(1-\cos t)}$ $\Rightarrow$ $\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = -1$.

Derivative of one function with respect to another

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

Solved Examples

Ex.23: Find derivative of $y = \ell n\, x$ with respect to $z = e^x$.

Sol. $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{1}{xe^x}$

SUCCESSIVE DIFFERENTIATION

If the first derivative $\frac{dy}{dx}$ of a function $y = f(x)$ is also a differentiable function, then it can be further differentiated w.r.t.x and this derivative is denoted by $\frac{d^2y}{dx^2}$ which is called the second derivative of y w.r.t. x. Further if $\frac{d^2y}{dx^2}$ is also differentiable then its derivative is called third derivative of y which is denoted by $\frac{d^3y}{dx^3}$. Similarly the nth derivative of y is denoted by $\frac{d^ny}{dx^n}$. All these derivatives are called as successive derivatives of y and this process is known as successive differentiation.

We also use the following symbols for the successive derivatives of $y = f(x)$:

y_1, y_2, y_3, ... y_n, ...

y', y'', y''', ... y^n, ...

$\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, ... $\frac{d^ny}{dx^n}$, ...

Dy, D^2y, D^3y, ... D^ny, ...

(where $D \equiv \frac{d}{dx}$)

The value of the nth derivative at x = a is expressed by the following symbols :

$$y_n(a),\ y^n(a),\ \left(\frac{d^ny}{dx^n}\right)_{x=a},\ D^ny(a)\ \&\ f^n(a)$$

Solved Examples

Ex.24 : If $y = x^3 \ell n\, x$, then find y'' and y'''

Sol. $y' = 3x^2 \ell n\, x + x^3 \frac{1}{x} = 3x^2 \ell n\, x + x^2$

$y'' = 6x\, \ell n\, x + 3x^2 . \frac{1}{x} + 2x = 6x\, \ell n\, x + 5x$

$y''' = 6\, \ell n\, x + 11$

Ex.25 : If $y = \left(\frac{1}{x}\right)^x$, then find $y''(1)$

Sol. Now taking $\log_e$ both sides, we get

$\ell n\, y = -x\, \ell n\, x$ when $x = 1$, then $y = 1$

$\ell n\, y = -x\, \ell n\, x$

$\Rightarrow \frac{y'}{y} = -(1 + \ell n\, x)$

$\Rightarrow y' = -y(1 + \ell n\, x)$ ......(i)

again diff. w.r.t. to x,

$y'' = -y'(1 + \ell n\, x) - y . \frac{1}{x}$

$\Rightarrow y'' = y(1 + \ell n\, x)^2 - \frac{y}{x}$ (using (i))

$\Rightarrow y''(1) = 0$

Ex.26 : If $x = t + 1$ and $y = t^2 + t^3$, then find $\frac{d^2y}{dx^2}$.

Sol. $\frac{dy}{dt} = 2t + 3t^2$; $\frac{dx}{dt} = 1$

$\Rightarrow \frac{dy}{dx} = 2t + 3t^2$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dt}(2t + 3t^2) . \frac{dt}{dx}$

$\frac{d^2y}{dx^2} = 2 + 6t.$

Ex.27 : Find second order derivative of y= sin x with respect to $z = e^x$.

Sol. $\Rightarrow \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{\cos x}{e^x}$

$\Rightarrow \frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{\cos x}{e^x}\right)$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{\cos x}{e^x}\right) . \frac{dx}{dz}$

$= \frac{-e^x \sin x - \cos x\, e^x}{(e^x)^2} \cdot \frac{1}{e^x}$

$\frac{d^2y}{dz^2} = -\frac{(\sin x + \cos x)}{e^{2x}}$

Ex.28 : $y = f(x)$ and $x = g(y)$ are inverse functions of each other, then express $g'(y)$ and $g''(y)$ in terms of derivative of $f(x)$.

Sol. $\frac{dy}{dx} = f'(x)$ and $\frac{dx}{dy} = g'(y)$

$\Rightarrow \quad g'(y) = \frac{1}{f'(x)}$ ....(i)

again differentiating w.r.t. to y

$$g''(y) = \frac{d}{dy}\left(\frac{1}{f'(x)}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{f'(x)}\right) \cdot \frac{dx}{dy}$$

$$= -\frac{f''(x)}{f'(x)^2} \cdot g'(y)$$

$\Rightarrow \quad g''(y) = -\frac{f''(x)}{f'(x)^3}$ ....(ii)

which can also be remembered as

$$\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}.$$

Ex.29 : $y = \sin(\sin x)$ then prove that $y'' + (\tan x)\, y' + y\cos^2 x = 0$

Sol. Such expression can be easily proved using implicit differentiation

$\Rightarrow \quad y' = \cos(\sin x)\cos x$

$\Rightarrow \quad \sec x . y' = \cos(\sin x)$

again differentiating w.r.t x, we get

$\sec x\, y'' + y' \sec x \tan x = -\sin(\sin x)\cos x$

$\Rightarrow \quad y'' + y' \tan x = -y\cos^2 x$

$\Rightarrow \quad y'' + (\tan x)\, y' + y\cos^2 x = 0$

N^{TH} DERIVATIVES OF SOME STANDARD FUNCTIONS

(1) $D^n(ax + b)^m = m(m-1)(m-2)$ .... $(m - n + 1)\, a^n (ax + b)^{m-n}$

(2) If $m \in N$ and $m > n$, then

$$D^n(ax + b)^m = \frac{m!}{(m-n)!}\, a^n (ax + b)^{m-n}$$

$$D^n(x^m) = \frac{m!}{(m-n)!} x^{m-n}$$

(3) $D^n(ax + b)^n = n!\, a^n$

$$D^n(x^n) = n!$$

(4) $D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$$D^n\left(\frac{1}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}$$

(5) $D^n\{\log(ax + b)\} = \frac{(-1)^{n-1}(n-1)!}{(ax+b)^n}\, a^n$

$$D^n(\log x) = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

(6) $D^n(e^{ax}) = a^n e^{ax}$

(7) $D^n(a^{mx}) = m^n (\log a)^n a^{mx}$

(8) $D^n\{\sin(ax + b)\} = a^n \sin(ax + b + n\frac{\pi}{2})$

$$D^n(\sin x) = \sin(x + n\frac{\pi}{2})$$

(9) $D^n\{\cos(ax + b)\} = a^n \cos(ax + b + n\frac{\pi}{2})$

$$D^n(\cos x) = \cos(x + n\frac{\pi}{2})$$

(10) $D^n\{e^{ax}\sin(bx + c)\} = (a^2 + b^2)^{n/2} e^{ax} \sin(bx + c + n\tan^{-1}\frac{b}{a})$

(11) $D^n\{e^{ax}\cos(bx + c)\} = (a^2 + b^2)^{n/2} e^{ax} \cos(bx + c + n\tan^{-1}\frac{b}{a})$

(12) $D^n(\tan^{-1}\frac{x}{a}) = \frac{(-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta}{a^n}$;

Where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$

(13) $D^n(\tan^{-1}x) = (-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta$,

Where $\theta = \tan^{-1}\left(\frac{1}{x}\right)$

DERIVATIVE OF A DETERMINANT :

If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h, l, m, n, u, v, w are differentiable functions of x, then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

L' HOSPITAL'S RULE :

If f(x) & g(x) are functions of x such that:

(i) $\underset{x\to a}{\text{Limit}}\ f(x) = \infty = \underset{x\to a}{\text{Limit}}\ g(x)$

OR

(ii) $\underset{x\to a}{\text{Limit}}\ f(x) = 0 = \underset{x\to a}{\text{Limit}}\ g(x)$, both f(x) and g(x) are continuous at x = a, both f(x) and g(x) are differentiable at x = a and both $f'(x)$ and $g'(x)$ are continuous at x = a,

then $\underset{x\to a}{\text{Limit}}\ \frac{f(x)}{g(x)} = \underset{x\to a}{\text{Limit}}\ \frac{f'(x)}{g'(x)}$

LEIBNITZ THEOREM

If u and v are two functions such that their nth derivatives exist, then the nth derivative of their product is given by

$$D^n(uv) = (D^n u)v + {}^nC_1 D^{n-1}u.\,Dv + {}^nC_2 D^{n-2}u.\,D^2v + \ldots.. + \ldots.. + {}^nC_r D^{n-r}u.\,D^r v + \ldots + u.D^n v$$

This theorem is used to find the nth derivative of a product of two functions. While using this theorem, the second function in the product should be taken that function whose successive derivatives start to vanish (if it is possible) after some steps and the first function be taken whose nth derivative is easily known.

Solved Examples

30 If $y = x^3 \cos x$ then find $D^n y$.

Sol. From Leibnitz theorem, choose cos x as first and x^3 as second function, then

$D^n(\cos x.\,x^3) = D^n(\cos x)(x^3) + {}^nC_1 D^{n-1}(\cos x)(Dx^3) + {}^nC_2 D^{n-2}(\cos x).(D^2x^3) + {}^nC_3 D^{n-3}(\cos x).(D^3x^3)$

$$= x^3 \cos\left(x + \frac{n\pi}{2}\right) + n.3x^2.\cos\left(x + \frac{(n-1)\pi}{2}\right) + \frac{n(n-1)}{1.2}.6x.\cos\left(x + \frac{(n-2)\pi}{2}\right) + \frac{n(n-1)(n-2)}{1.2.3}.6.\cos\left(x + \frac{(n-3)\pi}{2}\right)$$

$$= x^3\cos\left(x + \frac{n\pi}{2}\right) + 3nx^2\sin\left(x + \frac{n\pi}{2}\right) - 3n(n-1)x\cos\left(x + \frac{n\pi}{2}\right) - n(n-1)(n-2)\sin\left(x + \frac{n\pi}{2}\right)$$